「Econometrics II」 Homework No.1

Due: Bring your answer sheet in this classroom at 10:20AM, December 21, 2023, or put it into my mailbox by then.

1 Consider that n random variables $X_1, X_2, \cdots, X_n$ are mutually independently and identically distributed as the density function $f(x;\theta)$, where θ is an unknown parameter to be estimated. For simplicity, θ is a scalar. Let $s(X)$ be an unbiased estimator of θ for $X = (X_1, X_2, \cdots, X_n)$.

(1) Suppose that $L(\theta; x) = \prod_{i=1}^{n} f(x_i;\theta)$, which is called the likelihood function. Show the following two equalities:

$$\mathrm{E}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big) = 0.$$

$$-\mathrm{E}\Big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta^2}\Big) = \mathrm{E}\Big(\big(\frac{\partial \log L(\theta; X)}{\partial \theta}\big)^2\Big) = \mathrm{V}\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big).$$

(2) Show that the following inequality holds.

$$\mathrm{V}(s(X)) \geq (I(\theta))^{-1},$$

where $I(\theta) = -\mathrm{E}\big(\frac{\partial^2 \log L(\theta; X)}{\partial \theta^2}\big)$, which is called Fisher's information matrix.

(3) Suppose that $s(X) = \tilde{\theta}$ is a maximum likelihood estimator. Then, show that

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N(0, \sigma^2),$$

where $\sigma^2 = \lim_{n\to\infty}\big(\frac{I(\theta)}{n}\big)^{-1}$.

(4) Discuss about unbiasedness, consistency and efficiency of $\tilde{\theta}$.

2 Consider the following regression model:

$$y_i^* = X_i\beta + u_i,$$

where X_i is assumed to be exogenous and nonstochastic, and $u_1, u_2, \cdots, u_n$ are mutually independent errors.

Let $f(x)$ be the density function of u_i and $F(x)$ be the cumulative distribution function of u_i, i.e., $F(x) = \int_{-\infty}^{x} f(z)\mathrm{d}z$.

(a) Let us define:

$$y_i = \begin{cases} 1, & \text{if } y_i^* > 0, \\ 0, & \text{if } y_i^* \le 0, \end{cases}$$

i.e., y_i^* is not observed and we know the sign of y_i^* (i.e., positive or negative). $y_i$is assigned to be one when $y_i^* > 0$, while it is zero when $y_i \le 0$.

(1) What is $\mathrm{E}(y_i)$?
(2) Obtain the likelihood function.
(3) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.
(4) Discuss how to estimate β and σ^2.
(5) What is the asymptotic distribution of the maximum likelihood estimator of $\beta^* = \dfrac{\beta}{\sigma}$?

(b) Let us define:

$$y_i^* = y_i, \qquad \text{if } y_i > 0,$$

i.e., y_t^* is not observed when $y_t \le 0$ and $y_t^* = y_t$ is observed when $y_t > 0$.

(6) What is $\mathrm{E}(y_i | y_i > 0)$?
(7) Obtain the likelihood function.
(8) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.
(9) Discuss how to estimate β and σ^2.
(10) What are the asymptotic distributions of the maximum likelihood estimators of β and σ^2?

(c) Let us define:

$$y_i^* = \begin{cases} y_i, & \text{if } y_i > 0, \\ 0, & \text{if } y_i \le 0, \end{cases}$$

i.e., $y_t^* = 0$ is observed when $y_t \le 0$ and $y_t^* = y_t$ is observed when $y_t > 0$.

(11) Obtain the likelihood function.
(12) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.
(13) Discuss how to estimate β and σ^2.
(14) What are the asymptotic distributions of the maximum likelihood estimators of β and σ^2?

3 Suppose that the probability function of y_i is Poisson with parameter λ_i for $i = 1, 2, \cdots, n$.

(1) What is $\mathrm{E}(y_i)$?

(2) Assuming $\lambda_i = \exp(X_i\beta)$, obtain the likelihood function, where β is a unknown parameter vector to be estimated.

(3) Derive the first-order condition for maximization of the log-likelihood function.

(4) Discuss how to estimate β.

(5) What are the asymptotic distribution of the maximum likelihood estimator of β?

4 We want to estimate the following regression model:

$$y_{it} = X_{it}\beta + v_i + u_{it},$$

for $t = 1, 2, \cdots, T$ and $i = 1, 2, \cdots, n$. i denotes the ith individual and t denotes time t. v_i and u_{it}, $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$, are mutually independent with $\mathrm{E}(v_i) = 0$, $\mathrm{E}(u_{it}) = 0$, $\mathrm{V}(v_i) = \sigma_v^2$ and $\mathrm{V}(u_{it}) = \sigma_u^2$. Answer the following questions in the matrix form. Each matrix shoukd be defined.

(1) Derive the estimator of β, using the fixed effect model.

(2) Construct the likelihood function of y_{it}, $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$.

(3) Derive the estimator of β, using the random effect model.