5.3 MA Model

MA (Moving Average，移動平均) Model:

1. MA(q)

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

2. **Invertibility (反転可能性):**

The q solutions of x from $\theta(x) = 1 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_q x^q = 0$ の q are outside the unit circle.

$\Longrightarrow$ MA(q) model is rewritten as AR(∞) model.

Example: MA(1) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1}$

1. **Mean of MA(1) Process:**

$$\mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1}) = \mathrm{E}(\epsilon_t) + \theta_1 \mathrm{E}(\epsilon_{t-1}) = 0$$

2. **Autocovariance Function of MA(1) Process:**

$$\begin{aligned}\gamma(0) &= \mathrm{E}(y_t^2) = \mathrm{E}(\epsilon_t + \theta_1 \epsilon_{t-1})^2 = \mathrm{E}(\epsilon_t^2 + 2\theta_1 \epsilon_t \epsilon_{t-1} + \theta_1^2 \epsilon_{t-1}^2) \\ &= \mathrm{E}(\epsilon_t^2) + 2\theta_1 \mathrm{E}(\epsilon_t \epsilon_{t-1}) + \theta_1^2 \mathrm{E}(\epsilon_{t-1}^2) = (1 + \theta_1^2)\sigma_\epsilon^2\end{aligned}$$

$$\gamma(1) = \mathrm{E}(y_t y_{t-1}) = \mathrm{E}((\epsilon_t + \theta_1\epsilon_{t-1})(\epsilon_{t-1} + \theta_1\epsilon_{t-2})) = \theta_1\sigma_\epsilon^2$$

$$\gamma(2) = \mathrm{E}(y_t y_{t-2}) = \mathrm{E}((\epsilon_t + \theta_1\epsilon_{t-1})(\epsilon_{t-2} + \theta_1\epsilon_{t-3})) = 0$$

3. **Autocorrelation Function of MA(1) Process:**

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \begin{cases} \dfrac{\theta_1}{1+\theta_1^2}, & \text{for } \tau = 1, \\ 0, & \text{for } \tau = 2, 3, \cdots. \end{cases}$$

Let x be $\rho(1)$.

$$\frac{\theta_1}{1+\theta_1^2} = x, \qquad \text{i.e.,} \qquad x\theta_1^2 - \theta + x = 0.$$

θ_1 should be a real number.

$$1 - 4x^2 > 0, \qquad \text{i.e.,} \qquad -\frac{1}{2} \leq \rho(1) \leq \frac{1}{2}.$$

4. **Invertibility Condition of MA(1) Process:**

$$
\begin{aligned}
\epsilon_t &= -\theta_1 \epsilon_{t-1} + y_t \\
&= (-\theta_1)^2 \epsilon_{t-2} + y_t + (-\theta_1) y_{t-1} \\
&= (-\theta_1)^3 \epsilon_{t-3} + y_t + (-\theta_1) y_{t-1} + (-\theta_1)^2 y_{t-2} \\
&\qquad \vdots \\
&= (-\theta_1)^s \epsilon_{t-s} + y_t + (-\theta_1) y_{t-1} + (-\theta_1)^2 y_{t-2} + \cdots + (-\theta_1)^{t-s+1} y_{t-s+1}
\end{aligned}
$$

When $(-\theta_1)^s \epsilon_{t-s} \longrightarrow 0$, the MA(1) model is written as the AR(∞) model, i.e.,

$$
y_t = -(-\theta_1) y_{t-1} - (-\theta_1)^2 y_{t-2} - \cdots - (-\theta_1)^{t-s+1} y_{t-s+1} - \cdots + \epsilon_t
$$

That is, $|\theta_1| < 1$ represents the invertibility condition.

5. **Partial Autocorrelation Function of MA(1) Process:**

$$\phi_{1,1} = \rho(1) = \frac{\theta_1}{1+\theta_1^2} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}$$

$$\Longrightarrow \quad \phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{-\rho(1)^2}{1-\rho(1)^2} = \frac{-\theta_1^2}{1+\theta_1^2+\theta_1^4} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \end{pmatrix}$$

$$\implies \quad \phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{vmatrix}} = \frac{\rho(1)^3}{1 - 2\rho(1)^2} \neq 0$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) & \rho(3) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(1) \\ \rho(3) & \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{4,1} \\ \phi_{4,2} \\ \phi_{4,3} \\ \phi_{4,4} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \\ \phi(4) \end{pmatrix}$$

$$\Longrightarrow \quad \phi_{4,4} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(2) & \rho(1) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(3) \\ \rho(3) & \rho(2) & \rho(1) & \rho(4) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) & \rho(3) \\ \rho(1) & 1 & \rho(1) & \rho(2) \\ \rho(2) & \rho(1) & 1 & \rho(1) \\ \rho(3) & \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & 0 & \rho(1) \\ \rho(1) & 1 & \rho(1) & 0 \\ 0 & \rho(1) & 1 & 0 \\ 0 & 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 & 0 \\ \rho(1) & 1 & \rho(1) & 0 \\ 0 & \rho(1) & 1 & \rho(1) \\ 0 & 0 & \rho(1) & 1 \end{vmatrix}} \neq 0$$

As a result, $\phi_{k,k} \neq 0$ for all $k = 1, 2, \cdots$

6. **Likelihood Function of MA(1) Process:**

The autocovariance functions are: $\gamma(0) = (1+\theta_1^2)\sigma_\epsilon^2$, $\gamma(1) = \theta_1\sigma_\epsilon^2$, and $\gamma(\tau) = 0$ for $\tau = 2, 3, \cdots$.

