Then, $\tilde{\beta}_2$ is transformed into:

$$\begin{aligned}\tilde{\beta}_2 &= \sum_{i=1}^{n} c_i y_i = \sum_{i=1}^{n} (\omega_i + d_i)(\beta_1 + \beta_2 x_i + u_i) \\ &= \beta_1 \sum_{i=1}^{n} \omega_i + \beta_2 \sum_{i=1}^{n} \omega_i x_i + \sum_{i=1}^{n} \omega_i u_i + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} d_i u_i \\ &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i u_i + \sum_{i=1}^{n} d_i u_i.\end{aligned}$$

Equations (10) and (11) are used in the forth equality.

Taking the expectation on both sides of the above equation, we obtain:

$$\begin{aligned}\mathrm{E}(\tilde{\beta}_2) &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i + \sum_{i=1}^{n} \omega_i \mathrm{E}(u_i) + \sum_{i=1}^{n} d_i \mathrm{E}(u_i) \\ &= \beta_2 + \beta_1 \sum_{i=1}^{n} d_i + \beta_2 \sum_{i=1}^{n} d_i x_i.\end{aligned}$$

Note that $d_i$ is not a random variable and that $\mathrm{E}(u_i) = 0$.

Since $\tilde{\beta}_2$ is assumed to be unbiased, we need the following conditions:

$$\sum_{i=1}^{n} d_i = 0, \qquad \sum_{i=1}^{n} d_i x_i = 0.$$

When these conditions hold, we can rewrite $\tilde{\beta}_2$ as:

$$\tilde{\beta}_2 = \beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i.$$

The variance of $\tilde{\beta}_2$ is derived as:

$$\begin{aligned} \mathrm{V}(\tilde{\beta}_2) &= \mathrm{V}\Big(\beta_2 + \sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = \mathrm{V}\Big(\sum_{i=1}^{n} (\omega_i + d_i) u_i\Big) = \sum_{i=1}^{n} \mathrm{V}\Big((\omega_i + d_i) u_i\Big) \\ &= \sum_{i=1}^{n} (\omega_i + d_i)^2 \mathrm{V}(u_i) = \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + 2 \sum_{i=1}^{n} \omega_i d_i + \sum_{i=1}^{n} d_i^2) \\ &= \sigma^2 (\sum_{i=1}^{n} \omega_i^2 + \sum_{i=1}^{n} d_i^2). \end{aligned}$$

From unbiasedness of $\tilde{\beta}_2$, using $\sum_{i=1}^n d_i = 0$ and $\sum_{i=1}^n d_i x_i = 0$, we obtain:

$$\sum_{i=1}^n \omega_i d_i = \frac{\sum_{i=1}^n (x_i - \overline{x}) d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n x_i d_i - \overline{x} \sum_{i=1}^n d_i}{\sum_{i=1}^n (x_i - \overline{x})^2} = 0,$$

which is utilized to obtain the variance of $\tilde{\beta}_2$ in the third line of the above equation.

From (15), the variance of $\hat{\beta}_2$ is given by: $\mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^n \omega_i^2$.

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}_2) \geq \mathrm{V}(\hat{\beta}_2),$$

because of $\sum_{i=1}^n d_i^2 \geq 0$.

When $\sum_{i=1}^n d_i^2 = 0$, i.e., when $d_1 = d_2 = \cdots = d_n = 0$,

we have the equality: $\mathrm{V}(\tilde{\beta}_2) = \mathrm{V}(\hat{\beta}_2)$.

Thus, in the case of $d_1 = d_2 = \cdots = d_n = 0$, $\hat{\beta}_2$ is equivalent to $\tilde{\beta}_2$.

As shown above, the least squares estimator $\hat{\beta}_2$ gives us the **minimum variance linear unbiased estimator (最小分散線形不偏推定量)**, or equivalently the **best linear unbiased estimator (最良線形不偏推定量，BLUE)**, which is called the **Gauss-Markov theorem (ガウス・マルコフ定理)**.

**Asymptotic Properties (漸近的性質) of $\hat{\beta}_2$:** We assume that as $n$ goes to infinity we have the following:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 \longrightarrow m < \infty,$$

where $m$ is a constant value. From (12), we obtain:

$$n\sum_{i=1}^{n}\omega_i^2 = \frac{1}{(1/n)\sum_{i=1}^{n}(x_i - \overline{x})} \longrightarrow \frac{1}{m}.$$

Note that $f(x_n) \longrightarrow f(m)$ when $x_n \longrightarrow m$, called **Slutsky's theorem (スルツキー定理)**, where $m$ is a constant value and $f(\cdot)$ is a function.

