

Then, $V(Z'u) = E(Z'u(Z'u)') = E(Z'uu'Z) = Z'E(uu')Z = \sigma^2 Z'\Omega Z = W^{-1}$.

The following minimization problem should be solved.

$$\min_{\beta} (y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta).$$

The solution of β is given by the GMM estimator, denoted by β_{GMM} .

Remark: For the model: $y = X\beta + u$ and $u \sim (0, \sigma^2\Omega)$, solving the following minimization problem:

$$\min_{\beta} (y - X\beta)'\Omega^{-1}(y - X\beta),$$

GLS is given by:

$$b = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y.$$

Note that b is the best linear unbiased estimator.

Remark: The solution of the above minimization problem is equivalent to the GLE estimator of β in the following regression model:

$$Z'y = Z'X\beta + Z'u,$$

where Z , y , X , β and u are $n \times r$, $n \times 1$, $n \times k$, $k \times 1$ and $n \times 1$ matrices or vectors.

Note that $r > k$.

$y^* = Z'y$, $X^* = Z'X$ and $u^* = Z'u$ denote $r \times 1$, $r \times k$ and $r \times 1$ matrices or vectors, where $r > k$.

Rewrite as follows:

$$y^* = X^*\beta + u^*,$$

$\implies r$ is taken as the sample size.

u^* is a $r \times 1$ vector.

The elements of u^* are correlated with each other, because each element of u^* is a function of u_1, u_2, \dots, u_n .

The variance of u^* is:

$$V(u^*) = V(Z'u) = \sigma^2 Z'\Omega Z.$$

Go back to GMM:

$$\begin{aligned} & (y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta) \\ &= y'Z(Z'\Omega Z)^{-1}Z'y - \beta'X'Z(Z'\Omega Z)^{-1}Z'y - y'Z(Z'\Omega Z)^{-1}Z'X\beta + \beta'X'Z(Z'\Omega Z)^{-1}Z'X\beta \\ &= y'ZWZ'y - 2y'Z(Z'\Omega Z)^{-1}Z'X\beta + \beta'X'Z(Z'\Omega Z)^{-1}Z'X\beta. \end{aligned}$$

Note that $\beta'X'Z(Z'\Omega Z)^{-1}Z'y = y'Z(Z'\Omega Z)^{-1}Z'X\beta$ because both sides are scalars.

Remember that $\frac{\partial Ax}{x} = A'$ and $\frac{\partial x'Ax}{x} = (A + A')x$.

Then, we obtain the following derivation:

$$\begin{aligned}
 & \frac{\partial(y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta)}{\partial\beta} \\
 &= -2(y'Z(Z'\Omega Z)^{-1}Z'X)' + (X'Z(Z'\Omega Z)^{-1}Z'X + (X'Z(Z'\Omega Z)^{-1}Z'X)')\beta \\
 &= -2X'Z(Z'\Omega Z)^{-1}Z'y + 2X'Z(Z'\Omega Z)^{-1}Z'X\beta = 0
 \end{aligned}$$

The solution of β is denoted by β_{GMM} , which is:

$$\beta_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'y.$$

The mean of β_{GMM} is asymptotically obtained.

$$\begin{aligned}
 \beta_{GMM} &= (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'(X\beta + u) \\
 &= \beta + (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u \\
 &= \beta + \left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'X\right)^{-1}\left(\frac{1}{n}X'Z\right)\left(\frac{1}{n}Z'\Omega Z\right)^{-1}\left(\frac{1}{n}Z'u\right)
 \end{aligned}$$

We assume that

$$\frac{1}{n}X'Z \longrightarrow M_{xz} \quad \text{and} \quad \frac{1}{n}Z'\Omega Z \longrightarrow M_{z\Omega z},$$

which are $k \times r$ and $r \times r$ matrices.

From the assumption of $\frac{1}{n}Z'u \longrightarrow 0$, we have the following result:

$$\beta_{GMM} \longrightarrow \beta + (M_{xz}M_{z\Omega z}^{-1}M'_{xz})^{-1}M_{xz}M_{z\Omega z}^{-1} \times 0 = \beta.$$

Thus, β_{GMM} is a consistent estimator of β (i.e., asymptotically unbiased estimator).

