# Econometrics I

## (Wed., 8:50-10:20)

### Room # 1 (法経講義棟)

Face-to-Face and Online Class

Video Recoding in Zoom (but we do not open.)

- This class is based on **Statistics** (**統計**，Wed 2 and Fri 3, Spring-Summer Term, 『コア・テキスト　統計学』大屋 幸輔 著，新世社) and **Econometrics** (**計量経済**, Tue 1 and Thu 2, Fall-Winter Term, 『計量経済学』山本 拓 著，新世社), which are provided by Department of Economics, and **Basic Statistics** (**統計基礎**，Tue 5, Spring-Summer Term,『コア・テキスト　統計学』大屋 幸輔 著，新世社), provided by Graduate School of Economics.

Thus, Statistics and Econometrics of undergraduate level are prerequisites.

- Furthermore, **Special Lectures in Economics (Statistical Analysis)** or **Statistical Analysis** (**統計解析**，Wed 2, Spring-Summer Term), provided by Graduate School of Economics, should be studied with this class.

Or, do self-study using the lecture notes of

`http://www2.econ.osaka-u.ac.jp/~tanizaki/class/2012/econome1/index.htm`

(The notes are written in English with Japanese translation for econometrics terms).

# TA Session (Online):

⟶ Question-and-Answer Session，Not Lecture in This Term

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**Date:** Thur. 16:50-18:20

**Zoom Info. :** Meeting ID: `853 842 8522`
Passcode: `Keiryou1ta`

**Contents:** Basic Statistics, Matrix Algebra, and etc.

Ask TAs directly in the TA session
if you have questions about class, homework and etc.

# 1 Regression Analysis (回帰分析) — Review

## 1.1 Setup of the Model

When $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ are available, suppose that there is a linear relationship between $y$ and $x$, i.e.,

$$y_i = \beta_1 + \beta_2 x_i + u_i, \tag{1}$$

for $i = 1, 2, \cdots, n$. $x_i$ and $y_i$ denote the $i$th observations.

$\longrightarrow$ **Single (or simple) regression model (単回帰モデル)**

$y_i$ is called the **dependent variable (従属変数)** or the **explained variable (被説明変数)**, while $x_i$ is known as the **independent variable (独立変数)** or the **explanatory (or explaining) variable (説明変数)**.

$$\beta_1 = \textbf{Intercept (切片)}, \qquad \beta_2 = \textbf{Slope (傾き)}$$

$\beta_1$ and $\beta_2$ are unknown **parameters (パラメータ，母数)** to be estimated.

$\beta_1$ and $\beta_2$ are called the **regression coefficients (回帰係数)**.

$u_i$ is the unobserved **error term (誤差項)** assumed to be a random variable with mean zero and variance $\sigma^2$.
$\sigma^2$ is also a parameter to be estimated.

$x_i$ is assumed to be **nonstochastic (非確率的)**, but $y_i$ is **stochastic (確率的)** because $y_i$ depends on the error $u_i$.

The error terms $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed, which is called ***iid***. $\longrightarrow$ discussed later.

It is assumed that $u_i$ has a distribution with mean zero, i.e., $\mathrm{E}(u_i) = 0$ is assumed.

Taking the expectation on both sides of (1), the expectation of $y_i$ is represented as:

$$\begin{aligned} \mathrm{E}(y_i) &= \mathrm{E}(\beta_1 + \beta_2 x_i + u_i) = \beta_1 + \beta_2 x_i + \mathrm{E}(u_i) \\ &= \beta_1 + \beta_2 x_i, \end{aligned} \tag{2}$$

for $i = 1, 2, \cdots, n$. Using $\mathrm{E}(y_i)$ we can rewrite (1) as $y_i = \mathrm{E}(y_i) + u_i$.

(2) represents the true regression line.

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be estimates of $\beta_1$ and $\beta_2$.

Replacing $\beta_1$ and $\beta_2$ by $\hat{\beta}_1$ and $\hat{\beta}_2$, (1) turns out to be:

$$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_i + e_i, \tag{3}$$

for $i = 1, 2, \cdots, n$, where $e_i$ is called the **residual (残差)**.

The residual $e_i$ is taken as the experimental value (or realization) of $u_i$.

We define $\hat{y}_i$ as follows:

$$\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 x_i, \tag{4}$$

for $i = 1, 2, \cdots, n$, which is interpreted as the **predicted value (予測値)** of $y_i$.

