

6 Time Series Analysis (時系列分析)

6.1 Introduction

1. Stationarity (定常性) :

Let y_1, y_2, \dots, y_T be time series data.

(a) Weak Stationarity (弱定常性) :

$$E(y_t) = \mu,$$

$$E((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \quad \tau = 0, 1, 2, \dots$$

The first moment does not depend on time.

The second moment depends only on time difference.

(b) **Strong Stationarity (強定常性) :**

Let $f(y_{t_1}, y_{t_2}, \dots, y_{t_r})$ be the joint distribution of $y_{t_1}, y_{t_2}, \dots, y_{t_r}$.

$$f(y_{t_1}, y_{t_2}, \dots, y_{t_r}) = f(y_{t_1+\tau}, y_{t_2+\tau}, \dots, y_{t_r+\tau})$$

All the moments are same for all τ .

2. **Ergodicity (エルゴード性) :**

As time difference between two data is large, the two data become independent.

y_1, y_2, \dots, y_T is said to be ergodic in mean when \bar{y} converges in probability to $E(y_t)$.

3. **Auto-covariance Function (自己共分散関数) :**

$$E((y_t - \mu)(y_{t-\tau} - \mu)) = \gamma(\tau), \quad \tau = 0, 1, 2, \dots$$

$$\gamma(\tau) = \gamma(-\tau)$$

4. **Auto-correlation Function** (自己相関関数) :

$$\rho(\tau) = \frac{E((y_t - \mu)(y_{t-\tau} - \mu))}{\sqrt{\text{Var}(y_t)} \sqrt{\text{Var}(y_{t-\tau})}} = \frac{\gamma(\tau)}{\gamma(0)}$$

Note that $\text{Var}(y_t) = \text{Var}(y_{t-\tau}) = \gamma(0)$.

5. **Sample Mean** (標本平均) :

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^T y_t$$

6. **Sample Auto-covariance** (標本自己共分散) :

$$\hat{\gamma}(\tau) = \frac{1}{T} \sum_{t=\tau+1}^T (y_t - \hat{\mu})(y_{t-\tau} - \hat{\mu})$$

7. **Correlogram** (コレログラム, or 標本自己相関関数) :

$$\hat{\rho}(\tau) = \frac{\hat{\gamma}(\tau)}{\hat{\gamma}(0)}$$

8. Lag Operator (ラグ作要素) :

$$L^\tau y_t = y_{t-\tau}, \quad \tau = 1, 2, \dots$$

9. Likelihood Function (尤度関数) — Innovation Form :

The joint distribution of y_1, y_2, \dots, y_T is written as:

$$\begin{aligned} f(y_1, y_2, \dots, y_T) &= f(y_T | y_{T-1}, \dots, y_1) f(y_{T-1}, \dots, y_1) \\ &= f(y_T | y_{T-1}, \dots, y_1) f(y_{T-1} | y_{T-2}, \dots, y_1) f(y_{T-2}, \dots, y_1) \\ &\quad \vdots \\ &= f(y_T | y_{T-1}, \dots, y_1) f(y_{T-1} | y_{T-2}, \dots, y_1) \cdots f(y_2 | y_1) f(y_1) \\ &= f(y_1) \prod_{t=2}^T f(y_t | y_{t-1}, \dots, y_1). \end{aligned}$$

Therefore, the log-likelihood function is given by:

$$\log f(y_1, y_2, \dots, y_T) = \log f(y_1) + \sum_{t=2}^T \log f(y_t | y_{t-1}, \dots, y_1).$$

Under the normality assumption, $f(y_t | y_{t-1}, \dots, y_1)$ is given by the normal distribution with conditional mean $E(y_t | y_{t-1}, \dots, y_1)$ and conditional variance $\text{Var}(y_t | y_{t-1}, \dots, y_1)$.

