

5.3 Unit Root (単位根) Test (Dickey-Fuller (DF) Test)

1. The Case of $|\phi_1| < 1$:

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \quad y_0 = 0, \quad t = 1, \dots, T$$

Then, OLSE of ϕ_1 is:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2}.$$

We have the following asymptotic distribution:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{\frac{1}{\sqrt{T}} \sum_{t=1}^T y_{t-1} \epsilon_t}{\frac{1}{T} \sum_{t=1}^T y_{t-1}^2} \rightarrow N\left(0, \frac{\sigma_\epsilon^2}{\gamma(0)}\right) = N(0, 1 - \phi_1^2).$$

Note that $\gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$.

2. In the case of $\phi_1 = 1$, as expected, we have:

$$\sqrt{T}(\hat{\phi}_1 - 1) \longrightarrow 0.$$

That is, $\hat{\phi}_1$ has the distribution which converges in probability to $\phi_1 = 1$ (i.e., degenerated distribution).

Is this true?

3. **The Case of $\phi_1 = 1$:** \implies Random Walk Process

$y_t = y_{t-1} + \epsilon_t$ with $y_0 = 0$ is written as:

$$y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots + \epsilon_1.$$

Therefore, we can obtain:

$$y_t \sim N(0, \sigma_\epsilon^2 t).$$

The variance of y_t depends on time t . $\implies y_t$ is nonstationary.

4. Remember that $\hat{\phi}_1 = \phi_1 + \frac{\sum y_{t-1}\epsilon_t}{\sum y_{t-1}^2}$.

(a) First, consider the numerator $\sum y_{t-1}\epsilon_t$.

$$\text{We have } y_t^2 = (y_{t-1} + \epsilon_t)^2 = y_{t-1}^2 + 2y_{t-1}\epsilon_t + \epsilon_t^2.$$

Therefore, we obtain:

$$y_{t-1}\epsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \epsilon_t^2).$$

Taking into account $y_0 = 0$, we have:

$$\sum_{t=1}^T y_{t-1}\epsilon_t = \frac{1}{2}y_T^2 - \frac{1}{2}\sum_{t=1}^T \epsilon_t^2.$$

Divided by $\sigma_\epsilon^2 T$ on both sides, we have the following:

$$\frac{1}{\sigma_\epsilon^2 T} \sum_{t=1}^T y_{t-1}\epsilon_t = \frac{1}{2} \left(\frac{y_T}{\sigma_\epsilon \sqrt{T}} \right)^2 - \frac{1}{2\sigma_\epsilon^2} \frac{1}{T} \sum_{t=1}^T \epsilon_t^2.$$

From $y_t \sim N(0, \sigma_\epsilon^2 t)$, we obtain the following result:

$$\left(\frac{y_T}{\sigma_\epsilon \sqrt{T}} \right)^2 \sim \chi^2(1).$$

Moreover, the second term is derived from:

$$\frac{1}{T} \sum_{t=1}^T \epsilon_t^2 \rightarrow \mathbb{E}(\epsilon_t^2) = \sigma_\epsilon^2.$$

Therefore,

$$\frac{1}{\sigma_\epsilon^2 T} \sum_{t=1}^T y_{t-1} \epsilon_t = \frac{1}{2} \left(\frac{y_T}{\sigma_\epsilon \sqrt{T}} \right)^2 - \frac{1}{2\sigma_\epsilon^2} \frac{1}{T} \sum_{t=1}^T \epsilon_t^2 \rightarrow \frac{1}{2} (\chi^2(1) - 1).$$

(b) Next, consider $\sum y_{t-1}^2$.

$$\mathbb{E} \left(\sum_{t=1}^T y_{t-1}^2 \right) = \sum_{t=1}^T \mathbb{E}(y_{t-1}^2) = \sum_{t=1}^T \sigma_\epsilon^2 (t-1) = \sigma_\epsilon^2 \frac{T(T-1)}{2}.$$

Thus, we obtain the following result:

$$\frac{1}{T^2} \mathbb{E} \left(\sum_{t=1}^T y_{t-1}^2 \right) \rightarrow \text{a fixed value, i.e., } \frac{\sigma_\epsilon^2}{2}.$$

Therefore,

$$\frac{1}{T^2} \sum_{t=1}^T y_{t-1}^2 \rightarrow \text{a distribution.}$$

5. Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T) \sum y_{t-1} \epsilon_t}{(1/T^2) \sum y_{t-1}^2} \rightarrow \text{a distribution.}$$

6. Basic Concepts of Random Walk Process:

(a) Model: $y_t = y_{t-1} + \epsilon_t, \quad y_0 = 0, \quad \epsilon_t \sim N(0, 1).$

Then,

$$y_t = \epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1.$$

Therefore,

$$y_t \sim N(0, t).$$

\implies Nonstationary Process (i.e., variance depends on time t .)

