Solutions of Homework 1

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May 24, 2022

1 Question 1

1.1 Derive the ordinary least squares estimators of α and β, which should be denoted by $\hat{\alpha}$ and $\hat{\beta}$.

From the regression model, we have

$$u_t = y_t - \alpha - \beta X_t$$

Let J be the sum of residuals, then

$$J = \sum_t u_t^2 = \sum_t (y_t - \alpha - \beta X_t)^2 \tag{1}$$

The extreme value of J can be calculated by taking the first order condition(FOC)

$$\begin{cases} \dfrac{\partial J}{\partial \alpha} = -2\sum_t (y_t - \alpha - \beta X_t) \\ \dfrac{\partial J}{\partial \beta} = -2\sum_t X_t (y_t - \alpha - \beta X_t) \end{cases} \tag{2}$$

The second order condition(SOC) of equation (1) is

$$\mathbf{H} = \begin{pmatrix} \dfrac{\partial^2 J}{\partial^2 \alpha} & \dfrac{\partial^2 J}{\partial \alpha \partial \beta} \\ \dfrac{\partial^2 J}{\partial \beta \partial \alpha} & \dfrac{\partial^2 J}{\partial^2 \beta} \end{pmatrix} = \begin{pmatrix} 2T & 2\sum_t X_t \\ 2\sum_t X_t & 2\sum_t X_t^2 \end{pmatrix}$$

and the determinant of $\mathbf{H}$ is

$$\begin{aligned} |H| &= \left(2T \cdot 2\sum_t X_t^2\right) - \left(2\sum_t X_t \cdot 2\sum_t X_t\right) \\ &= 4T^2 \left(\frac{1}{T}\sum_t X_t^2 - \left(\sum_t \frac{X_t}{T}\right)^2\right) \\ &= 4T^2 V(X_t) > 0, \end{aligned}$$

therefore J has the minimum value when equation (3) equals to 0

$$\begin{cases} \dfrac{\partial J}{\partial \alpha} = -2\sum_t \left(y_t - \hat{\alpha} - \hat{\beta}X_t\right) = 0 \\ \dfrac{\partial J}{\partial \beta} = -2\sum_t X_t \left(y_t - \hat{\alpha} - \hat{\beta}X_t\right) = 0 \end{cases}$$

$$\begin{cases} \hat{\alpha} = \bar{y}_t - \hat{\beta}\bar{X} \\ \hat{\beta} = \dfrac{\sum_t X_t \left(y_t - \bar{y}\right)}{\sum_t X_t \left(X_t - \bar{X}\right)} \end{cases} \tag{3}$$

Moreover, we can rewrite the estimator of $\hat{\beta}$. The covariance between y_t and X_t, $Cov(X_t, y_t)$, is

$$Cov(X_t, y_t) = E\left[(X_t - E(X_t))(y_t - E(y_t))\right],$$

so the sample covariance between y_t and X_t, $S_{X,y}$, is

$$\begin{aligned} S_{X,y} &= \frac{1}{T-1}\sum_t (X_t - \bar{X})(y_t - \bar{y}) \\ &= \frac{1}{T-1}\sum_t (X_t y_t - X_t\bar{y} - \bar{X}y_t + \bar{X}\bar{y}) \\ &= \frac{1}{T-1}(\sum_t X_t y_t - \bar{y}\sum_t X_t - \bar{X}\sum_t y_t + T\bar{X}\bar{y}) \\ &= \frac{1}{T-1}(\sum_t X_t y_t - \bar{y}\sum_t X_t) \\ &= \frac{1}{T-1}\sum_t X_t(y_t - \bar{y}) \end{aligned}$$

Similarly, the sample variance of X_t, S_{XX}, is

$$\begin{aligned} S_{X,X} &= \frac{1}{T-1}\sum_t (X_t - \bar{X})^2 \\ &= \frac{1}{T-1}(\sum_t X_t^2 - 2\bar{X}\sum_t X_t + T\bar{X}^2) \\ &= \frac{1}{T-1}(\sum_t X_t^2 - \bar{X}\sum_t X_t) \\ &= \frac{1}{T-1}\sum_t X_t(X_t - \bar{X}) \end{aligned}$$

