Example: Demand function using STATA

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year
y　＝ 実収入（一月当たり，実質データ）
q1 ＝ 穀類支出額（一年当たり，実質データ）
p1 ＝ 穀類価格　（相対価格＝穀類 CPI ／総合 CPI）
p2 ＝ 魚介類価格（相対価格＝魚介類 CPI ／総合 CPI）
p3 ＝ 肉類価格　（相対価格＝肉類 CPI ／総合 CPI）

year	y	q1	p1	p2	p3
2000	567865	7087.0	1.043390	0.884965	0.818365
2001	561722	6993.1	1.032520	0.886179	0.822154
2002	553768	6934.4	1.031800	0.891282	0.834872
2003	539928	6816.8	1.050410	0.876543	0.843621
2004	547006	6651.6	1.089510	0.865226	0.868313
2005	541367	6615.8	1.020640	0.862745	0.887513
2006	540863	6523.7	1.000000	0.878601	0.891975
2007	543994	6680.5	0.994856	0.886831	0.908436
2008	541821	6494.7	1.043610	0.894523	0.932049
2009	533154	6477.3	1.066870	0.898148	0.934156
2010	539577	6458.2	1.040420	0.889119	0.924352
2011	529750	6448.4	1.025960	0.894081	0.925234
2012	538988	6377.6	1.057170	0.904366	0.917879
2013	542018	6360.7	1.047620	0.909938	0.916149
2014	523953	6174.6	1.016130	0.971774	0.960685
2015	525669	6268.0	1.000000	1.000000	1.000000
2016	527501	6244.8	1.018020	1.019020	1.017020

```
2017 531693 6106.6 1.027890 1.066730 1.025900


. tsset year
        time variable:  year, 2000 to 2017
                delta:  1 unit

. reg q1 y p1 p2 p3 if year>2000.5

      Source |       SS           df       MS      Number of obs   =        17
-------------+----------------------------------   F(4, 12)        =     25.83
       Model |  913640.443         4  228410.111   Prob > F        =    0.0000
    Residual |  106100.077        12  8841.67308   R-squared       =    0.8960
-------------+----------------------------------   Adj R-squared   =    0.8613
       Total |  1019740.52        16  63733.7825   Root MSE        =     94.03

------------------------------------------------------------------------------
          q1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |   .0067843   .0045443     1.49   0.161     -.003117    .0166856
          p1 |  -1128.834   998.7698    -1.13   0.280    -3304.966    1047.299
          p2 |   356.8095   806.2301     0.44   0.666    -1399.815    2113.434
          p3 |  -3442.221   1130.078    -3.05   0.010    -5904.448   -979.9931
       _cons |   6850.563   3179.316     2.15   0.052    -76.57278     13777.7
------------------------------------------------------------------------------

. gmm (q1-{b0}-{b1}*y-{b2}*p1-{b3}*p2-{b4}*p3) if year>2000.5, instruments(y p1
> p2 p3)
```

```
Step 1
Iteration 0:   GMM criterion Q(b) =   42400764
Iteration 1:   GMM criterion Q(b) =  6.781e-12
Iteration 2:   GMM criterion Q(b) =  6.781e-12  (backed up)

Step 2
Iteration 0:   GMM criterion Q(b) =  1.966e-15
Iteration 1:   GMM criterion Q(b) =  1.963e-15  (backed up)
convergence not achieved
    The Gauss-Newton stopping criterion has been met but missing standard errors
    indicate some of the parameters are not identified.

GMM estimation

Number of parameters =   5
Number of moments    =   5
Initial weight matrix: Unadjusted                 Number of obs   =         17
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         /b0 |   6850.563   17645.71     0.39   0.698     -27734.4    41435.53
         /b1 |   .0067843   .0282325     0.24   0.810    -.0485504     .062119
         /b2 |  -1128.834   1057.915    -1.07   0.286    -3202.309    944.6415
         /b3 |   356.8095    1565.86     0.23   0.820    -2712.219    3425.838
         /b4 |  -3442.221   5085.561    -0.68   0.498    -13409.74    6525.296
------------------------------------------------------------------------------
```

```
Instruments for equation 1: y p1 p2 p3 _cons
Warning: convergence not achieved

. gmm (q1-{b0}-{b1}*y-{b2}*p1-{b3}*p2-{b4}*p3) if year>2000.5, instruments(p1 p2
> p3 l.p1 l.p2 l.p3)

Step 1
Iteration 0:   GMM criterion Q(b) =   42404066
Iteration 1:   GMM criterion Q(b) =  2790.3146
Iteration 2:   GMM criterion Q(b) =  2790.3146

Step 2
Iteration 0:   GMM criterion Q(b) =   .3201826
Iteration 1:   GMM criterion Q(b) =   .2469289
Iteration 2:   GMM criterion Q(b) =   .2469289

GMM estimation

Number of parameters =   5
Number of moments    =   7
Initial weight matrix: Unadjusted                 Number of obs   =         17
GMM weight matrix:     Robust

------------------------------------------------------------------------------
             |               Robust
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         /b0 |  -1192.466   4669.012    -0.26   0.798    -10343.56     7958.63
         /b1 |   .0186312   .0067682     2.75   0.006     .0053657    .0318967
```

```
         /b2 |  -1016.864    780.979    -1.30   0.193    -2547.554    513.8271
         /b3 |  -905.5585   598.0885    -1.51   0.130     -2077.79    266.6734
         /b4 |  -499.8064   1147.985    -0.44   0.663    -2749.815    1750.202
------------------------------------------------------------------------------
Instruments for equation 1: p1 p2 p3 L.p1 L.p2 L.p3 _cons

. estat overid

  Test of overidentifying restriction:

  Hansen's J chi2(2) = 4.19779 (p = 0.1226)
```