Let T be the sample size.

$$\frac{y_t}{\sqrt{T}} \sim N\left(0, \frac{t}{T}\right)$$

Let us define $r = \frac{t}{T}$.

Keeping $r = \frac{t}{T}$, as $T \rightarrow \infty$,

$$\frac{y_t}{\sqrt{T}} \sim N\left(0, \frac{t}{T}\right) \rightarrow N(0, r) = W(r)$$

The limit when $T \rightarrow \infty$ is a **continuous time** (連続時間) process known as **standard Brownian motion** or **Wiener process**.

$W(r)$ is a normal random variable with mean zero and variance r and it is a continuous function in r .

(b) **Examples:**

$$W(0) = 0$$

$$W(1) = N(0, 1)$$

$$W(1)^2 = \chi^2(1)$$

$$\sigma W(r) \sim N(0, \sigma^2 r)$$

$$W(r)^2 = \sqrt{r} W(1)^2 \sim r \times \chi^2(1)$$

(c) Suppose that $y_t = y_{t-1} + \epsilon_t$, $y_0 = 0$, $\epsilon_t \sim N(0, \sigma_\epsilon)$.

$$y_t \sim N(0, \sigma_\epsilon^2 t), \text{ i.e.,}$$

$$\frac{y_t}{\sqrt{T}} \sim N(0, \sigma_\epsilon^2 \frac{t}{T}), \text{ i.e.,}$$

$$\frac{y_t}{\sigma_\epsilon \sqrt{T}} \sim N(0, \frac{t}{T}) \longrightarrow N(0, r) = W(r).$$

$$\bullet \sum_{t=1}^T \frac{1}{T} \frac{y_t}{\sigma_\epsilon \sqrt{T}} \longrightarrow \int_0^1 W(r) dr.$$

$$\left(\frac{y_t}{\sigma_\epsilon \sqrt{T}} \right)^2 \longrightarrow W(r)^2.$$

$$\bullet \sum_{t=1}^T \frac{1}{T} \left(\frac{y_t}{\sigma_\epsilon \sqrt{T}} \right)^2 \longrightarrow \int_0^1 W(r)^2 dr.$$

(*) **Continuous Mapping Theorem** (連続写像定理):

if $x_T \rightarrow x$ (convergence in distribution) and $g(\cdot)$ is a continuous function, then $g(x_T) \rightarrow g(x)$ (convergence in distribution).

Therefore, we have the following result:

$$\frac{1}{T} \sum_{t=1}^T \left(\frac{y_t}{\sqrt{T}} \right)^2 = \frac{1}{T^2} \sum_{t=1}^T y_t^2 \rightarrow \sigma_\epsilon^2 \int_0^1 W(r)^2 dr.$$

7. Asymptotic Distribution of AR(1) Model:

(a) $H_0 : y_t = y_{t-1} + \epsilon_t$ and $H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t$ for $|\phi_1| < 1$

OLSE of ϕ_1 , denoted by $\hat{\phi}_1$, is given by:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2} = \phi_1 + \frac{\sum_{t=1}^T y_{t-1} \epsilon_t}{\sum_{t=1}^T y_{t-1}^2}$$

Using $\phi_1 = 1$ and some formulas shown above, we obtain:

$$T(\hat{\phi}_1 - 1) = \frac{T^{-1} \sum_{t=1}^T y_{t-1} \epsilon_t}{T^{-2} \sum_{t=1}^T y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}(W(1))^2 - 1}{\int_0^1 W(r)^2 dr}$$

Remember that

$$T^{-1} \sum_{t=1}^T y_{t-1} \epsilon_t \longrightarrow \frac{1}{2} \sigma_\epsilon^2 (W(1))^2 - 1$$

and

$$T^{-2} \sum_{t=1}^T y_{t-1}^2 \longrightarrow \sigma_\epsilon^2 \int_0^1 W(r)^2 dr,$$

where $(W(1))^2 = \chi^2(1)$.

We say that $\hat{\phi}_1$ is **super-consistent** (超一致性) or **T -consistent**.

Remember that when $|\phi_1| < 1$ we have $\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$, and in this case we say that $\hat{\phi}_1$ is **\sqrt{T} -consistent**.

