Solutions of Homework 3

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1 Question 1

1.1 Derive the restricted OLS $\tilde{\beta}$.

Under this restriction, we can rewrite the question as

$$\min_{\beta} \ (y - X\beta)'(y - X\beta)$$
$$s.t. \ R\beta = r$$

Applying the Lagrange multiplier, we can have

$$L = (y - X\beta)'(y - X\beta) - 2\lambda'(R\beta - r)$$

When the FOC equals to 0, $\tilde{\lambda}$ and $\tilde{\beta}$ can be obtained to minimize the above equation.

$$\begin{cases} \dfrac{\partial L}{\partial \tilde{\beta}} = -2X'(y - X\tilde{\beta}) - 2R'\tilde{\lambda} = 0 \\ \\ \dfrac{\partial L}{\partial \tilde{\lambda}} = -2(R\tilde{\beta} - r) = 0 \end{cases}$$

$$\tilde{\beta} = (X'X)^{-1}X'y + (X'X)^{-1}R'\tilde{\lambda} = \hat{\beta} + (X'X)^{-1}R'\tilde{\lambda}$$

Multiply R by both side, we have

$$R\tilde{\beta} = r = R\hat{\beta} + R(X'X)^{-1}R'\tilde{\lambda}$$

$$\tilde{\lambda} = (R(X'X)^{-1}R')^{-1}(r - R\hat{\beta})$$

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(r - R\hat{\beta}) \tag{1}$$

1.2 Show the following: $\dfrac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(G, T-k)$.

In the previous section we have learned that:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(T-k)} \sim F(G, T-k)$$

where $G = rank(R)$.

From equation (1), we can derive

$$\hat{\beta} - \tilde{\beta} = (X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)$$

Multiply R by both sides, the following expression can be obtained:

$$R(\hat{\beta} - \tilde{\beta}) = R(X'X)^{-1}R'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = R\hat{\beta} - r$$

Therefore, the numerator can be simplified as

$$\begin{aligned}(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) &= (\hat{\beta} - \tilde{\beta})'R'(R(X'X)^{-1}R')^{-1}R(\hat{\beta} - \tilde{\beta}) \\ &= (\hat{\beta} - \tilde{\beta})'(X'X)(\hat{\beta} - \tilde{\beta})\end{aligned}$$

Moreover, since

$$\begin{aligned}(y - X\tilde{\beta})'(y - X\tilde{\beta}) &= (y - X\hat{\beta} + X\hat{\beta} - X\tilde{\beta})'(y - X\hat{\beta} + X\hat{\beta} - X\tilde{\beta}) \\ &= (y - X\hat{\beta})'(y - X\hat{\beta}) + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta}) \\ &- (y - X\hat{\beta})'X(\tilde{\beta} - \hat{\beta}) - (\tilde{\beta} - \hat{\beta})'X'(y - X\hat{\beta})\end{aligned}$$

and $X'(y - X\hat{\beta}) = X'\hat{u} = 0$,

$$\begin{aligned}&(y - X\tilde{\beta})'(y - X\tilde{\beta}) = \tilde{u}'\tilde{u} \\ &= \hat{u}'\hat{u} + (\tilde{\beta} - \hat{\beta})'X'X(\tilde{\beta} - \hat{\beta})\end{aligned}$$

$$\begin{aligned}(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) &= (\hat{\beta} - \tilde{\beta})'(X'X)(\hat{\beta} - \tilde{\beta}) \\ &= \tilde{u}'\tilde{u} - \hat{u}'\hat{u}\end{aligned}$$

Summarizing, we can obtain

$$\frac{(\tilde{u}'\tilde{u} - \hat{u}'\hat{u})/G}{\hat{u}'\hat{u}/(T-k)} \sim F(G, T-k) \tag{2}$$

1.3 Show the following: $\dfrac{(\hat{R}^2 - \tilde{R}^2)/G}{(1-\hat{R}^2)/(n-k)} \sim F(G, T-k)$

Since the coefficients of determination of the restricted model and unrestricted model are

$$\tilde{R}^2 = 1 - \frac{\tilde{u}'\tilde{u}}{y'My}, \quad \hat{R}^2 = 1 - \frac{\hat{u}'\hat{u}}{y'My}$$

Hence,

$$\tilde{u}'\tilde{u} = (1-\tilde{R}^2)y'My, \quad \hat{u}'\hat{u} = (1-\hat{R}^2)y'My$$