5　Time Series Analysis (時系列分析)

5.1　Introduction

代表的テキスト：

・J.D. Hamilton (1994) *Time Series Analysis*

沖本・井上訳 (2006)『時系列解析 (上・下)』

・A.C. Harvey (1981) *Time Series Models*

国友・山本訳 (1985)『時系列モデル入門』

・沖本竜義 (2010)『経済・ファイナンスデータの計量時系列分析』

1. **Stationarity (定常性) :**

Let $y_1, y_2, \cdots, y_T$ be time series data.

(a) **Weak Stationarity (弱定常性) :**

$$\mathrm{E}(y_t) = \mu,$$

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

The first and second moments do not depend on time.

The second moment depends on time difference, not time itself.

(b) **Strong Stationarity (強定常性) :**

Let $f(y_{t_1}, y_{t_2}, \cdots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \cdots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \cdots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \cdots, y_{t_r+\tau})$$

All the moments are same for all τ.

2. **Ergodicity (エルゴード性) :**

As time difference between two randopm variables is large, the correlation between the two random variables becomes zero.

$y_1, y_2, \cdots, y_T$ is said to be ergodic in mean when $\bar{y}$ converges in probability to $\mathrm{E}(y_t)$.

3. **Auto-covariance Function (自己共分散関数) :** In the case of stationary process,

$$\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \qquad \tau = 0, 1, 2, \cdots$$

$$\gamma(\tau) = \gamma(-\tau)$$

4. **Auto-correlation Function (自己相関関数) :** In the case of stationary process,

$$\rho(\tau) = \frac{\mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\mathrm{Var}(y_t)}\sqrt{\mathrm{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-\tau}) = \gamma(0)$.