Substitute $\hat{\beta}$ in equation (3) with $S_{X,y}$ and $S_{X,X}$,

$$\hat{\beta} = \frac{S_{X,y}}{S_{X,X}} = \frac{\sum_t (X_t - \bar{X})(y_t - \bar{y})}{\sum_t (X_t - \bar{X})^2} \tag{4}$$

Finally, we have

$$(\hat{\alpha}, \hat{\beta}) = (\bar{y_t} - \hat{\beta}\bar{X}, \frac{\sum_t (X_t - \bar{X})(y_t - \bar{y})}{\sum_t (X_t - \bar{X})^2}) \tag{5}$$

1.2 Obtain mean and variance of $\hat{\beta}$.

From equation (4) and the process of $S_{X,y}$ and $S_{X,X}$, we can obtain

$$\begin{aligned}
\hat{\beta} &= \frac{\sum_t (X_t - \bar{X})(y_t - \bar{y})}{\sum_t (X_t - \bar{X})^2} = \frac{\sum_t y_t (X_t - \bar{X}) - \bar{y} \sum_t (X_t - \bar{X})}{\sum_t (X_t - \bar{X})^2} \\
&= \frac{\sum_t y_t (X_t - \bar{X})}{\sum_t (X_t - \bar{X})^2} = \frac{\sum_t (X_t - \bar{X})(\alpha + \beta X_t + u_t)}{\sum_t (X_t - \bar{X})^2} \\
&= \frac{\alpha \sum_t (X_t - \bar{X}) + \beta \sum_t X_t (X_t - \bar{X}) + \sum_t u_t (X_t - \bar{X}))}{\sum_t (X_t - \bar{X})^2} \\
&= \beta + \frac{\sum_t u_t (X_t - \bar{X})}{\sum_t (X_t - \bar{X})^2} \\
&= \beta + \sum_t \omega_t u_t
\end{aligned}$$

where $\omega_t = (X_t - \bar{X}) / \sum_t (X_t - \bar{X})^2$

Since $u_t \overset{\text{iid}}{\sim} (0, \sigma^2)$,

$$E(\hat{\beta}) = \beta + \sum_t \omega_t E(u_t) = \beta \tag{6}$$

Besides, the variance of u_t can be written as

$$V(u_t) = E(u_t^2) - E(u_t)^2 = E(u_t^2) = \sigma^2$$

The variance of $\hat{\beta}$ is

$$\begin{aligned}
V(\hat{\beta}) &= E[(\hat{\beta} - E(\hat{\beta}))^2] = E[(\hat{\beta} - \beta))^2] = E[(\sum_t \omega_t u_t)^2] \\
&= E[\sum_t \omega_t^2 u_t^2 + \sum_t \sum_{s \neq t} \omega_t u_t \omega_s u_s] \\
&= \sum_t \omega_t^2 E(u_t^2) + \sum_t \sum_{s \neq t} \omega_t \omega_s E(u_t u_s) \\
&= \sigma^2 \frac{\sum_t (X_t - \bar{X})^2}{(\sum_t (X_t - \bar{X})^2)^2} + \sum_t \sum_{s \neq t} \omega_t \omega_s E(u_t u_s)
\end{aligned}$$

Since u_t are identically independently distributed, $E(u_t u_s) = E(u_t)E(u_s) = 0$.