Substitute for equation (2), we have

$$\frac{((1-\tilde{R}^2)y'My - (1-\hat{R}^2)y'My)/G}{(1-\hat{R}^2)y'My/(T-k)} = \frac{(\hat{R}^2 - \tilde{R}^2)/G}{(1-\hat{R}^2)/(n-k)} \sim F(G, T-k)$$

2 Question 2

2.1 Test $H_0 : \alpha_1 = \alpha_2 = 0$.

From the null hypothesis $H_0 : \alpha_1 = \alpha_2 = 0$, the alternative hypothesis can be derived as

$$H_1 : \alpha_1 \neq 0 \; or \; \alpha_2 \neq 0$$

and restrictions are

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}$$

where $R_1\alpha = r_1$. Besides, $k-1 = 3-1 = 2$, $T-k = (1997-1968)-3 = 26$

Therefore, we can easily derive F score from R^2

$$F = \frac{R^2/(k-1)}{(1-R^2)/(T-k)} = \frac{0.986684/2}{(1-0.986684)/26} \approx 963.2691$$

which is much greater than the test statistic 5.526 under 1% significant level in $F \sim (2, 26)$.

Hence, we can reject the null hypothesis that $\alpha_1 = \alpha_2 = 0$.

2.2 Test whether the production function is homogeneous.

If the production function is homogeneous, then we can hypothesize

$$H_0 : \alpha_1 + \alpha_2 = 1, \; H_1 : \alpha_1 + \alpha_2 \neq 1$$

and the restrictions are

$$R_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}, \qquad r_2 = 1$$

where $R_2\alpha = r_2$.

Notice that the second model can be rewrite

$$\log(Y_t/L_t) = \beta_0 + \beta_1 \log(K_t/L_t) + u_t$$

$$\log Y_t - \log L_t = \beta_0 + \beta_1 \log K_t - \beta_1 L_t + u_t$$

$$\log Y_t = \beta_0 + \beta_1 \log K_t + (1 - \beta_1) \log L_t + u_t$$

We can treat α_1 as β_1 and α_2 as $(1 - \beta_1)$. Since $\beta_1 + (1 - \beta_1) = 1$, the second model is the corresponding restricted model with H_0.

Under these conditions, the degree of freedoms are $G = rank(R_2) = 1$, $T - k = 26$.

Then we can obtain the F score:

$$F = \frac{(\hat{R}^2 - \tilde{R}^2)/G}{(1 - \hat{R}^2)/(T - k)} = \frac{0.986684 - 0.934448}{(1 - 0.986684)/26} \approx 101.9928$$

which is greater than the test statistic 7.721 under 1% significant level in $F \sim (1, 26)$.

Thus, we should reject the hypothesis that the production function is homogeneous.

2.3 Test whether the structural change occurred after 1991.

Assume that there is no structural change after 1991, the hypotheses can be expressed as

$$H_0 : \gamma_3 = \gamma_4 = \gamma_5 = 0$$

Similarly, we derive the restrictions as

$$R_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix}$$

where $R_3\gamma = r_3$.

Thus, we can find that the first model is the corresponding restricted model with the above hypotheses.

$$G = rank(R_3) = 3, \; T - k = 29 - 6 = 23$$

$$F = \frac{(\hat{R}^2 - \tilde{R}^2)/G}{(1 - \hat{R}^2)/(T - k)} = \frac{(0.987960 - 0.986684)/3}{(1 - 0.987960)/23} \approx 0.8125$$

which is less than the test statistic 2.339 under 10% significant level in $F \sim (3, 23)$.

Accordingly, we can not reject the hypothesis that no structural change occurred after 1991.

3 Question 3

3.1 Show that there exists P, such that $\Omega = PP'$ when Ω is a positive definite matrix.

When Ω is positive definite, all its eigenvalues are positive. In general, Ω is symmetric and diagonalizable. Using the matrix of eigenvectors of Ω denoted by A and the diagonal matrix Λ where the elements are eigenvalues Λ_i, Ω can be decomposed as

$$\Omega = A\Lambda A' = A\sqrt{\Lambda}\sqrt{\Lambda}A' = (A\Lambda^{\frac{1}{2}})(A\Lambda^{\frac{1}{2}})',$$

and we obtain $P = A\Lambda^{\frac{1}{2}}$.

