$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \sigma^2} = -\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4/(1-\phi_1^2)}y_1^2 + \frac{1}{2\sigma^4}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2 = 0$$

$$\frac{\partial \log f(y_T, \cdots, y_1)}{\partial \phi_1} = -\frac{\phi_1}{1-\phi_1^2} + \frac{\phi_1}{\sigma^2}y_1^2 + \frac{1}{\sigma^2}\sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})y_{t-1} = 0$$

The MLE of $\phi_1$ and $\sigma^2$ satisfies the above two equation.

$$\tilde{\sigma}^2 = \frac{1}{T}\left((1-\tilde{\phi}_1^2)y_1^2 + \sum_{t=2}^{T}(y_t - \tilde{\phi}_1 y_{t-1})^2\right)$$

$$\tilde{\phi}_1 = \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} + \left(\tilde{\phi}_1 y_1^2 - \frac{\tilde{\sigma}^2 \tilde{\phi}_1}{1-\tilde{\phi}_1^2}\right)\Big/\sum_{t=2}^{T} y_{t-1}^2$$

$$\approx \frac{\sum_{t=2}^{T} y_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2}, \qquad \text{when } T \text{ is large.}$$

(b) Ordinary Least Squares (OLS) Method

$$S(\phi_1) = \sum_{t=2}^{T}(y_t - \phi_1 y_{t-1})^2$$

is minimized with respect to $\phi_1$.

$$\hat{\phi}_1 = \frac{\sum_{t=2}^{T} y_{t-1}y_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=2}^{T} y_{t-1}\epsilon_t}{\sum_{t=2}^{T} y_{t-1}^2} = \phi_1 + \frac{(1/T)\sum_{t=2}^{T} y_{t-1}\epsilon_t}{(1/T)\sum_{t=2}^{T} y_{t-1}^2}$$

$$\longrightarrow \phi_1 + \frac{E(y_{t-1}\epsilon_t)}{E(y_{t-1}^2)} = \phi_1$$

OLSE of $\phi_1$ is a consistent estimator.

OLSE of $\phi_1$ is equal to MLE when $T$ is large.

The following equations are utilized.

$$E(y_{t-1}\epsilon_t) = 0, \qquad E(y_{t-1}^2) = \text{Var}(y_{t-1}) = \gamma(0)$$

8. Asymptotic distribution of OLSE $\hat{\phi}_1$:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$$

**Proof:**

$y_{t-1}\epsilon_t$, $t = 1, 2, \cdots, T$, are distributed with mean zero and variance $\dfrac{\sigma_\epsilon^4}{1 - \phi_1^2}$.

From the central limit theorem,

$$\frac{(1/T) \sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sqrt{\sigma_\epsilon^4/(1 - \phi_1^2)}/\sqrt{T}} \longrightarrow N(0, 1)$$

Rewriting,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0, \frac{\sigma_\epsilon^4}{1 - \phi_1^2}).$$

Next,

$$\frac{1}{T} \sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2}$$

yields:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{(1/\sqrt{T}) \sum_{t=1}^{T} y_{t-1}\epsilon_t}{(1/T) \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N(0, 1 - \phi_1^2)$$

9. Some formulas:

   (a) Central Limit Theorem

   Random variables $x_1$, $x_2$, $\cdots$, $x_T$ are mutually independently distributed with mean $\mu$ and variance $\sigma^2$.

   Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

   Then,

   $$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} = \frac{\overline{x} - \mu}{\sigma/\sqrt{T}} \longrightarrow N(0, 1)$$

   (b) Central Limit Theorem II

   Random variables $x_1$, $x_2$, $\cdots$, $x_T$ are distributed with mean $\mu$ and variance $\sigma^2$.

88

Define $\overline{x} = (1/T) \sum_{t=1}^{T} x_t$.

Then,

$$\frac{\overline{x} - \mathrm{E}(\overline{x})}{\sqrt{\mathrm{V}(\overline{x})}} \longrightarrow N(0, 1)$$

(c) Let $x$ and $y$ be random variables.

$y$ converges in distribution to a distribution, and $x$ converges in probability to a fixed value.

