TA session #8

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Econometrics I

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Theorem 1 Chebyshev's inequality.

Let X(integrable) be a random variable with finite expected value μ and finite non-zero variance σ^2. Then for any real number $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}, \qquad \mu = E(X), \sigma^2 = V(X). \tag{1}$$

Proof. Set $I = \{x : |x - \mu| \geq k\sigma\}$,

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\ &\geq \int_I (x-\mu)^2 f(x)dx \\ &\geq (k\sigma^2) \int_I f(x)dx \\ &= (k\sigma^2) \cdot P(|X - \mu| \geq k\sigma).\end{aligned}$$

Then, divide both sides by $(k\sigma)^2$ we can get (1). □

Theorem 2 Positive Definiteness of a Matrix.

Let $n = 1$; we say that a square matrix A is symmetric if $A \in \mathbb{R}^{n\times n}$ is equal to its transpose($A^\top = A$), and we say that A is nonnegative definite (positive semi-definite) if a matrix $B \in \mathbb{R}^{n\times n}$ exists such that $A = B^\top B$.

Proposition 1 (nonnegative definite matrix). The following three conditions are equivalent for a symmetric matrix $A \in \mathbb{R}^{n\times n}$.

1. A matrix $B \in \mathbb{R}^{n\times n}$ exists such that $A = B^\top B$.
2. $x^\top Ax \geq 0$ for any $x \in \mathbb{R}^n$.
3. The eigenvalues[1] of A are nonnegative.

Proof. $(a) \Rightarrow (b)$, $\quad A = B^\top B \Rightarrow x^\top Ax \Rightarrow x^\top B^\top Bx \Rightarrow \|Bx\|^2 \geq 0$.
$(b) \Rightarrow (c)$, $\quad x^\top Ax \geq 0, x \in \mathbb{R}^n \Rightarrow \forall y \in \mathbb{R}^n, y^\top Ay = y^\top \lambda y = \lambda\|y\|^2 \geq 0$
$(c) \Rightarrow (a)$, $\qquad \text{since}\, \lambda_1, ..., \lambda_n \geq 0\, \text{and} A\, \text{is a symmetric matrix} \Rightarrow A = PDP^\top = P\sqrt{D}\sqrt{D}P^\top = (\sqrt{D}P^\top)^\top\sqrt{D}P^\top$,

[1] An element v is called an eigenvector of A if there exists a number λ such that $Av = \lambda v$, i.e. $(A - \lambda I)v = 0$. If $v \neq 0$ then λ is uniquely determined. In this case, we say that λ is an eigenvalue of A belonging to the eigenvector v. We also say that v is an eigenvector with the eigenvalue λ. Instead of eigenvector and eigenvalue, on also usesthe terms characteristic vector and characteristic value.

where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ and $\sqrt{D} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$ and P is the corresponding orthogonal matrix[2]. □

Lemma 1. Real matrix A is symmetric $\iff$ A is diagonalizable:

$$A = \underbrace{U}_{\text{orthogonal matrix}} \overbrace{D}^{\text{diagonal matrix}} U^{-1}$$

Lemma 2. If A is diagonalizable, there exists unitary matrix P such that $P^{-1}AP = D$ where $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ and $p = \begin{pmatrix} p_1 & \cdots & p_n \end{pmatrix}$. Note that $\lambda_1 \cdots \lambda_n$ are eigenvalues and $p_1 \cdots p_n$ are eigenvectors with λs.

[2]For real square matrix U, U is orthogonal matrix $\iff U^\top U = UU^\top = I_n$.This leads to the equivalent characterization: a matrix U is orthogonal if its transpose is equal to its inverse i.e. $U^\top = U^{-1}$.