3.2 What is the variance-covariance matrix, denoted by $\boldsymbol{\sigma^2\Omega}$?

Since $\{u_t\}$ is mutually independent, $\mathrm{Cov}(u_t, u_s) = 0$ for $t \neq s$. Then the variance-covariance matrix is denoted by

$$\sigma^2\Omega = \sigma^2 \begin{pmatrix} z_1^2 & 0 & \cdots & 0 \\ 0 & z_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_T^2 \end{pmatrix}$$

3.3 What is the variance-covariance matrix, denoted by $\boldsymbol{\sigma^2\Omega}$?

Since $\{\epsilon_t\}$ is mutually independent with homogeneous variance σ^2, $\mathrm{V}(u_t) = \rho^2\mathrm{V}(u_{t-1}) + \sigma^2$. In general, $|\rho| < 1$, then

$$\mathrm{V}(u_t) = (\rho^2)^2\mathrm{V}(u_{t-2}) + (1+\rho^2)\sigma^2 = \cdots = (1 + \rho^2 + \cdots + \rho^{T-1})\sigma^2 = \frac{1}{1-\rho^2}\sigma^2.$$

Similarly,

$$\mathrm{Cov}(u_t, u_{t-1}) = \mathrm{Cov}(\rho u_{t-1} + \epsilon_t, u_{t-1}) = \rho\mathrm{V}(u_{t-1}) + 0 = \rho\frac{\sigma^2}{1-\rho^2}.$$

$$\mathrm{Cov}(u_t, u_{t-k}) = \rho^k\frac{\sigma^2}{1-\rho^2}, \qquad k = 0, 1, \cdots, T-1.$$

Thus,

$$\sigma^2\Omega = \frac{\sigma^2}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{pmatrix}.$$

3.4 Derive b.

From 3.1, we know that there exists P such that $\Omega = PP'$. Multiply P^{-1} on both sides of $y = X\beta + u$, we have

$$y^* = X^*\beta + u^*,$$

where $y^* = P^{-1}y$, $X^* = P^{-1}X$, and $u^* = P^{-1}u$.
The variance of u^* is

$$\mathrm{V}(u^*) = \sigma^2 P^{-1}\Omega P'^{-1} = \sigma^2 I_T.$$

Then the original regressio model can be rewritten as

$$y^* = X^*\beta + u^*,\ u^* \sim N(0, \sigma^2 I_T).$$

By application of OLS to the transformed regression model, we derive b as

$$b = (X^{*\prime}X^*)^{-1}(X^{*\prime}y^*) = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

3.5 Show that $\mathrm{V}(\boldsymbol{\beta}) - \mathrm{V}(\boldsymbol{b})$ is a positive definite matrix by considering model in 3.3.

When we apply OLS directly to the regression model in 3.3, the $\hat{\beta}$ is denoted as

$$\hat{\beta} = (X'X)^{-1}X'y = \beta + (X'X)^{-1}X'u.$$

The expectation and variance of $\hat{\beta}$ is given by

$$\mathrm{E}(\hat{\beta}) = \beta,$$
$$\mathrm{V}(\hat{\beta}) = \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1}.$$

The GLS yields

$$b = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}u,$$

of which the expectation and variance is

$$\mathrm{E}(b) = \beta,$$

$$\mathrm{V}(b) = \sigma^2(X'\Omega^{-1}X)^{-1}.$$

Compare OLS to GLS,

$$\begin{aligned}
\mathrm{V}(\beta) - \mathrm{V}(b) &= \sigma^2(X'X)^{-1}X'\Omega X(X'X)^{-1} - \sigma^2(X'\Omega^{-1}X)^{-1} \\
&= \sigma^2((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1})\Omega((X'X)^{-1}X' - (X'\Omega^{-1}X)^{-1}X'\Omega^{-1})' \\
&= \sigma^2 \underbrace{A}_{k\times T}\underbrace{\Omega}_{T\times T} A'.
\end{aligned}$$

Since Ω is a positive definite matrix, $A\Omega A'$ is also positive definite matrix for any $x \in \mathbb{R}^k$, $x \neq 0$, $x'(A\Omega A')x > 0$. Then $\mathrm{V}(\beta) - \mathrm{V}(b)$ is positive definite matrix, indicating that GLS estimator is more efficient than OLS estimator in this case.