We show both **consistency (一致性)** of $\hat{\beta}_2$ and **asymptotic normality (漸近正規性)** of $\sqrt{n}(\hat{\beta}_2 - \beta_2)$.

● First, we prove that $\hat{\beta}_2$ is a consistent estimator of $\beta_2$.

**[Review] Chebyshev's inequality (チェビシェフの不等式)** is given by:

$$P(|X-\mu| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2}, \qquad \text{where } \mu = \mathrm{E}(X),\ \sigma^2 = \mathrm{V}(X) \text{ and any } \epsilon > 0.$$

**[End of Review]**

Replace $X$, E($X$) and V($X$) by:

$$\hat{\beta}_2, \qquad \mathrm{E}(\hat{\beta}_2) = \beta_2, \quad \text{and} \quad \mathrm{V}(\hat{\beta}_2) = \sigma^2 \sum_{i=1}^{n} \omega_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})}.$$

Then, when $n \longrightarrow \infty$, we obtain the following result:

$$P(|\hat{\beta}_2 - \beta_2| > \epsilon) \leq \frac{\sigma^2 \sum_{i=1}^{n} \omega_i^2}{\epsilon^2} = \frac{\sigma^2 n \sum_{i=1}^{n} \omega_i^2}{n\epsilon^2} \longrightarrow 0,$$

where $\sum_{i=1}^{n} \omega_i^2 \longrightarrow 0$ because $n \sum_{i=1}^{n} \omega_i^2 \longrightarrow \dfrac{1}{m}$ from the assumption.

Thus, we obtain the result that $\hat{\beta}_2 \longrightarrow \beta_2$ as $n \longrightarrow \infty$.

Therefore, we can conclude that $\hat{\beta}_2$ is a **consistent estimator (一致推定量)** of $\beta_2$.

● Next, we want to show that $\sqrt{n}(\hat{\beta}_2 - \beta_2)$ is asymptotically normal.

**[Review]** The **Central Limit Theorem (中心極限定理, CLT)** is: for random variables $X_1, X_2, \cdots, X_n$,

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\sum_{i=1}^n X_i - \mathrm{E}(\sum_{i=1}^n X_i)}{\sqrt{\mathrm{V}(\sum_{i=1}^n X_i)}} \longrightarrow N(0, 1), \qquad \text{as} \qquad n \longrightarrow \infty,$$

where $\overline{X} = \dfrac{1}{n}\sum_{i=1}^n X_i$.

$X_1, X_2, \cdots, X_n$ are not necessarily iid, if $\lim_{n\to\infty} n\mathrm{V}(\overline{X})$ is finite in this case.

**[End of Review]**

Note that $\hat{\beta}_2 = \beta_2 + \sum_{i=1}^n \omega_i u_i$ as in (13), and $X_i$ is replaced by $\omega_i u_i$.

From the central limit theorem, asymptotic normality is shown as follows:

$$\frac{\sum_{i=1}^n \omega_i u_i - \mathrm{E}(\sum_{i=1}^n \omega_i u_i)}{\sqrt{\mathrm{V}(\sum_{i=1}^n \omega_i u_i)}} = \frac{\sum_{i=1}^n \omega_i u_i}{\sigma \sqrt{\sum_{i=1}^n \omega_i^2}} = \frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0, 1),$$

where

- $\mathrm{E}(\sum_{i=1}^n \omega_i u_i) = 0$,
- $\mathrm{V}(\sum_{i=1}^n \omega_i u_i) = \sigma^2 \sum_{i=1}^n \omega_i^2$, and
- $\sum_{i=1}^n \omega_i u_i = \hat{\beta}_2 - \beta_2$

are substituted in the first and second equalities.

Moreover, we can rewrite as follows:

$$\frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} = \frac{\sqrt{n}(\hat{\beta}_2 - \beta_2)}{\sigma / \sqrt{(1/n)\sum_{i=1}^n (x_i - \overline{x})^2}}.$$

Replacing $(1/n)\sum_{i=1}^n (x_i - \overline{x})^2$ by its converged value $m$, we have:

$$\frac{\sqrt{n}(\hat{\beta}_2 - \beta_2)}{\sigma / \sqrt{m}} \longrightarrow N(0, 1),$$

which implies

$$\sqrt{n}(\hat{\beta}_2 - \beta_2) \longrightarrow N(0, \frac{\sigma^2}{m}).$$

Thus, the asymptotic normality of $\sqrt{n}(\hat{\beta}_2 - \beta_2)$ is shown.