Conventional t test statistic is given by:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi},$$

where

$$s_\phi = \left(s^2 / \sum_{t=1}^T y_{t-1}^2 \right)^{1/2} \quad \text{and} \quad s^2 = \frac{1}{T-1} \sum_{t=1}^T (y_t - \hat{\phi}_1 y_{t-1})^2.$$

Next, consider t statistic.

The t test statistic, denoted by t , is represented as follows:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi} = \frac{T(\hat{\phi}_1 - 1)}{T s_\phi}$$

The denominator is:

$$\begin{aligned} T s_\phi &= \left(s^2 / \frac{1}{T^2} \sum_{t=1}^T y_{t-1}^2 \right)^{1/2} \\ &\rightarrow \left(\sigma_\epsilon^2 / \left(\sigma_\epsilon^2 \int_0^1 (W(r))^2 dr \right) \right)^{1/2} = \left(\int_0^1 (W(r))^2 dr \right)^{-1/2}, \end{aligned}$$

where $s^2 \rightarrow \sigma_\epsilon^2$ is utilized.

Therefore, we have the following asymptotic distribution:

$$\begin{aligned} t = \frac{\hat{\phi}_1 - 1}{s_\phi} &\rightarrow \frac{\frac{1}{2}((W(1))^2 - 1)}{\int_0^1 (W(r))^2 dr} \left/ \left(\int_0^1 (W(r))^2 dr \right)^{-1/2} \right. \\ &= \frac{\frac{1}{2}((W(1))^2 - 1)}{\left(\int_0^1 (W(r))^2 dr \right)^{1/2}}. \end{aligned}$$

Therefore, the distribution of the t statistic shown above is different from the t distribution.

8. The distributions of the t statistic: $\frac{\hat{\phi}_1 - 1}{s_\phi}$

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-2.49	-2.06	-1.71	-1.32	1.32	1.71	2.06	2.49
50	-2.40	-2.01	-1.68	-1.30	1.30	1.68	2.01	2.40
100	-2.36	-1.98	-1.66	-1.29	1.29	1.66	1.98	2.36
250	-2.34	-1.97	-1.65	-1.28	1.28	1.65	1.97	2.34
500	-2.33	-1.96	-1.65	-1.28	1.28	1.65	1.96	2.33
∞	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33

$$(a) H_0 : y_t = y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1 \text{ or } -1 < \phi_1$$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-2.66	-2.26	-1.95	-1.60	0.92	1.33	1.70	2.16
50	-2.62	-2.25	-1.95	-1.61	0.91	1.31	1.66	2.08
100	-2.60	-2.24	-1.95	-1.61	0.90	1.29	1.64	2.03
250	-2.58	-2.23	-1.95	-1.62	0.89	1.29	1.63	2.01
500	-2.58	-2.23	-1.95	-1.62	0.89	1.28	1.62	2.00
∞	-2.58	-2.23	-1.95	-1.62	0.89	1.28	1.62	2.00

(b) $H_0 : y_t = y_{t-1} + \epsilon_t$

$H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t$ for $\phi_1 < 1$ or $-1 < \phi_1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-3.75	-3.33	-3.00	-2.63	-0.37	0.00	0.34	0.72
50	-3.58	-3.22	-2.93	-2.60	-0.40	-0.03	0.29	0.66
100	-3.51	-3.17	-2.89	-2.58	-0.42	-0.05	0.26	0.63
250	-3.46	-3.14	-2.88	-2.57	-0.42	-0.06	0.24	0.62
500	-3.44	-3.13	-2.87	-2.57	-0.43	-0.07	0.24	0.61
∞	-3.43	-3.12	-2.86	-2.57	-0.44	-0.07	0.23	0.60

$$(d) H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t$$

$$H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t \text{ for } \phi_1 < 1 \text{ or } -1 < \phi_1$$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	-4.38	-3.95	-3.60	-3.24	-1.14	-0.80	-0.50	-0.15
50	-4.15	-3.80	-3.50	-3.18	-1.19	-0.87	-0.58	-0.24
100	-4.04	-3.73	-3.45	-3.15	-1.22	-0.90	-0.62	-0.28
250	-3.99	-3.69	-3.43	-3.13	-1.23	-0.92	-0.64	-0.31
500	-3.98	-3.68	-3.42	-3.13	-1.24	-0.93	-0.65	-0.32
∞	-3.96	-3.66	-3.41	-3.12	-1.25	-0.94	-0.66	-0.33