Therefore,

$$V(\hat{\beta}) = \frac{\sigma^2}{\sum_t (X_t - \bar{X})^2} \tag{7}$$

1.3 Obtain mean and variance of $\hat{\alpha}$.

From the regression model and equation (3), we have

$$\bar{y} = \alpha + \beta \bar{X} + \bar{u}$$

$$\hat{\alpha} = \alpha + \beta \bar{X} + \bar{u} - \hat{\beta} \bar{X}$$

$$E(\hat{\alpha}) = \alpha + (\beta - E(\hat{\beta}))\bar{X} + E(\bar{u}) = \alpha \tag{8}$$

$$\begin{aligned} V(\hat{\alpha}) &= E[(\hat{\alpha} - E(\hat{\alpha}))^2] = E[(\hat{\alpha} - \alpha^2)] \\ &= E[(\bar{u} - (\hat{\beta} - \beta)\bar{X})^2] \\ &= E[\bar{u}^2 + (\hat{\beta} - \beta)^2 \bar{X}^2 - 2(\hat{\beta} - \beta)\bar{X}\bar{u}] \\ &= E(\bar{u}^2) + \bar{X}^2 E[(\hat{\beta} - \beta)^2] - 2\bar{X} E((\hat{\beta} - \beta)\bar{u}) \end{aligned}$$

Similar to the process of the variance of $\hat{\beta}$, we can derive

$$\begin{aligned} E(\bar{u}^2) &= E[(\frac{\sum_t u_t}{T})^2] = \frac{1}{T^2} E[(\sum_t u_t)^2] \\ &= \frac{1}{T^2} E[\sum_t u_t^2 + \sum_t \sum_{s \neq t} u_t u_s] \\ &= \frac{1}{T^2} [\sum_t E(u_t^2) + \sum_t \sum_{s \neq t} E(u_t u_s)] \\ &= \frac{1}{T^2}(T\sigma^2) = \frac{\sigma^2}{T} \end{aligned}$$

$$\bar{X}^2 E[(\hat{\beta} - \beta)^2] = \bar{X}^2 V(\hat{\beta}) = \frac{\sigma^2 \bar{X}^2}{\sum_t (X_t - \bar{X})^2} \tag{9}$$

$$
\begin{aligned}
\bar{X}E((\hat{\beta}-\beta)\bar{u}) &= \bar{X}E[\frac{\sum_t (X_t-\bar{X})\bar{u}}{\sum_t (X_t-\bar{X})^2}\frac{\sum_s u_s}{T}] \\
&= \frac{\bar{X}}{T\sum_t (X_t-\bar{X})^2}E[\sum_t (X_t-\bar{X})u_t \sum_s u_s] \\
&= \frac{\bar{X}}{T\sum_t (X_t-\bar{X})^2}E[\sum_t (X_t-\bar{X})(u_t^2+\sum_{s\neq t} u_t u_s)] \\
&= \frac{\bar{X}}{T\sum_t (X_t-\bar{X})^2}[\sum_t (X_t-\bar{X})(E(u_t^2)+\sum_{s\neq t} E(u_t u_s))] \\
&= \frac{\bar{X}}{T\sum_t (X_t-\bar{X})^2}[\sigma^2\sum_t (X_t-\bar{X})] \\
&= \frac{\sigma^2\bar{X}\sum_t (X_t-\bar{X})}{T\sum_t (X_t-\bar{X})^2} = 0
\end{aligned}
$$

Therefore, the variance of $\hat{\alpha}$ is

$$
V(\hat{\alpha}) = \frac{\sigma^2}{T} + \frac{\sigma^2\bar{X}^2}{\sum_t (X_t-\bar{X})^2} = \frac{\sigma^2\sum_t X_t^2}{T\sum_t (X_t-\bar{X})^2} \tag{10}
$$

1.4 Prove that $\hat{\beta}$ is a linear estimator of β.

According to the process of 1.2, $\hat{\beta}$ can be derived as

$$
\hat{\beta} = \sum_t \omega_t y_t \tag{11}
$$

where $\omega_t = (X_t-\bar{X})/\sum_t (X_t-\bar{X})^2$.

Since X_t are nonstochastic and there is a linear relationship between X_t and y_t, $\hat{\beta}$ is a linear estimator of β

1.5 Prove that $\hat{\beta}$ is a linear unbiased estimator of β.

From equation (6) in 1.2, we have

$$
E(\hat{\beta}) = \beta \tag{12}
$$

It shows that $\hat{\beta}$ is an unbiased estimator of β.

