[Review]

Central Limit Theorem (中心極限定理, CLT):

For random variables $X_1, X_2, \cdots, X_n$ with $\mathrm{E}(X_i) = \mu$ and $\mathrm{V}(X_i) = \sigma^2 < \infty$ for all i,

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0,1), \quad \text{as} \quad n \longrightarrow \infty,$$

where $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$.

Note that $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{V}(\overline{X}) = \dfrac{\sigma^2}{n}$.

Or equivalently,

$$\sqrt{n}(\overline{X} - \mu) \longrightarrow N(0, \sigma^2), \quad \text{as} \quad n \longrightarrow \infty.$$

Note that $\mathrm{V}(\sqrt{n}(\overline{X} - \mu)) = \sigma^2$.

Central Limit Theorem (中心極限定理, CLT) II:

For random variables $X_1, X_2, \cdots, X_n$ with $\mathrm{E}(X_i) = \mu$ and $\mathrm{V}(X_i) = \sigma_i^2 < \infty$ for all i,

$$\sqrt{n}(\overline{X} - \mu) \longrightarrow N(0, \sigma^2), \qquad \text{as} \qquad n \longrightarrow \infty,$$

where $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2$ is defined.

Central Limit Theorem (中心極限定理, CLT) III:

$X_1, X_2, \cdots, X_n$ are not necessarily iid, if $\lim_{n\to\infty} \mathrm{V}(\sqrt{n}(\overline{X} - \mathrm{E}(\overline{X}))) = \lim_{n\to\infty} n\mathrm{V}(\overline{X})$ is finite in this case.

$$\frac{\overline{X} - \mathrm{E}(\overline{X})}{\sqrt{\mathrm{V}(\overline{X})}} \longrightarrow N(0, 1), \qquad \text{as} \qquad n \longrightarrow \infty,$$

[End of Review]

In the case of regression model: $y_i = \beta_1 + \beta_2 x_i + u_i$, note that
$\hat{\beta}_2 = \beta_2 + \sum_{i=1}^n \omega_i u_i = \beta_2 + \dfrac{\sum_{i=1}^n (x_i - \overline{x})u_i}{\sum_{j=1}^n (x_j - \overline{x})^2}$.
$(x_i - \overline{x})u_i$ is taken as one random variable with mean zero and variance $\sigma^2(x_i - \overline{x})^2$.

$\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})u_i$ is applied to CLT II.

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n (x_i - \overline{x})u_i \longrightarrow N(0, \sigma_*^2), \qquad \text{as} \qquad n \longrightarrow \infty,$$

where $\sigma_*^2 = \sigma^2 \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2 = \sigma^2 m$ and $m = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2$.

$$\hat{\beta} - \beta = \sum_{i=1}^n \omega_i u_i = \frac{\sum_{i=1}^n (x_i - \overline{x})u_i}{\sum_{j=1}^n (x_j - \overline{x})^2} = \frac{\frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})u_i}{\frac{1}{n}\sum_{j=1}^n (x_j - \overline{x})^2}$$

$$\sqrt{n}(\hat{\beta} - \beta) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (x_i - \overline{x})u_i}{\frac{1}{n}\sum_{j=1}^n (x_j - \overline{x})^2} \longrightarrow N(0, \frac{\sigma^2}{m})$$