The joint distribution of $y_1, y_2, \cdots, y_T$ is:

$$f(y_1, y_2, \cdots, y_T) = \frac{1}{(2\pi)^{T/2}}|V|^{-1/2} \exp\left(-\frac{1}{2}Y'V^{-1}Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \qquad V = \sigma_\epsilon^2 \begin{pmatrix} 1+\theta_1^2 & \theta_1 & 0 & \cdots & 0 \\ \theta_1 & 1+\theta_1^2 & \theta_1 & \ddots & \vdots \\ 0 & \theta_1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1+\theta_1^2 & \theta_1 \\ 0 & \cdots & 0 & \theta_1 & 1+\theta_1^2 \end{pmatrix}.$$

7. **MA(1) +drift:** $y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}$

Mean of MA(1) Process:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L$.

Taking the expectation,

$$\mathrm{E}(y_t) = \mu + \theta(L)\mathrm{E}(\epsilon_t) = \mu.$$

Example: MA(2) Model: $y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$

1. Autocovariance Function of MA(2) Process:

$$\gamma(\tau) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma_\epsilon^2, & \text{for } \tau = 0, \\ (\theta_1 + \theta_1\theta_2)\sigma_\epsilon^2, & \text{for } \tau = 1, \\ \theta_2\sigma_\epsilon^2, & \text{for } \tau = 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. let $-1/\beta_1$ and $-1/\beta_2$ be two solutions of x from $\theta(x) = 0$.

   For invertibility condition, both β_1 and β_2 should be less than one in absolute value.

   Then, the MA(2) model is represented as:

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

$$= (1 + \theta_1 L + \theta_2 L^2)\epsilon_t$$

$$= (1 + \beta_1 L)(1 + \beta_2 L)\epsilon_t$$

AR(∞) representation of the MA(2) model is given by:

$$\begin{aligned}
\epsilon_t &= \frac{1}{(1 + \beta_1 L)(1 + \beta_2 L)} y_t \\
&= \left(\frac{\beta_1/(\beta_1 - \beta_2)}{1 + \beta_1 L} + \frac{-\beta_2/(\beta_1 - \beta_2)}{1 + \beta_2 L} \right) y_t
\end{aligned}$$

3. **Likelihood Function:**

$$f(y_1, y_2, \cdots, y_T) = \frac{1}{(2\pi)^{T/2}} |V|^{-1/2} \exp\left(-\frac{1}{2} Y' V^{-1} Y\right)$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix}, \quad V = \sigma_\epsilon^2 \begin{pmatrix} 1+\theta_1^2+\theta_2^2 & \theta_1+\theta_1\theta_2 & \theta_2 & & 0 \\ \theta_1+\theta_1\theta_2 & 1+\theta_1^2+\theta_2^2 & \theta_1+\theta_1\theta_2 & \ddots & \\ \theta_2 & \theta_1+\theta_1\theta_2 & \ddots & \ddots & \theta_2 \\ & \ddots & \ddots & 1+\theta_1^2+\theta_2^2 & \theta_1+\theta_1\theta_2 \\ 0 & & \theta_2 & \theta_1+\theta_1\theta_2 & 1+\theta_1^2+\theta_2^2 \end{pmatrix}$$

4. **MA(2) +drift:** $y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2$.

Therefore,

$$\mathrm{E}(y_t) = \mu + \theta(L)\mathrm{E}(\epsilon_t) = \mu$$

Example: MA(q) Model: $y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

1. **Mean of MA(q) Process:**

$$\mathrm{E}(y_t) = \mathrm{E}(\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}) = 0$$

2. **Autocovariance Function of MA(q) Process:**

$$\gamma(\tau) = \begin{cases} \sigma_\epsilon^2(\theta_0\theta_\tau + \theta_1\theta_{\tau+1} + \cdots + \theta_{q-\tau}\theta_q) = \sigma_\epsilon^2 \sum_{i=0}^{q-\tau} \theta_i\theta_{\tau+i}, & \tau = 1, 2, \cdots, q, \\ 0, & \tau = q+1, q+2, \cdots, \end{cases}$$

where $\theta_0 = 1$.

3. MA(q) process is stationary.

4. **MA(q) +drift:** $y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}$

Mean:

$$y_t = \mu + \theta(L)\epsilon_t,$$

where $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$.

Therefore, we have:

$$\mathrm{E}(y_t) = \mu + \theta(L)\mathrm{E}(\epsilon_t) = \mu.$$