1.6 Prove that $\hat{\beta}$ has minimum variance within a class of linear unbiased estimators.

Consider the alternative linear unbiased estimator $\widetilde{\beta}$ as follows:

$$
\widetilde{\beta} = \sum_t c_t y_t = \sum_t (\omega_t + d_t) y_t,
$$

where $c_t = \omega_t + d_t$ is defined and d_t is nonstochastic. Then, $\widetilde{\beta}$ is transformed into:

$$\begin{aligned}
\widetilde{\beta} &= \sum_t (\omega_t + d_t)(\alpha + \beta X_t + u_t) \\
&= \alpha \sum_t \omega_t + \beta \sum_t \omega_t X_t + \sum_t \omega_t u_t + \alpha \sum_t d_t + \beta \sum_t d_t X_t + \sum_t d_t u_t \\
&= \beta + \alpha \sum_t d_t + \beta \sum_t d_t X_t + \sum_t \omega_t u_t + \sum_t d_t u_t.
\end{aligned}$$

Taking the expectation on both sides of the above equation, we obtain:

$$\begin{aligned}
E(\widetilde{\beta}) &= \beta + \alpha \sum_t d_t + \beta \sum_t d_t X_t + \sum_t \omega_t E(u_t) + \sum_t d_t E(u_t) \\
&= \beta + \alpha \sum_t d_t + \beta \sum_t d_t X_t.
\end{aligned}$$

Note that $E(u_t) = 0$. Since $\widetilde{\beta}$ is assumed to be unbiased, we need the following conditions:

$$\sum_t d_t = 0, \qquad \sum_t d_t X_t = 0, \tag{13}$$

where $E(\widetilde{\beta}) = \beta$. When these conditions hold, we can rewrite $\widetilde{\beta}$ as:

$$\widetilde{\beta} = \beta + \sum_t (\omega_t + d_t) u_t.$$

The variance of $\widetilde{\beta}$ is derived as :

$$\begin{aligned}
V(\widetilde{\beta}) &= V(\beta + \sum_t (\omega_t + d_t) u_t) = V(\sum_t (\omega_t + d_t) u_t) = \sum_t V((\omega_t + d_t) u_t) \\
&= \sum_t (\omega_t + d_t)^2 V(u_t) = \sigma^2 (\sum_t \omega_t^2 + 2 \sum_t \omega_t d_t + \sum_t d_t^2).
\end{aligned}$$

Note that $V(u_t) = \sigma^2$. From the unbiasedness of $\widetilde{\beta}$, using result (13) we obtain:

$$\sum_t \omega_t d_t = \frac{\sum_t (X_t - \bar{X}) d_t}{\sum_t (X_t - \bar{X})^2} = \frac{\sum_t x_t d_t - \bar{X} \sum_t d_t}{\sum_t (X_t - \bar{X})^2} = 0,$$

from which we can obtain that:

$$\begin{aligned}
V(\widetilde{\beta}) &= \sigma^2 (\sum_t \omega_t^2 + \sum_t d_t^2) \\
&= \sigma^2 \sum_t \omega_t^2 + \sigma^2 \sum_t d_t^2 \\
&= V(\widehat{\beta}) + \sigma^2 \sum_t d_t^2 \\
&\geq V(\widehat{\beta}),
\end{aligned}$$

for the reason that $\sum_t d_t^2 \geq 0$. Thus, the OLS estimator $\widehat{\beta}$ gives us the minimum variance linear unbiased estimator.

1.7 Prove that $\hat{\beta}$ is a consistent estimator of β.

From equation (7), we have $V(\hat{\beta}) = \frac{\sigma^2}{\sum_t (X_t - \bar{X})^2}$ then according to the assumption that

$$\frac{1}{T}\sum_t (X_t - \bar{X})^2 \xrightarrow{p} m < \infty,$$

when $T \to \infty$, we obtain that:

$$P(|\widehat{\beta} - \beta| > \epsilon) \leq \frac{\sigma^2 \sum_t \omega_t^2}{\epsilon^2} = \frac{\sigma^2 T \sum_t \omega_t^2}{T\epsilon^2} \to 0,$$