6.2 Time Series Models (時系列モデル)

Autoregressive Model (自己回帰モデル or AR モデル): $AR(p)$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \epsilon_t$$

Moving Average Model (移動平均モデル or MA モデル): $MA(q)$

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}$$

ARMA Model: ARMA(p, q)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

ARIMA Model: ARIMA(p, d, q)

$$\Delta y_t = y_t - y_{t-1} = (1 - L)y_t,$$

$$\Delta^2 y_t = \Delta y_t - \Delta y_{t-1} = (1 - L)^2 y_t,$$

\vdots

$$\Delta^d y_t = (1 - L)^d y_t.$$

$$\Delta^d y_t \sim \text{ARMA}(p, q) \iff y_t \sim \text{ARIMA}(p, d, q)$$

$$\Delta^d y_t = \phi_1 \Delta^d y_{t-1} + \phi_2 \Delta^d y_{t-2} + \cdots + \phi_p \Delta^d y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$$

SARIMA Model: SARIMA(p, d, q)

${}^s\Delta y_t = y_t - y_{t-s}$, $s = 4$ for quarterly data $s = 12$ for monthly data

${}^s\Delta\Delta^d y_t \sim \text{ARMA}(p, q) \iff y_t \sim \text{SARIMA}(p, d, q)$

${}^s\Delta\Delta^d y_t = \phi_1 {}^s\Delta\Delta^d y_{t-1} + \phi_2 {}^s\Delta\Delta^d y_{t-2} + \cdots + \phi_p {}^s\Delta\Delta^d y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q}$

6.3 Autoregressive Model (自己回帰モデル or AR モデル)

1. AR(p) Model :

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

which is rewritten as:

$$\phi(L)y_t = \epsilon_t,$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p.$$

2. Stationarity (定常性) :

Suppose that all the p solutions of x from $\phi(x) = 0$ are real numbers

When the p solutions are greater than one in absolute value, y_t is stationary.

Suppose that the p solutions include imaginary numbers.

When the p solutions are outside unit circle, y_t is stationary.

3. **Partial Autocorrelation Coefficient** (偏自己相関係数), $\phi_{k,k}$:

The partial autocorrelation coefficient between y_t and y_{t-k} , denoted by $\phi_{k,k}$, is a measure of strength of the relationship between y_t and y_{t-k} , after removing influence of $y_{t-1}, \dots, y_{t-k+1}$.

$$\phi_{1,1} = \rho(1)$$

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \end{pmatrix}$$

⋮

$$\begin{pmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{k,1} \\ \phi_{k,2} \\ \vdots \\ \phi_{k,k-1} \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{pmatrix}$$

Use Cramer's rule (クラメールの公式) to obtain $\phi_{k,k}$.

$$\phi_{k,k} = \frac{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & \rho(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \cdots & \rho(k-2) & \rho(k-1) \\ \rho(1) & 1 & & \rho(k-3) & \rho(k-2) \\ \vdots & \vdots & & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \cdots & \rho(1) & 1 \end{vmatrix}}$$

Example: AR(1) Model: $y_t = \phi_1 y_{t-1} + \epsilon_t$

1. The stationarity condition is: the solution of $\phi(x) = 1 - \phi_1 x = 0$, i.e., $x = 1/\phi_1$, is greater than one in absolute value, or equivalently, $|\phi_1| < 1$.

2. Rewriting the AR(1) model,

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \epsilon_t \\&= \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\&= \phi_1^3 y_{t-3} + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} \\&\vdots \\&= \phi_1^s y_{t-s} + \epsilon_t + \phi_1 \epsilon_{t-1} + \cdots + \phi_1^{s-1} \epsilon_{t-s+1}.\end{aligned}$$

As s is large, ϕ_1^s approaches zero. \implies Stationarity condition

3. For stationarity, $y_t = \phi_1 y_{t-1} + \epsilon_t$ is rewritten as:

$$y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \cdots$$

MA representation of AR model.

(MA will be discussed later.)