(4) indicates the estimated regression line, which is different from (2).

Moreover, using $\hat{y}_i$ we can rewrite (3) as $y_i = \hat{y}_i + e_i$.

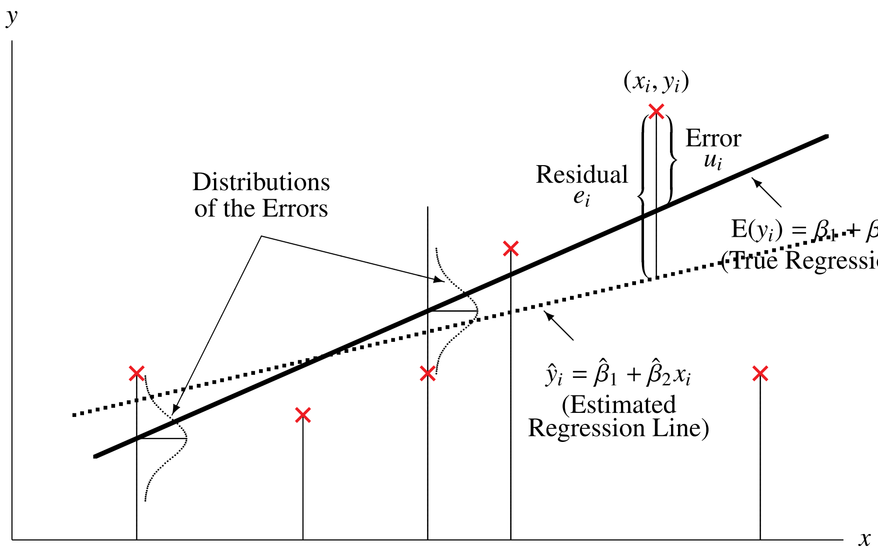
(2) and (4) are displayed in Figure 1.

Consider the case of $n = 6$ for simplicity. $\times$ indicates the observed data series.

The true regression line (2) is represented by the solid line, while the estimated regression line (4) is drawn with the dotted line.

Based on the observed data, $\beta_1$ and $\beta_2$ are estimated as: $\hat{\beta}_1$ and $\hat{\beta}_2$.

**Figure 1. True and Estimated Regression Lines (回帰直線)**




In the next section, we consider how to obtain the estimates of $\beta_1$ and $\beta_2$, i.e., $\hat{\beta}_1$ and $\hat{\beta}_2$.

## 1.2 Ordinary Least Squares Estimation

Suppose that $(x_1, y_1)$, $(x_2, y_2)$, $\cdots$, $(x_n, y_n)$ are available.

For the regression model (1), we consider estimating $\beta_1$ and $\beta_2$.

Replacing $\beta_1$ and $\beta_2$ by their estimates $\hat{\beta}_1$ and $\hat{\beta}_2$, remember that the residual $e_i$ is given by:

$$e_i = y_i - \hat{y}_i = y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i.$$

The sum of squared residuals is defined as follows:

$$S(\hat{\beta}_1, \hat{\beta}_2) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2.$$

It might be plausible to choose the $\hat{\beta}_1$ and $\hat{\beta}_2$ which minimize the sum of squared residuals, i.e., $S(\hat{\beta}_1, \hat{\beta}_2)$.

This method is called the **ordinary least squares estimation (最小二乗法，OLS)**.

To minimize $S(\hat{\beta}_1, \hat{\beta}_2)$ with respect to $\hat{\beta}_1$ and $\hat{\beta}_2$, we set the partial derivatives equal to zero:

$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0,$$
$$\frac{\partial S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2} = -2 \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i) = 0.$$

The second order condition for minimization is:

$$\begin{pmatrix} \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1^2} & \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_1 \partial \hat{\beta}_2} \\ \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2 \partial \hat{\beta}_1} & \frac{\partial^2 S(\hat{\beta}_1, \hat{\beta}_2)}{\partial \hat{\beta}_2^2} \end{pmatrix} = \begin{pmatrix} 2n & 2\sum_{i=1}^{n} x_i \\ 2\sum_{i=1}^{n} x_i & 2\sum_{i=1}^{n} x_i^2 \end{pmatrix}$$

should be a positive definite matrix.

The diagonal elements $2n$ and $2\sum_{i=1}^{n} x_i^2$ are positive.