where $\sum_t \omega_t^2 \to 0$ because $T\sum_t \omega_t^2 \to \frac{1}{m}$ from the assumption.
Thus we have:

$$\widehat{\beta} \to \beta \, as \, T \to \infty.$$

1.8 Derive an asymptotic distribution of $\sqrt{T}(\hat{\beta} - \beta)$. Note that a distribution of ut is not assumed.

Note that $\widehat{\beta} = \beta + \sum_t \omega_t u_t$. From the Central Limit Theorem, we have

$$\frac{\widehat{\beta} - E(\widehat{\beta})}{\sqrt{V(\widehat{\beta})}} = \frac{\sum_t \omega_t u_t}{\sigma\sqrt{\sum_t \omega_t^2}} = \frac{\widehat{\beta} - \beta}{\sigma/\sqrt{\sum_t (X_t - \bar{X})^2}} \to N(0,1),$$

which can be rewritten as

$$\frac{\sqrt{T}(\widehat{\beta} - \beta)}{\sigma/\sqrt{(1/T)\sum_t (X_t - \bar{X})^2}}.$$

Replacing $(1/T)\sum_t (X_t - \bar{X})^2$ by its converged value m, we have

$$\frac{\sqrt{T}(\widehat{\beta} - \beta)}{\sigma/\sqrt{m}} \to N(0,1),$$

which implies that

$$\sqrt{T}(\widehat{\beta} - \beta) \to N(0, \frac{\sigma^2}{m}).$$

1.9 As an extra assumption, suppose that u_t is normally distributed for all t. Derive an exact distribution of $\hat{\beta}$, using the moment-generating function.

The moment generating function of $\widehat{\beta}$ is

$$M_{\widehat{\beta}}(\theta) = E(exp\{(\beta + \sum_t \omega_t u_t)\theta\}) \\ = e^{\beta\theta} \prod_{t=1}^{T} E(e^{\theta\omega_t u_t}).$$

Since the moment generating function of $u_t \sim N(0, \sigma^2)$ is

$$M_{u_t}(\theta) = E(e^{\theta u_t}) = exp\{\frac{\sigma^2\theta^2}{2}\},$$

we can rewrite

$$M_{\widehat{\beta}}(\theta) = e^{\beta\theta} \prod_{t=1}^{T} E(e^{(\theta\omega_t)u_t}) \\ = e^{\beta\theta} \prod_{t=1}^{T} exp\{\frac{\sigma^2(\theta\omega_t)^2}{2}\} \\ = exp\{\beta\theta + \frac{\theta^2\sigma^2\sum_t \omega_t^2}{2}\},$$

which implies that the exact distribution of $\widehat{\beta}$ is

$$\widehat{\beta} \sim N(\beta, \sigma^2 \sum_t \omega_t^2).$$

1.10

Set $Z = \frac{\widehat{\beta}-\beta}{\sigma\sqrt{\sum_t \omega_t^2}}$. Then $Z \sim N(0, 1)$ since $\widehat{\beta} \sim N(\beta, \sigma^2 \sum_t \omega_t^2)$.

From the definition of χ^2 distribution we know that

$$\frac{(T-2)s^2}{\sigma^2} \sim \chi^2(T-2)$$

, where $(T-2)$ is the degree of freedom.

Since the t distribution is defined as $\frac{Z}{\sqrt{V/k}} \sim t(k)$ for $z \sim N(0, 1)$, $V \sim \chi^2(k)$ and Z is independent of V, in this condition, $V = \frac{(T-2)s^2}{\sigma^2}$ and $k = T-2$. Thus we obtain that

$$\frac{Z}{\sqrt{V/k}} = \frac{\widehat{\beta}-\beta}{\sigma\sqrt{\sum_t \omega_t^2}} \Bigg/ \sqrt{\frac{(T-2)s^2}{\sigma^2} \Bigg/ (T-2)} \\ = \frac{\widehat{\beta}-\beta}{s\sqrt{\sum_t \omega_t^2}} \sim t(T-2).$$