第14章　Unit Root (単位根) Test (Dickey-Fuller (DF) Test)

自己回帰（AutoRegressive, AR）モデル：

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + u_t$$

$\longrightarrow$ AR(p) モデル

移動平均（Moving Average, MA）モデル

$$y_t = u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q}$$

$\longrightarrow$ MA(q) モデル

自己回帰移動平均（ARMA）モデル
$\longrightarrow$ ARMA(p, q) モデル

以下では，AR(1) モデルを考える。

1. Why is a unit root problem important?

   (a) In nonstationary time series, the unit root is the most important.

   In the case of unit root, OLSE of the first-order autoregressive coefficient is consistent.

   OLSE is $\sqrt{T}$-consistent in the case of stationary AR(1) process, but OLSE is T-consistent in the case of nonstationay AR(1) process.

   (b) A lot of economic variables increase over time.

   It is important to check an economic variable is trend stationary (i.e., $y_t = a_0 + a_1 t + \epsilon_t$) or difference stationary (i.e., $y_t = b_0 + y_{t-1} + \epsilon_t$).

   Consider k-step ahead prediction for both cases.

   (Trend Stationarity) $\qquad y_{t+k|t} = a_0 + a_1(t + k)$

   (Difference Stationarity) $\qquad y_{t+k|t} = b_0 k + y_t$

2. **The Case of $|\phi_1| < 1$:**

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \qquad y_0 = 0, \qquad t = 1, \cdots, T$$

Then, OLSE of ϕ_1 is:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2}.$$

In the case of $|\phi_1| < 1$,

$$\hat{\phi}_1 = \phi_1 + \frac{\dfrac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\dfrac{1}{T}\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1.$$

Note as follows:

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow \mathrm{E}(y_{t-1}\epsilon_t) = 0.$$

By the central limit theorem,

$$\frac{\overline{y\epsilon} - \mathrm{E}(\overline{y\epsilon})}{\sqrt{\mathrm{V}(\overline{y\epsilon})}} \longrightarrow N(0,1)$$

where

$$\overline{y\epsilon} = \frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t.$$

$$\begin{aligned}
\mathrm{E}(\overline{y\epsilon}) &= 0, \\
\mathrm{V}(\overline{y\epsilon}) &= \mathrm{V}(\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t) = \mathrm{E}\Big((\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t)^2\Big) \\
&= \frac{1}{T^2}\mathrm{E}\Big(\sum_{t=1}^{T}\sum_{s=1}^{T} y_{t-1}y_{s-1}\epsilon_t\epsilon_s\Big) = \frac{1}{T^2}\mathrm{E}\Big(\sum_{t=1}^{T} y_{t-1}^2\epsilon_t^2\Big) = \frac{1}{T}\sigma^2\gamma(0).
\end{aligned}$$

Therefore,

$$\frac{\overline{y\epsilon}}{\sqrt{\sigma^2\gamma(0)/T}} = \frac{1}{\sigma_\epsilon\sqrt{\gamma(0)}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0,1),$$

which is rewritten as:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0,\sigma_\epsilon^2\gamma(0)).$$

Using $\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0)$, we have the following asymptotic distribution:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N\left(0, \frac{\sigma_\epsilon^2}{\gamma(0)}\right) = N\left(0, 1-\phi_1^2\right).$$

Note that $\gamma(0) = \dfrac{\sigma_\epsilon^2}{1-\phi_1^2}$.

3. In the case of $\phi_1 = 1$, as expected, we have:

$$\sqrt{T}(\hat{\phi}_1 - 1) \longrightarrow 0.$$

That is, $\hat{\phi}_1$ has the distribution which converges in probability to $\phi_1 = 1$ (i.e., degenerated distribution).

Is this true?

4. **The Case of $\phi_1 = 1$:** $\Longrightarrow$ Random Walk Process

$y_t = y_{t-1} + \epsilon_t$ with $y_0 = 0$ is written as:

$$y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots + \epsilon_1.$$

Therefore, we can obtain:

$$y_t \sim N(0, \sigma_\epsilon^2 t).$$

The variance of y_t depends on time t. $\Longrightarrow$ y_t is nonstationary.