Graph between τ and $\rho(\tau)$ (or $\hat{\rho}(\tau)$) $\longrightarrow$ **Correlogram (コレログラム)**

5. **Sample Mean (標本平均) :**

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} y_t$$

6. **Sample Auto-covariance (標本自己共分散) :**

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^{T} (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

7. **Sample Auto-correlation (標本自己相関関数) :**

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

8. **Lag Operator (ラグ作要素) :**

$$L^{\tau} y_t = y_{t-\tau}, \qquad \tau = 1, 2, \cdots$$

9. **Likelihood Function (尤度関数) —** Innovation Form :

The joint distribution of $y_1, y_2, \cdots, y_T$ is written as:

$$\begin{aligned}
f(y_1, , y_2, \cdots, y_T) &= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1}, \cdots, y_1) \\
&= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1} | y_{T-2}, \cdots, y_1) f(y_{T-2}, \cdots, y_1) \\
&\ \ \vdots \\
&= f(y_T | y_{T-1}, \cdots, y_1) f(y_{T-1} | y_{T-2}, \cdots, y_1) \ \cdots \ f(y_2 | y_1) f(y_1) \\
&= f(y_1) \prod_{t=2}^{T} f(y_t | y_{t-1}, \cdots, y_1).
\end{aligned}$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \cdots, y_T) = \log f(y_1) + \sum_{t=2}^{T} \log f(y_t|y_{t-1}, \cdots, y_1).$$

Under the normality assumption, $f(y_t|y_{t-1}, \cdots, y_1)$ is given by the normal distribution with conditional mean $\mathrm{E}(y_t|y_{t-1}, \cdots, y_1)$ and conditional variance $\mathrm{Var}(y_t|y_{t-1}, \cdots, y_1)$.

5.2 Autoregressive Model (自己回帰モデル or AR モデル)

1. **AR(p) Model :**

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L) y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. **Stationarity (定常性) :**

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one, y_t is stationary.

Suppose that the p solutions include imaginary numbers.

When the p solutions are outside unit circle, y_t is stationary.

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

(a) The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

(b) Rewriting the AR(1) model,

$$
\begin{aligned}
y_t &= \phi_1 y_{t-1} + \epsilon_t \\
&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\
&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\
&\quad \vdots \\
&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.
\end{aligned}
$$

As s is large, ϕ_1^s approaches zero. $\Longrightarrow$ Stationarity condition

(c) For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$
y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots
$$

MA representation of AR model.

(d) (*) **MA (Moving Average，移動平均) Model:** MA(q)

$$y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q},$$

which is rewritten as:

$$y_t = \theta(L)\epsilon_t,$$

where

$$\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q.$$

(e) Mean of AR(1) process, μ

$$\begin{aligned}\mu = \mathrm{E}(y_t) &= \mathrm{E}(\epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + \cdots) \\ &= \mathrm{E}(\epsilon_t) + \phi_1\mathrm{E}(\epsilon_{t-1}) + \phi_1^2\mathrm{E}(\epsilon_{t-2}) + \cdots = 0\end{aligned}$$

(f) Variance of AR(1) process, $\gamma(0)$

$$\gamma(0) = \mathrm{V}(y_t) = \mathrm{V}(\epsilon_t + \phi_1\epsilon_{t-1} + \phi_1^2\epsilon_{t-2} + \cdots)$$

$$= \mathrm{V}(\epsilon_t) + \mathrm{V}(\phi_1 \epsilon_{t-1}) + \mathrm{V}(\phi_1^2 \epsilon_{t-2}) + \cdots$$

$$= \mathrm{V}(\epsilon_t) + \phi_1^2 \mathrm{V}(\epsilon_{t-1}) + \phi_1^4 \mathrm{V}(\epsilon_{t-2}) + \cdots$$

$$= \sigma^2 (1 + \phi_1^2 + \phi_1^4 + \cdots) = \frac{\sigma^2}{1 - \phi_1^2}$$

(g) Autocovariance and autocorrelation functions of the AR(1) process:

Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}
\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\
&= \mathrm{E}\Big((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}) y_{t-\tau}\Big) \\
&= \phi_1^\tau \mathrm{E}(y_{t-\tau} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) + \phi_1 \mathrm{E}(\epsilon_{t-1} y_{t-\tau}) + \cdots + \phi_1^{\tau-1} \mathrm{E}(\epsilon_{t-\tau+1} y_{t-\tau}) \\
&= \phi_1^\tau \gamma(0) = \frac{\sigma^2 \phi_1^\tau}{1 - \phi_1^2}.
\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^{\tau}.$$

(h) **Another Derivation of $\gamma(\tau)$:**

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$\mathrm{E}(y_t y_{t-\tau}) = \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2 = \phi_1^2 \gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1 \gamma(0)$.

