Homework Solutions

Royou Kiku

January 24th, 2022

1.

(1)

When $E(X'u) \neq 0$, the OLSE can be written as:

$$\hat{\beta} = \beta + (\frac{1}{n}X'X)^{-1}\frac{1}{n}X'u \longrightarrow \beta + M_{xx}^{-1}M_{xu} \neq \beta$$

Where：

$$\frac{1}{n}X'X \longrightarrow M_{xx}, \quad \frac{1}{n}X'u \longrightarrow M_{xu} \neq 0$$

Which imply that plim$\hat{\beta} \neq \beta$. In such case OLSE is inconsistent.

(2)

$$\hat{\beta} = \beta + (X'X)^{-1}X'u$$

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u$$

We assume that

$$\frac{1}{n}\sum_{i=1}^{n} x'_i x_i = \frac{1}{n}X'X \longrightarrow M_{xx}$$

Applying Central limit Theorem

$$\frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx})$$

Therefore:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n}X'X\right)^{-1}\frac{1}{\sqrt{n}}X'u \longrightarrow N(0, \sigma^2 M_{xx}^{-1})$$

(3)

Multiplying Z' on both sides of the regression model $y = X\beta + u$, we obtain:

$$Z'y = Z'X\beta + Z'u$$

The matrix Z is not correlated with u, we can get $\frac{1}{n}Z'u \longrightarrow M_{zu} = 0$.

Then, take plim on both sides.

$$plim\left(\frac{1}{n}Z'y\right) = plim\left(\frac{1}{n}Z'X\right)\beta + plim\left(\frac{1}{n}Z'u\right) = plim\left(\frac{1}{n}Z'X\right)\beta$$

$$\beta = \left(plim\left(\frac{1}{n}Z'X\right)\right)^{-1} plim\left(\frac{1}{n}Z'y\right)$$

Therefore,

$$\tilde{\beta} = (Z'X)^{-1}Z'y$$

(4)

$$\tilde{\beta} = (Z'X)^{-1}Z'y = (Z'X)^{-1}Z'(X\beta + u) = \beta + (Z'X)^{-1}Z'u$$

$$\sqrt{n}(\tilde{\beta} - \beta) = \left(\frac{1}{n}Z'X\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right)$$

We assume that

$$\frac{1}{n}\sum_{i=1}^{n} z'_i x_i = \frac{1}{n}Z'X \longrightarrow M_{zx} \quad and \quad \frac{1}{n}\sum_{i=1}^{n} z'_i z_i = \frac{1}{n}Z'Z \longrightarrow M_{zz}$$

Applying the Central limit Theorem

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz})$$

Therefore:

$$\sqrt{n}(\tilde{\beta} - \beta) = \left(\frac{1}{n}Z'X\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right) \longrightarrow N(0, \sigma^2 M_{zx}^{-1} M_{zz} M'^{-1}_{zx})$$

(5)

$$H_0: X\ and\ u\ are\ independent.$$

$$H_1: X\ and\ u\ are\ not\ independent.$$

$\hat{\beta}$ is consistent and efficient under H_0, but is not consistent under H_1.So, under H_0, we choose $\hat{\beta}$.
$\tilde{\beta}$ is consistent under H_0 and H_1, but is not efficient under H_0. So, under H_1, we choose $\tilde{\beta}$.

2.

(6)

For simplicity, we assume that $v_i + u_{it} \sim N(0, \Omega)$,

The likelihood function is

$$L(\beta, \sigma_u^2, \sigma_v^2) = (2\pi)^{-\frac{nT}{2}}|\Omega|^{-\frac{1}{2}} \times exp\left\{-\frac{1}{2}(y - X\beta)'\Omega^{-1}(y - X\beta)\right\}$$

The log-likelihood function is

$$logL(\beta, \sigma_u^2, \sigma_v^2) = \frac{-nT}{2}\log(2\pi) - \frac{1}{2}\log|\Omega| - \frac{1}{2}(y - X\beta)'\Omega^{-1}(y - X\beta)$$

The MLE of β, denoted by $\tilde{\beta}$, is

$$\tilde{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

(7)

We define that $\bar{y}_i = \frac{1}{T}\sum_{t=1}^{T} y_{it}$, $\bar{x}_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}$, $\bar{u}_i = \frac{1}{T}\sum_{t=1}^{T} u_{it}$, $\tilde{y}_{it} = y_{it} - \bar{y}_i$, $\tilde{x}_{it} = x_{it} - \bar{x}_i$, $\tilde{u}_{it} = u_{it} - \bar{u}_i$.

By (1)-(2), we can get (3).

$$y_{it} = X_{it}\beta + v_i + u_{it} \qquad (1)$$

$$\bar{y}_i = \bar{x}_i\beta + v_i + \bar{u}_i \qquad (2)$$

$$y_{it} - \bar{y}_i = (x_{it} - \bar{x}_i)\beta + u_{it} - \bar{u}_i$$

$$\tilde{y}_{it} = \tilde{x}_{it}\beta + \tilde{u}_{it} \qquad (3)$$

Finally, we get a consistent estimator of β, denoted by $\hat{\beta}$.

$$\hat{\beta} = \left(\sum_{i=1}^{N}\sum_{t=1}^{T} \tilde{x}_{it}\tilde{x}'_{it}\right)^{-1} \sum_{i=1}^{N}\sum_{t=1}^{T} \tilde{x}_{it}\tilde{y}_{it}$$

(8)

$$H_0: v_i \text{ is uncorrelated with } X_{it}.$$
$$H_1: v_i \text{ is correlated with } X_{it}.$$

Under H_0, $\tilde{\beta}$ is consistent and efficient. Thus, we should use $\tilde{\beta}$. Note that $\hat{\beta}$ is also consistent and not efficient.

Under H_1, only $\hat{\beta}$ is consistent. Thus, we should use $\hat{\beta}$.

3.

(9)

We assume that $u \sim N(0, \sigma^2 I)$, minimize $(X - Z\Gamma)'(X - Z\Gamma)$, we can get

$$\hat{\Gamma} = (Z'Z)^{-1}Z'X$$
$$\hat{X} = Z(Z'Z)^{-1}Z'X$$

Utilizing $\hat{X}$, we obtain a consistent estimator β.

$$\hat{\beta} = \left(\hat{X}'X\right)^{-1}\hat{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$$

(10)

$$\hat{\beta} = \beta + (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'u$$

We assume that

$$\frac{1}{n}X'Z \longrightarrow M_{xz} \quad \text{and} \quad \frac{1}{n}Z'Z \longrightarrow M_{zz}$$

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz})$$

Therefore:

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{n}Z'X\right)\right)^{-1} \times \left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'u\right)$$
$$\longrightarrow N(0, \sigma^2(M_{xz}M_{zz}^{-1}M'_{xz})^{-1})$$