The determinant:

$$\begin{vmatrix} 2n & 2\sum_{i=1}^{n} x_i \\ 2\sum_{i=1}^{n} x_i & 2\sum_{i=1}^{n} x_i^2 \end{vmatrix} = 4n \sum_{i=1}^{n} x_i^2 - 4(\sum_{i=1}^{n} x_i)^2 = 4n \sum_{i=1}^{n} (x_i - \overline{x})^2$$

is positive. $\Longrightarrow$ The second-order condition is satisfied.

The first two equations yield the following two equations:

$$\overline{y} = \hat{\beta}_1 + \hat{\beta}_2 \overline{x}, \tag{5}$$

$$\sum_{i=1}^{n} x_i y_i = n\overline{x}\hat{\beta}_1 + \hat{\beta}_2 \sum_{i=1}^{n} x_i^2, \tag{6}$$

where $\overline{y} = \dfrac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

Multiplying (5) by $n\overline{x}$ and subtracting (6), we can derive $\hat{\beta}_2$ as follows:

$$\hat{\beta}_2 = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y}}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{7}$$

From (5), $\hat{\beta}_1$ is directly obtained as follows:

$$\hat{\beta}_1 = \overline{y} - \hat{\beta}_2 \overline{x}. \tag{8}$$

When the observed values are taken for $y_i$ and $x_i$ for $i = 1, 2, \cdots, n$, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimate**s (or simply the **least squares estimate**s, **最小二乗推定値**) of $\beta_1$ and $\beta_2$.

When $y_i$ for $i = 1, 2, \cdots, n$ are regarded as the random sample, we say that $\hat{\beta}_1$ and $\hat{\beta}_2$ are called the **ordinary least squares estimator**s (or the **least squares estimator**s, **最小二乗推定量**) of $\beta_1$ and $\beta_2$.

## 1.3 Properties of Least Squares Estimator

Equation (7) is rewritten as:

$$\begin{aligned}\hat{\beta}_2 &= \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x}) y_i}{\sum_{i=1}^n (x_i - \overline{x})^2} - \frac{\overline{y} \sum_{i=1}^n (x_i - \overline{x})}{\sum_{i=1}^n (x_i - \overline{x})^2} \\ &= \sum_{i=1}^n \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2} y_i = \sum_{i=1}^n \omega_i y_i. \end{aligned} \tag{9}$$

In the third equality, $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$ is utilized because of $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

In the fourth equality, $\omega_i$ is defined as: $\omega_i = \frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$.

$\omega_i$ is nonstochastic because $x_i$ is assumed to be nonstochastic.

$\omega_i$ has the following properties:

$$\sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} \frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 0, \tag{10}$$

$$\sum_{i=1}^{n} \omega_i x_i = \sum_{i=1}^{n} \omega_i (x_i - \overline{x}) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = 1, \tag{11}$$

$$\sum_{i=1}^{n} \omega_i^2 = \sum_{i=1}^{n} \left(\frac{x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2}\right)^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\left(\sum_{i=1}^{n}(x_i - \overline{x})^2\right)^2} = \frac{1}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{12}$$

The first equality of (11) comes from (10).

From now on, we focus only on $\hat{\beta}_2$, because usually $\beta_2$ is more important than $\beta_1$ in the regression model (1).

In order to obtain the properties of the least squares estimator $\hat{\beta}_2$, we rewrite (9) as:

$$\begin{aligned}\hat{\beta}_2 &= \sum_{i=1}^{n} \omega_i y_i = \sum_{i=1}^{n} \omega_i(\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i = \beta_2 + \sum_{i=1}^{n} \omega_i u_i. \end{aligned} \tag{13}$$

In the fourth equality of (13), (10) and (11) are utilized.

**[Review] Random Variables**:

Let $X_1$, $X_2$, $\cdots$, $X_n$ be $n$ random variables, which are mutually independently and identically distributed.

**mutually independent** $\Longrightarrow$ $f(x_i, x_j) = f_i(x_i)f_j(x_j)$ for $i \neq j$.

$f(x_i, x_j)$ denotes a joint distribution of $X_i$ and $X_j$.

$f_i(x)$ indicates a marginal distribution of $X_i$.

**identical** $\Longrightarrow$ $f_i(x) = f_j(x)$ for $i \neq j$.

**[End of Review]**

**[Review] Mean and Variance**:

Let $X$ and $Y$ be random variables (continuous type), which are independently distributed.