Autocovariance function $\gamma(\tau)$ is:

$$\gamma(\tau) = \phi_1 \gamma(\tau - 1) = \phi_1^2 \gamma(\tau - 2) = \cdots = \phi_1^\tau \gamma(0).$$

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2}$$

(i) Estimation of AR(1) model:

i. Likelihood function

$$\begin{aligned}\log f(y_T, \cdots, y_1) &= \log f(y_1) + \sum_{t=2}^{T} \log f(y_t | y_{t-1}, \cdots, y_1) \\ &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log\left(\frac{\sigma^2}{1 - \phi_1^2}\right) - \frac{1}{\sigma^2/(1 - \phi_1^2)} y_1^2 \\ &\quad - \frac{T-1}{2} \log(2\pi) - \frac{T-1}{2} \log(\sigma^2) - \frac{1}{\sigma^2} \sum_{t=2}^{T} (y_t - \phi_1 y_{t-1})^2\end{aligned}$$

$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log(\sigma^2) - \frac{1}{2}\log\left(\frac{1}{1-\phi_1^2}\right)$$

$$-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2 - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

Note as follows:

$$f(y_1) = \frac{1}{\sqrt{2\pi\sigma^2/(1-\phi_1^2)}}\exp\left(-\frac{1}{2\sigma^2/(1-\phi_1^2)}y_1^2\right)$$

$$f(y_t|y_{t-1},\cdots,y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_t - \phi_1 y_{t-1})^2\right)$$

$$\frac{\partial \log f(y_T,\cdots,y_1)}{\partial\sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T,\cdots,y_1)}{\partial\phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of ϕ_1 and σ^2 satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2\tilde{\phi}_1}{1-\tilde{\phi}_1^2}\right)\Big/\sum_{t=2}^{T} y_{t-1}^2$$

ii. Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to ϕ_1.

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} y_{t-1}y_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^{T} y_{t-1}\epsilon_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{(1/T)\sum_{t=2}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=2}^{T} y_{t-1}^2}$$

$$\longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1$$

OLSE of ϕ_1 is a consistent estimator.

The following equations are utilized.

$$\mathrm{E}(y_{t-1}\epsilon_t) = 0$$

$$\mathrm{E}(y_{t-1}^2) = \mathrm{Var}(y_{t-1}) = \gamma(0)$$

(j) Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

Proof:

$y_{t-1}\epsilon_t$, $t = 1, 2, \cdots, T$, are distributed with mean zero and variance $\dfrac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T)\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1 - \phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0, \frac{\sigma_\epsilon^4}{1-\phi_1^2}).$$

Next,

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1-\phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T}) \sum_{t=1}^{T} y_{t-1}\epsilon_t}{(1/T) \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N(0, 1-\phi_1^2)$$

(k) Some formulas:

i. Central Limit Theorem

Random variables $x_1,\ x_2,\ \cdots,\ x_T$ are mutually independently distributed with mean μ and variance σ^2.

Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} = \frac{\overline{x} - \mu}{\sigma / \sqrt{T}} \longrightarrow N(0, 1)$$

ii. Central Limit Theorem II

Random variables x_1, x_2, $\cdots$, x_T are distributed with mean μ and variance σ^2.

Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} \longrightarrow N(0, 1)$$

iii. Let x and y be random variables.

y converges in distribution to a distribution, and x converges in probability to a fixed value.

Then, xy converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \qquad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$