single regression model : $y_i = \alpha + \beta x_i + u_i$

OLS : $y_i = \hat{\alpha} + \hat{\beta} x_i + e_i$

$$S(\alpha, \beta) = \sum e_i^2$$

$$\frac{\partial S}{\partial \hat{\alpha}} = -2\sum(y_i - \hat{\alpha} - \hat{\beta} x_i) = 0$$

$$\Rightarrow \sum e_i = 0$$

$$\frac{\partial S}{\partial \hat{\beta}} = -2\sum x_i (y_i - \hat{\alpha} - \hat{\beta} x_i) = 0$$

$$\Rightarrow \quad \sum x_i e_i = 0$$

$$e_i = y_i - \hat{y}_i = \alpha + \beta x_i + u_i - \hat{\alpha} - \hat{\beta} x_i$$

$$= u_i - (\hat{\alpha} - \alpha) - (\hat{\beta} - \beta) x_i$$

$$u_i = (\hat{\alpha} - \alpha) + (\hat{\beta} - \beta) x_i + e_i$$

$$u_i^2 = (\hat{\alpha} - \alpha)^2 + (\hat{\beta} - \beta)^2 x_i^2 + e_i^2 + 2(\hat{\alpha} - \alpha)(\hat{\beta} - \beta) x_i$$
$$+ 2(\hat{\alpha} - \alpha) e_i + 2(\hat{\beta} - \beta) x_i e_i$$

$$\sum u_i^2 = n(\hat{\alpha} - \alpha)^2 + (\hat{\beta} - \beta)^2 \sum x_i^2 + \sum e_i^2 + 2n\bar{x}(\hat{\alpha} - \alpha)(\hat{\beta} - \beta)$$

$$E(\sum u_i^2) = n \underbrace{E(\hat{\alpha} - \alpha)^2}_{V(\hat{\alpha})} + \underbrace{E(\hat{\beta} - \beta)^2}_{V(\hat{\beta})} \sum x_i^2 + E(\sum e_i^2) + 2n\bar{x} \underbrace{E((\hat{\alpha} - \alpha)(\hat{\beta} - \beta))}_{Cov(\hat{\alpha}, \hat{\beta})}$$

$$V(\hat{\beta}) = E(\hat{\beta} - \beta)^2 = E\left[\frac{\sum(x_i - \bar{x}) u_i}{\sum(x_i - \bar{x})^2}\right]^2 = E\left[\frac{\sum(x_i - \bar{x})^2 u_i^2 + \sum\sum_{i \neq j} (x_i - \bar{x})(x_j - \bar{x}) u_i u_j}{(\sum(x_i - \bar{x})^2)^2}\right]$$

$$= \frac{\sum(x_i - \bar{x})^2 E(u_i^2) + \sum\sum_{i \neq j} (x_i - \bar{x})(x_j - \bar{x}) E(u_i u_j)}{(\sum(x_i - \bar{x})^2)^2}$$

$$= \frac{\sigma^2}{\sum(x_i - \bar{x})^2}$$

$$V(\hat{\alpha}) = E(\hat{\alpha}-\alpha)^2 = E(-(\hat{\beta}-\beta)\bar{x}+\bar{u})^2$$

$$= E((\hat{\beta}-\beta)^2\bar{x}^2 - 2(\hat{\beta}-\beta)\bar{x}\bar{u} + \bar{u}^2)$$

$$= \bar{x}^2 V(\hat{\beta}) + E(\bar{u}^2) - 2E((\hat{\beta}-\beta)\bar{x}\bar{u})$$

$$V(\bar{u}) = E(\bar{u}^2) - [E(\bar{u})]^2$$

$$= E(\bar{u}^2) = \frac{\sigma^2}{n}$$

$$E((\hat{\beta}-\beta)\bar{u}) = E\left[\frac{\sum(x_i-\bar{x})u_i \frac{1}{n}\sum u_i}{\sum(x_i-\bar{x})^2}\right]$$

$$= \frac{1}{n\sum(x_i-\bar{x})^2} E\Big[\sum_i (x_i-\bar{x})u_i \sum_j u_j\Big]$$

$$= \frac{1}{n\sum(x_i-\bar{x})^2} \underbrace{\sum_i\sum_j (x_i-\bar{x})E(u_iu_j)}_{0}$$

$$= 0$$

$$V(\hat{\alpha}) = \frac{\sigma^2}{n} + \bar{x}V(\hat{\beta}) = \frac{\sigma^2\sum x_i^2}{n\sum(x_i-\bar{x})^2}$$

$$\mathrm{cov}(\hat{\alpha},\hat{\beta}) = E(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)$$

$$= E(-(\hat{\beta}-\beta)\bar{x}+\bar{u})(\hat{\beta}-\beta))]$$

$$= -E(\hat{\beta}-\beta)^2\bar{x} + E(\bar{u}(\hat{\beta}-\beta))$$

$$= -\bar{x}V(\hat{\beta}) = \frac{-\sigma^2\bar{x}}{\sum(x_i-\bar{x})^2}$$

$$E(\sum u_i^2) = n\sigma^2 = n\left(\frac{\sigma^2 \sum x_i^2}{n\sum(x_i-\bar{x})^2}\right) + \frac{\sigma^2 \sum x_i^2}{\sum(x_i-\bar{x})^2} + E(\sum e_i^2)$$

$$+ 2n\bar{x}\left(\frac{-\sigma^2 \bar{x}}{\sum(x_i-\bar{x})^2}\right)$$

$$n\sigma^2 = 2\sigma^2\left(\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\bar{x})^2}\right) + E(\sum e_i^2)$$

$$(n-2)\sigma^2 = E(\sum e_i^2)$$

$$\sigma^2 = E\left(\frac{\sum e_i^2}{n-2}\right) = E(s^2)$$