Then, $xy$ converges in distribution.

For example, consider:

$$y \longrightarrow N(\mu, \sigma^2), \qquad x \longrightarrow c.$$

Then, we obtain:

$$xy \longrightarrow N(c\mu, c^2\sigma^2)$$

89

10. **AR(1) +drift:**  $y_t = \mu + \phi_1 y_{t-1} + \epsilon_t$

Mean:

Using the lag operator,

$$\phi(L)y_t = \mu + \epsilon_t$$

where $\phi(L) = 1 - \phi_1 L$.

Multiply $\phi(L)^{-1}$ on both sides. Then, when $|\phi_1| < 1$, we have:

$$y_t = \phi(L)^{-1}\mu + \phi(L)^{-1}\epsilon_t.$$

Taking the expectation on both sides,

$$\begin{aligned}
\mathrm{E}(y_t) &= \phi(L)^{-1}\mu + \phi(L)^{-1}\mathrm{E}(\epsilon_t) \\
&= \phi(1)^{-1}\mu = \frac{\mu}{1 - \phi_1}
\end{aligned}$$

**Example: AR(2) Model:**   Consider $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$.

1. The stationarity condition is: two solutions of $x$ from $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0$ are outside the unit circle.

2. Rewriting the AR(2) model,

$$(1 - \phi_1 L - \phi_2 L^2) y_t = \epsilon_t.$$

Let $1/\alpha_1$ and $1/\alpha_2$ be the solutions of $\phi(x) = 0$.

Then, the AR(2) model is written as:

$$(1 - \alpha_1 L)(1 - \alpha_2 L) y_t = \epsilon_t,$$

which is rewritten as:

$$y_t = \frac{1}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \epsilon_t$$

91

$$= \left( \frac{\alpha_1/(\alpha_1 - \alpha_2)}{1 - \alpha_1 L} + \frac{-\alpha_2/(\alpha_1 - \alpha_2)}{1 - \alpha_2 L} \right) \epsilon_t$$

3. **Mean of AR(2) Model:**

   When $y_t$ is stationary, i.e., $\alpha_1$ and $\alpha_2$ are inside the unit circle,

   $$\mu = \mathrm{E}(y_t) = \mathrm{E}(\phi(L)\epsilon_t) = 0$$

4. **Autocovariance Function of AR(2) Model:**

$$
\begin{aligned}
\gamma(\tau) &= \mathrm{E}((y_t - \mu)(y_{t-\tau} - \mu)) = \mathrm{E}(y_t y_{t-\tau}) \\
&= \mathrm{E}\big((\phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) y_{t-\tau}\big) \\
&= \phi_1 \mathrm{E}(y_{t-1} y_{t-\tau}) + \phi_2 \mathrm{E}(y_{t-2} y_{t-\tau}) + \mathrm{E}(\epsilon_t y_{t-\tau}) \\
&= \begin{cases} \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2), & \text{for } \tau \neq 0, \\ \phi_1 \gamma(\tau - 1) + \phi_2 \gamma(\tau - 2) + \sigma_\epsilon^2, & \text{for } \tau = 0. \end{cases}
\end{aligned}
$$

The initial condition is obtained by solving the following three equations:

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2,$$

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1),$$

$$\gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0).$$

Therefore, the initial conditions are given by:

$$\gamma(0) = \left(\frac{1 - \phi_2}{1 + \phi_2}\right)\frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2},$$

$$\gamma(1) = \frac{\phi_1}{1 - \phi_2}\gamma(0) = \left(\frac{\phi_1}{1 - \phi_2}\right)\left(\frac{1 - \phi_2}{1 + \phi_2}\right)\frac{\sigma_\epsilon^2}{(1 - \phi_2)^2 - \phi_1^2}.$$

Given $\gamma(0)$ and $\gamma(1)$, we obtain $\gamma(\tau)$ as follows:

$$\gamma(\tau) = \phi_1\gamma(\tau - 1) + \phi_2\gamma(\tau - 2), \qquad \text{for } \tau = 2, 3, \cdots.$$