The variance of β_{GMM} is asymptotically obtained as follows:

$$\begin{aligned}
V(\beta_{GMM}) &= E\left((\beta_{GMM} - E(\beta_{GMM}))(\beta_{GMM} - E(\beta_{GMM}))'\right) \approx E\left((\beta_{GMM} - \beta)(\beta_{GMM} - \beta)'\right) \\
&= E\left((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u)'\right) \\
&= E\left((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'uu'Z(Z'\Omega Z)^{-1}Z'X(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}\right) \\
&\approx (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'E(uu')Z(Z'\Omega Z)^{-1}Z'X(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1} \\
&= \sigma^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}.
\end{aligned}$$

Note that $\beta_{GMM} \rightarrow \beta$ implies $E(\beta_{GMM}) \rightarrow \beta$ in the 1st line.

\approx in the 4th line indicates that Z and X are treated as exogenous variables although they are stochastic.

We assume that $E(uu') = \sigma^2\Omega$ from the 4th line to the 5th line.

- We derive the asymptotic distribution of β_{GMM} .

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{z\Omega z}).$$

Accordingly, β_{GMM} is asymptotically distributed as:

$$\begin{aligned} \sqrt{n}(\beta_{GMM} - \beta) &= \left(\left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{\sqrt{n}}Z'u \right) \\ &\longrightarrow N(0, \sigma^2 (M_{xz} M_{z\Omega z}^{-1} M'_{xz})^{-1}). \end{aligned}$$

Practically, we use: $\beta_{GMM} \sim N(\beta, s^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1})$,

where $s^2 = \frac{1}{n-k}(y - X\beta_{GMM})'\Omega^{-1}(y - X\beta_{GMM})$.

We may use n instead of $n - k$.

Identically and Independently Distributed Errors:

- If u_1, u_2, \dots, u_n are mutually independent and u_i is distributed with mean zero and variance σ^2 , the mean and variance of u^* are given by:

$$E(u^*) = 0 \quad \text{and} \quad V(u^*) = E(u^*u^{*'}) = \sigma^2 Z'Z.$$

Using GLS, GMM is obtained as:

$$\beta_{GMM} = (X^{*'}(Z'Z)^{-1}X^*)^{-1}X^{*'}(Z'Z)^{-1}y^* = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

- We derive the asymptotic distribution of β_{GMM} .

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Accordingly, β_{GMM} is distributed as:

$$\begin{aligned}\sqrt{n}(\beta_{GMM} - \beta) &= \left(\left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{n} Z'X \right) \right)^{-1} \left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{\sqrt{n}} Z'u \right) \\ &\longrightarrow N\left(0, \sigma^2 (M_{xz} M_{zz}^{-1} M'_{xz})^{-1}\right).\end{aligned}$$

Practically, for large n we use the following distribution:

$$\beta_{GMM} \sim N\left(\beta, s^2 (X'Z(Z'Z)^{-1}Z'X)^{-1}\right),$$

where $s^2 = \frac{1}{n-k} (y - X\beta_{GMM})'(y - X\beta_{GMM})$.

- The above GMM is equivalent to 2SLS.

$X: n \times k, \quad Z: n \times r, \quad r > k.$

Assume:

$$\frac{1}{n} X' u = \frac{1}{n} \sum_{i=1}^n x_i' u_i \longrightarrow E(x' u) \neq 0,$$

$$\frac{1}{n} Z' u = \frac{1}{n} \sum_{i=1}^n z_i' u_i \longrightarrow E(z' u) = 0.$$

Regress X on Z , i.e., $X = Z\Gamma + V$ by OLS, where Γ is a $r \times k$ unknown parameter matrix and V is an error term,

Denote the predicted value of X by $\hat{X} = Z\hat{\Gamma} = Z(Z'Z)^{-1}Z'X$, where $\hat{\Gamma} = (Z'Z)^{-1}Z'X$.

Review — IV estimator: Consider the regression model is:

$$y = X\beta + u,$$

Assumption: $E(X'u) \neq 0$ and $E(Z'u) = 0$.

The $n \times k$ matrix Z is called the instrumental variable (IV).

The IV estimator is given by:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

- Note that 2SLS is equivalent to IV in the case of $Z = \hat{X}$, where this Z is different from the previous Z .

This Z is a $n \times k$ matrix, while the previous Z is a $n \times r$ matrix.

Z in the IV estimator is replaced by \hat{X} .

Then,

$$\beta_{2SLS} = (\hat{X}'X)^{-1}\hat{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y = \beta_{GMM}.$$

GMM is interpreted as the GLS applied to MM.