5. Remember that $\hat{\phi}_1 = \phi_1 + \dfrac{\sum y_{t-1}\epsilon_t}{\sum y_{t-1}^2}$.

   (a) First, consider the numerator $\sum y_{t-1}\epsilon_t$.

   We have $y_t^2 = (y_{t-1} + \epsilon_t)^2 = y_{t-1}^2 + 2y_{t-1}\epsilon_t + \epsilon_t^2$.

   Therefore, we obtain:

$$y_{t-1}\epsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \epsilon_t^2).$$

   Taking into account $y_0 = 0$, we have:

$$\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}y_T^2 - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t^2.$$

   Divided by $\sigma_\epsilon^2 T$ on both sides, we have the following:

$$\frac{1}{\sigma_\epsilon^2 T}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma_\epsilon\sqrt{T}}\right)^2 - \frac{1}{2\sigma_\epsilon^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2.$$

From $y_t \sim N(0, \sigma_\epsilon^2 t)$, we obtain the following result:

$$\left(\frac{y_T}{\sigma_\epsilon \sqrt{T}}\right)^2 \sim \chi^2(1).$$

Moreover, the second term is derived from:

$$\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \sigma_\epsilon^2.$$

Therefore,

$$\frac{1}{\sigma_\epsilon^2 T}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma\sqrt{T}}\right)^2 - \frac{1}{2\sigma_\epsilon^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \frac{1}{2}(\chi^2(1) - 1).$$

(b) Next, consider $\sum y_{t-1}^2$.

$$\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) = \sum_{t=1}^{T}\mathrm{E}(y_{t-1}^2) = \sum_{t=1}^{T}\sigma_\epsilon^2(t-1) = \sigma_\epsilon^2\frac{T(T-1)}{2}.$$

Thus, we obtain the following result:

$$\frac{1}{T^2}\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) \longrightarrow \text{ a fixed value.}$$

Therefore,

$$\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \text{ a distribution.}$$

6. Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T)\sum y_{t-1}\epsilon_t}{(1/T^2)\sum y_{t-1}^2} \longrightarrow \text{ a distribution.}$$

The distributions of the t statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$, where s_ϕ denotes the standard error of $\hat{\phi}_1$.

$\Longrightarrow$ Compare t distribution with (a) – (c).

$\Longrightarrow$ **Unit Root Test (単位根検定, or Dickey-Fuller (DF) Test)**

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

Test $H_0 : \ \phi_1 = 1$ against $H_1 : \ \phi_1 < 1$.

Equivalently,

$$\Delta y_t = \rho y_{t-1} + \epsilon_t.$$

Test $H_0 : \ \rho = 0$ against $H_1 : \ \rho < 0$.

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.49	−2.06	−1.71	−1.32	1.32	1.71	2.06	2.49
50	−2.40	−2.01	−1.68	−1.30	1.30	1.68	2.01	2.40
100	−2.36	−1.98	−1.66	−1.29	1.29	1.66	1.98	2.36
250	−2.34	−1.97	−1.65	−1.28	1.28	1.65	1.97	2.34
500	−2.33	−1.96	−1.65	−1.28	1.28	1.65	1.96	2.33
∞	−2.33	−1.96	−1.64	−1.28	1.28	1.64	1.96	2.33

(a) $\boldsymbol{H_0: \ y_t = y_{t-1} + \epsilon_t}$
$\boldsymbol{H_1: \ y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

To test $H_0: \ \rho = 0$ against $H_1: \ \rho < 0$, estimate $\Delta y_t = \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

(b) $\boldsymbol{H_0 : y_t = y_{t-1} + \epsilon_t}$
$\boldsymbol{H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60

To test $H_0 : \rho = 0$ against $H_1 : \rho < 0$, estimate $\Delta y_t = \alpha_0 + \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

(c) $\boldsymbol{H_0 :\ y_t = \alpha_0 + y_{t-1} + \epsilon_t}$
$\boldsymbol{H_1 :\ y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

To test $H_0 :\ \rho = 0$ against $H_1 :\ \rho < 0$, estimate $\Delta y_t = \alpha_0 + \alpha_1 t + \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.