Econometrics I TA Session

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Contents

1 Brief review about Moment-generating Function (MGF)

1.1 Definition

Let x be a random variable with cdf $F(x)$, the MGF of x, denoted by $M_X(\theta)$, is

$$M_X(\theta) = E[e^{\theta x}] = \int_{-\infty}^{+\infty} e^{\theta x} dF(x) \tag{1}$$

The series expansion of $e^{\theta x}$ is

$$e^{\theta x} = \frac{\theta x}{1!} + \frac{\theta^2 x^2}{2!} + \frac{\theta^3 x^3}{3!} + \cdots + \frac{\theta^n x^n}{n!} + \cdots \tag{2}$$

Hence, MGF can be rewritten as

$$M_X(\theta) = \frac{\theta E(x)}{1!} + \frac{\theta^2 E(x^2)}{2!} + \cdots + \frac{\theta^n E(x^n)}{n!} + \cdots \tag{3}$$

1.2 An Important Property

If $\mathbf{X_i}$ and $\mathbf{Y_i}$ are two random variables and for all values of i,

$$M_X(\theta) = M_Y(\theta) \tag{4}$$

then, $\mathbf{X_i}$ and $\mathbf{Y_i}$ have the same distribution:

$$M_X(x) = M_Y(x) \tag{5}$$

1.3 MGF of $\mathbf{X_i} \sim \mathcal{N}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\} \tag{6}$$

$$M_\theta(x) = \int_{-\infty}^{+\infty} \exp(\theta x) f(x) dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{2\sigma^2\theta x - (x^2+\mu^2-2x\mu)}{2\sigma^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2+(\mu+\sigma^2\theta)^2-2x\mu-2\sigma^2\theta x-(\sigma^2\theta)^2-2\mu\sigma^2\theta}{2\sigma^2}\right] dx$$

$$= \exp\left[\mu\theta + \frac{\sigma^2\theta^2}{2}\right] \boxed{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[x-(\mu+\sigma^2\theta)]^2}{2\sigma^2}\right\} dx} = 1$$

$$= \exp\left[\mu\theta + \frac{\sigma^2\theta^2}{2}\right]$$

↳ the integral of $N(\mu+\sigma^2\theta, \sigma^2)$

When $\mu = 0$, $\sigma^2 = 1$,

$$M_X(\theta) = \exp\left(\frac{\theta^2}{2}\right)$$

2 Central Limit Theorem

2.1 Definition

Let $\mathbf{X_i} \overset{\text{iid}}{\sim} (\mu, \sigma^2)$. Suppose n samples are drawn from X_i,

$$\bar{X_n} \equiv \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n} \tag{7}$$

Then when n goes to infinity, the random variable $\sqrt{n}(\bar{X_n} - \mu)/\sigma$ converge to $\mathcal{N}(\mu, \sigma^2)$

2.2 Deduction

First, to normalize X_i, we have

$$Z_i = (X_i - \mu)/\sigma, \quad E(Z_i) = 0, \quad V(Z_i) = 1$$

$$\therefore e^{\theta z} = 1 + \frac{\theta z}{1!} + \frac{\theta^2 z^2}{2!} + \frac{\theta^3 z^3}{3!} e^{sz}, \quad s \in (0, \theta)$$

$$\because E(Z_i) = 0, \quad E(Z_i^2) = V(Z_i) + E^2(Z_i) = 1$$

$$\therefore M_\theta(Z_i) = 1 + \theta E(Z_i) + \frac{\theta^2 E(Z_i^2)}{2} + \frac{\theta^3}{6} E(Z_i^3 e^{sZ_i})$$

$$= 1 + \frac{\theta^2}{2} + \frac{\theta^3}{6} E(Z_i^3 e^{sZ_i})$$

Then we normalize the sample average $\bar{X}_n$ ($\bar{X}_n \sim (\mu, \frac{\sigma^2}{n})$) and denote them as Z_n^*.

$$Z_n^* = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{(X_i - \mu)}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i$$

$$\therefore M_{Z_n^*}(\theta) = E[e^{\theta Z_n^*}] = E[\exp(\frac{\theta}{\sqrt{n}}\sum_{i=1}^{n} Z_i)] = E[\prod_{i=1}^{n}\exp(\frac{\theta Z_i}{\sqrt{n}})]$$

$$= \prod_{i=1}^{n} E[\exp(\frac{\theta Z_i}{\sqrt{n}})] = \boxed{\prod_{i=1}^{n} M_{Z_i}(\frac{\theta}{\sqrt{n}}) = [M_{Z_i}(\frac{\theta}{\sqrt{n}})]^n}$$

Since Z_i are iid, the MGFs of Z_i don't vary from the subscript i.

$$= [1 + \frac{\theta^2}{2n} + \frac{\theta^3}{6n\sqrt{n}} E(Z_i^3 e^{sZ_i})]^n$$

$$= [1 + \underline{\frac{\theta^2}{2n} + o(n^{-1})}]^n$$

Let $t_n = \frac{\theta^2}{2n} + o(n^{-1})$

$$= (1 + t_n)^n = [(1 + t_n)^{1/t_n}]^{n t_n}$$

$$\because t_n = \frac{\theta^2}{2n} + o(n^{-1}) \qquad \therefore n t_n = \frac{\theta^2}{2} + o(1) \quad \text{when } n \to \infty$$

$$\because e = \lim_{n\to\infty}(1+n)^{\frac{1}{n}} = \lim_{n\to 0}(1 + \frac{1}{n})^n$$

$$\therefore \lim_{n\to\infty} M_{Z_n^*}(\theta) = \lim_{n\to\infty}[(1+t_n)^{1/t_n}]^{\frac{\theta^2}{2} + o(1)} = \exp(\frac{\theta^2}{2})$$

$$\therefore Z_n^* = \sqrt{n}(\bar{X}_n - \mu)/\sigma^2 \text{ converges to } N(0, 1)$$