2020/05/26

Econometrics I's Homework #2

Deadline: June 14, 2022, PM23:59:59

- The answer should be written in English or Japanese.
- Your name and student ID number should be included in your answer sheet.
- The file size of your answer sheet must be less than **2MB** to reduce a network traffic.
- Send your answer to the email address: `tanizaki@econ.osaka-u.ac.jp`.
- The subject should be **Econome** or **計量**. Otherwise, your mail may go to the **trash box**.

1 Consider the following regression model:

$$y = X\beta + u$$

where y, X, β and u denote $n \times 1$, $n \times k$, $k \times 1$ and $n \times 1$ matrices. k and n are the numberof explanatory variables and the sample size. $u_1, u_2, \cdots, u_n$ are mutually independently and **normally** distributed with mean zero and variance σ^2, i.e., $u \sim N(0, \sigma^2 I_n)$ for $u = (u_1,\ u_2,\ \cdots,\ u_n)'$. β is a vector of unknown parameters to be estimated. Let $\hat{\beta}$ be the ordinary least squares estimator of β.

(1) Derive $\hat{\beta}$.

(2) Derive mean and variance of $\hat{\beta}$.

(3) Derive a distribution of $\hat{\beta}$, using the moment-generating function.

(4) Show that $s^2 = \dfrac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta})$ is an unbiased estimator of σ^2.

(5) Show that $\dfrac{(n-k)s^2}{\sigma^2}$ is distributed as a χ^2 random variavble with $n-k$ degrees of freedom.

(6) Show that $\hat{\beta}$ is a best linear unbiased estimator.

(7) $\hat{\beta}$ is normally distributed with mean β and variance $\sigma^2(X'X)^{-1}$.
Then, show that $\dfrac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)}{\sigma^2} \sim \chi^2(k)$.
Especially, why is the degree of freedom equal to k?

(8) Show that $\hat{\beta}$ is independent of $s^2 = \dfrac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta})$.

(9) Show that $\dfrac{(\hat{\beta}-\beta)X'X(\hat{\beta}-\beta)/k}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(k, n-k)$.

(10) Show that $\sum_{i=1}^{n}(y_i - \overline{y})^2 = y'(I_n - \dfrac{1}{n}ii')y$, where $y = (y_1, y_2, \cdots, y_n)'$ and $i = (1, 1, \cdots, 1)'$.

(11) Show that $I_n - \dfrac{1}{n}ii'$ is symmetric and idempotent.

2 (Statistics Questions) n random variables $X_1,\ X_2,\ \cdots,\ X_n$ are assumed to be mutually independently, identically and normally distributed with mean μ and variance σ^2. We want to prove $\dfrac{\sum_{j=1}^{n}(X_j - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$, which is shown in standard statistics textbooks, where $\overline{X} = \dfrac{1}{n}\sum_{j=1}^{n} X_j$.

Define $X = (X_1,\ X_2,\ \cdots,\ X_n)'$ and $i = (1, 1, \cdots, 1)'$, which are $n \times 1$ vectors. Then, we can rewrite as follows:

$$X \sim N(\mu i, \sigma^2 I_n).$$

Answer the following questions.

(12) What is the distribution of $\dfrac{(X - \mu i)'(X - \mu i)}{\sigma^2}$?

(13) Show that

$$(X - \mu i)'(I_n - \frac{1}{n}ii')(X - \mu i) = \sum_{j=1}^{n}(X_j - \overline{X})^2.$$

(14) Show that $\dfrac{(X - \mu i)'(I_n - \frac{1}{n}ii')(X - \mu i)}{\sigma^2} \sim \chi^2(n-1)$. (That is, $\dfrac{\sum_{j=1}^{n}(X_j - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$ is obtained.)