**Definition and Formulas:**

- $\mathrm{E}(g(X)) = \displaystyle\int g(x)f(x)\mathrm{d}x$ for a function $g(\cdot)$ and a density function $f(\cdot)$.
- $\mathrm{V}(X) = \mathrm{E}((X-\mu)^2) = \displaystyle\int (x-\mu)^2 f(x)\mathrm{d}x$ for $\mu = \mathrm{E}(X)$.
- $\mathrm{E}(aX+b) = a\mathrm{E}(X) + b$ and $\mathrm{V}(aX+b) = \mathrm{V}(aX) = a^2\mathrm{V}(X)$ for constant $a$ and $b$.
- $\mathrm{E}(X \pm Y) = \mathrm{E}(X) \pm \mathrm{E}(Y)$ and $\mathrm{V}(X \pm Y) = \mathrm{V}(X) + \mathrm{V}(Y)$.

**[End of Review]**

**Mean and Variance of $\hat{\beta}_2$:** $u_1, u_2, \cdots, u_n$ are assumed to be mutually independently and identically distributed with mean zero and variance $\sigma^2$, but they are not necessarily normal.

Remember that we do not need normality assumption to obtain mean and variance but the normality assumption is required to test a hypothesis.

From (13), the expectation of $\hat{\beta}_2$ is derived as follows:

$$\mathrm{E}(\hat{\beta}_2) = \mathrm{E}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \mathrm{E}(\sum_{i=1}^{n} \omega_i u_i) = \beta_2 + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) = \beta_2. \qquad (14)$$

It is shown from (14) that the ordinary least squares estimator $\hat{\beta}_2$ is an unbiased estimator of $\beta_2$.

From (13), the variance of $\hat{\beta}_2$ is computed as:

$$\mathrm{V}(\hat{\beta}_2) = \mathrm{V}(\beta_2 + \sum_{i=1}^{n} \omega_i u_i) = \mathrm{V}(\sum_{i=1}^{n} \omega_i u_i) = \sum_{i=1}^{n} \mathrm{V}(\omega_i u_i) = \sum_{i=1}^{n} \omega_i^2 \mathrm{V}(u_i)$$

$$= \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n} (x_i - \overline{x})^2}. \tag{15}$$

The third equality holds because $u_1, u_2, \cdots, u_n$ are mutually independent.

The last equality comes from (12).

Thus, $\mathrm{E}(\hat{\beta}_2)$ and $\mathrm{V}(\hat{\beta}_2)$ are given by (14) and (15).

**[Review] Three Good Properties on Estimator:**

$\theta$ : Parameter

$\hat{\theta}$ : Estimator of $\theta$, i.e., $\hat{\theta} = \hat{\theta}(X_1, X_2, \cdots, X_n)$,

where $X_1, X_2, \cdots, X_n$ are mutually independent random variables.

(*) Estimate of $\theta$: $\hat{\theta} = \hat{\theta}(x_1, x_2, \cdots, x_n)$, where $x_i$ denotes the observed data of $X_i$.

- **Unbiasedness (不偏性)**: $\mathrm{E}(\hat{\theta}) = \theta$.
- **Efficiency (有効性)**:

The minimum variance estimator within all the unbiased estimators.

(*) It is not easy to check efficiency in general. Instead, consider the **best linear unbiased estimator (BLUE, 最良線型不偏推定量)**.

- **Consistency (一致性)**: $\hat{\theta} \longrightarrow \theta$ as $n \longrightarrow \infty$. Note that $\hat{\theta}$ depends on # of obs.

**[End of Review]**

**Gauss-Markov Theorem (ガウス・マルコフ定理):** It has been discussed above that $\hat{\beta}_2$ is represented as (9), which implies that $\hat{\beta}_2$ is a linear estimator, i.e., linear in $y_i$.

In addition, (14) indicates that $\hat{\beta}_2$ is an unbiased estimator.

Therefore, summarizing these two facts, it is shown that $\hat{\beta}_2$ is a **linear unbiased estimator (線形不偏推定量)**.

Furthermore, here we show that $\hat{\beta}_2$ has minimum variance within a class of the linear unbiased estimators.

Consider the alternative linear unbiased estimator $\tilde{\beta}_2$ as follows:

$$\tilde{\beta}_2 = \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i) y_i,$$

where $c_i = \omega_i + d_i$ is defined and $d_i$ is nonstochastic.