We can use either of the following two:

$$\frac{\hat{\beta}_2 - \beta_2}{\sigma / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0, 1),$$

$$\sqrt{n}(\hat{\beta}_2 - \beta_2) \longrightarrow N(0, \frac{\sigma^2}{m}), \qquad \text{where } m = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2.$$

Finally, replacing $\sigma^2$ by its consistent estimator $s^2$, it is known as follows:

$$\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}} \longrightarrow N(0,1), \tag{16}$$

where $s^2$ is defined as:

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n} e_i^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2, \tag{17}$$

which is a consistent and unbiased estimator of $\sigma^2$. $\longrightarrow$ Proved later.

Thus, using (16), in large sample we can construct the confidence interval and test the hypothesis.

**[Review] Confidence Interval (信頼区間，区間推定)):**

Suppose $X_1, X_2, \cdots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$. $\longrightarrow$ No N assumption

From CLT, $\dfrac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0, 1).$

Replacing $\sigma^2$ by $S^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \overline{X})^2$, we have: $\dfrac{\overline{X} - \mu}{S/\sqrt{n}} \longrightarrow N(0, 1).$

That is, for large $n$,

$$P\Big(-1.96 < \frac{\overline{X} - \mu}{S/\sqrt{n}} < 1.96\Big) = 0.95, \text{ i.e., } P\Big(\overline{X} - 1.96\frac{S}{\sqrt{n}} < \mu < \overline{X} + 1.96\frac{S}{\sqrt{n}}\Big) = 0.95.$$

Note that 1.96 is obtained from the normal distribution table.

Then, replacing the estimators $\overline{X}$ and $S^2$ by the estimates $\overline{x}$ and $s^2$, we obtain the 95% confidence interval of $\mu$ as follows:

$$(\overline{x} - 1.96\frac{s}{\sqrt{n}},\ \overline{x} + 1.96\frac{s}{\sqrt{n}}).$$

**[End of Review]**

Going back to OLS, we have:

$$\frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \longrightarrow N(0, 1).$$

Therefore,

$$P\Big(-2.576 < \frac{\hat{\beta}_2 - \beta_2}{s/\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} < 2.576\Big) = 0.99,$$

i.e.,

$$P\Big(\hat{\beta}_2 - 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} < \beta_2 < \hat{\beta}_2 + 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}\Big) = 0.99.$$

Note that 2.576 is 0.005 value of $N(0, 1)$, which comes from the statistical table.

Thus, the 99% confidence interval of $\beta_2$ is:

$$\Big(\hat{\beta}_2 - 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}},\ \hat{\beta}_2 + 2.576\frac{s}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}\Big),$$

where $\hat{\beta}_2$ and $s^2$ should be replaced by the observed data.

**[Review] Testing the Hypothesis (仮説検定)**:

Suppose that $X_1, X_2, \cdots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$.

From CLT, $\dfrac{\overline{X} - \mu}{S/\sqrt{n}} \longrightarrow N(0,1)$, where $S^2 = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \overline{X})^2$, which is known as the unbiased estimator of $\sigma^2$.

- The null hypothesis $H_0: \ \mu = \mu_0$, where $\mu_0$ is a fixed number.
- The alternative hypothesis $H_1: \ \mu \neq \mu_0$

Under the null hypothesis, in large sample we have the following distribution:

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim N(0,1).$$

Replacing $\overline{X}$ and $S^2$ by $\overline{x}$ and $s^2$, compare $\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$ and $N(0,1)$.

$H_0$ is rejected at significance level 0.05 when $\left|\dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}\right| > 1.96$.

**[End of Review]**

In the case of OLS, the hypotheses are as follows:

- The null hypothesis $H_0 : \beta_2 = \beta_2^*$
- The alternative hypothesis $H_1 : \beta_2 \neq \beta_2^*$

Under $H_0$, in large sample,

$$\frac{\hat{\beta}_2 - \beta_2^*}{s / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} \sim N(0, 1).$$

Replacing $\hat{\beta}_2$ and $s^2$ by the observed data, compare $\dfrac{\hat{\beta}_2 - \beta_2^*}{s / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}$ and $N(0, 1)$.

$H_0$ is rejected at significance level 0.05 when $\left|\dfrac{\hat{\beta}_2 - \beta_2^*}{s / \sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}}\right| > 1.96$.