4. Mean of AR(1) process, μ

$$\begin{aligned}\mu &= E(y_t) = E(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \dots) \\ &= E(\epsilon_t) + \phi_1 E(\epsilon_{t-1}) + \phi_1^2 E(\epsilon_{t-2}) + \dots = 0\end{aligned}$$

5. Autocovariance and autocorrelation functions of the AR(1) process:

Rewriting the AR(1) process, we have:

$$y_t = \phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \dots + \phi_1^{\tau-1} \epsilon_{t-\tau+1}.$$

Therefore, the autocovariance function of AR(1) process is:

$$\begin{aligned}\gamma(\tau) &= E((y_t - \mu)(y_{t-\tau} - \mu)) = E(y_t y_{t-\tau}) \\ &= E\left((\phi_1^\tau y_{t-\tau} + \epsilon_t + \phi_1 \epsilon_{t-1} + \dots + \phi_1^{\tau-1} \epsilon_{t-\tau+1})y_{t-\tau}\right) \\ &= \phi_1^\tau E(y_{t-\tau} y_{t-\tau}) + E(\epsilon_t y_{t-\tau}) + \phi_1 E(\epsilon_{t-1} y_{t-\tau}) + \dots + \phi_1^{\tau-1} E(\epsilon_{t-\tau+1} y_{t-\tau}) \\ &= \phi_1^\tau \gamma(0).\end{aligned}$$

The autocorrelation function of AR(1) process is:

$$\rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \phi_1^\tau.$$

Multiply $y_{t-\tau}$ on both sides of the AR(1) process and take the expectation:

$$E(y_t y_{t-\tau}) = \phi_1 E(y_{t-1} y_{t-\tau}) + E(\epsilon_t y_{t-\tau})$$

$$\gamma(\tau) = \begin{cases} \phi_1 \gamma(\tau - 1), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \sigma^2, & \text{for } \tau = 0. \end{cases}$$

Using $\gamma(\tau) = \gamma(-\tau)$, $\gamma(\tau)$ for $\tau = 0$ is given by:

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2 = \phi_1^2 \gamma(0) + \sigma^2.$$

Note that $\gamma(1) = \phi_1 \gamma(0)$.

Therefore, $\gamma(0)$ is given by:

$$\gamma(0) = \frac{\sigma^2}{1 - \phi_1^2}$$

6. Partial autocorrelation function of AR(1) process:

$$\phi_{1,1} = \rho(1) = \phi_1$$

$$\phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\rho(2) - \rho(1)^2}{1 - \rho(1)^2} = 0$$

7. Estimation of AR(1) model:

(a) Likelihood function

$$\log f(y_T, \dots, y_1) = \log f(y_1) + \sum_{t=1}^T \log f(y_t | y_{t-1}, \dots, y_1)$$

$$\begin{aligned}
&= -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log\left(\frac{\sigma^2}{1 - \phi_1^2}\right) - \frac{1}{\sigma^2/(1 - \phi_1^2)} y_1^2 \\
&\quad - \frac{T-1}{2} \log(2\pi) - \frac{T-1}{2} \log(\sigma^2) - \frac{1}{\sigma^2} \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2 \\
&= -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(\sigma^2) - \frac{1}{2} \log\left(\frac{1}{1 - \phi_1^2}\right) \\
&\quad - \frac{1}{2\sigma^2/(1 - \phi_1^2)} y_1^2 - \frac{1}{2\sigma^2} \sum_{t=2}^T (y_t - \phi_1 y_{t-1})^2
\end{aligned}$$

Note as follows:

$$\begin{aligned}
f(y_1) &= \frac{1}{\sqrt{2\pi\sigma^2/(1 - \phi_1^2)}} \exp\left(-\frac{1}{2\sigma^2/(1 - \phi_1^2)} y_1^2\right) \\
f(y_t|y_{t-1}, \dots, y_1) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} (y_t - \phi_1 y_{t-1})^2\right)
\end{aligned}$$