Therefore,

$$\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \text{ a distribution.}$$

6. Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T)\sum y_{t-1}\epsilon_t}{(1/T^2)\sum y_{t-1}^2} \longrightarrow \text{ a distribution.}$$

The distributions of the t statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$, where s_ϕ denotes the standard error of $\hat{\phi}_1$.

$\Longrightarrow$ Compare t distribution with (a) – (c).

$\Longrightarrow$ **Unit Root Test (単位根検定, or Dickey-Fuller (DF) Test)**

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

Test $H_0 : \phi_1 = 1$ against $H_1 : \phi_1 < 1$.

Equivalently,

$$\Delta y_t = \rho y_{t-1} + \epsilon_t.$$

Test $H_0 : \rho = 0$ against $H_1 : \rho < 0$.

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.49	−2.06	−1.71	−1.32	1.32	1.71	2.06	2.49
50	−2.40	−2.01	−1.68	−1.30	1.30	1.68	2.01	2.40
100	−2.36	−1.98	−1.66	−1.29	1.29	1.66	1.98	2.36
250	−2.34	−1.97	−1.65	−1.28	1.28	1.65	1.97	2.34
500	−2.33	−1.96	−1.65	−1.28	1.28	1.65	1.96	2.33
∞	−2.33	−1.96	−1.64	−1.28	1.28	1.64	1.96	2.33

(a) $H_0:\ y_t = y_{t-1} + \epsilon_t$

$H_1:\ y_t = \phi_1 y_{t-1} + \epsilon_t$ for $\phi_1 < 1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

To test $H_0:\ \rho = 0$ against $H_1:\ \rho < 0$, estimate $\Delta y_t = \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

(b) $H_0: \; y_t = y_{t-1} + \epsilon_t$

$H_1: \; y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t$ **for** $\phi_1 < 1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60

To test $H_0: \; \rho = 0$ against $H_1: \; \rho < 0$, estimate $\Delta y_t = \alpha_0 + \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

(c) $\boldsymbol{H_0: \ y_t = \alpha_0 + y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1: \ y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

To test $H_0 : \rho = 0$ against $H_1 : \rho < 0$, estimate $\Delta y_t = \alpha_0 + \alpha_1 t + \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

8.2 Unit Root (More Formally)

Consider $y_t = y_{t-1} + \epsilon_t$ and $y_0 = 0$.

$$y_t = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_t \sim N(0, t\sigma^2)$$

$$\frac{1}{\sqrt{T}} y_t = \frac{1}{\sqrt{T}} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t) \sim N(0, \frac{t}{T}\sigma^2) \longrightarrow N(0, r\sigma^2)$$

where $0 \le r \le 1$ and $r = \dfrac{t}{T}$.

Note that time interval $(1,\ T)$ is transformed into (0, 1), divided by T.

$$\frac{1}{\sqrt{T}\sigma} y_t = \frac{1}{\sqrt{T}\sigma} (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_t) \sim N(0, \frac{t}{T}) \longrightarrow N(0, r) \equiv W(r)$$

As T (t at the same time) goes to infinity keeping $r = \dfrac{t}{T}$, $W(r)$ results in a continuous function of r where r takes any number between zero and one.

$W(r)$ is a normal random variable with mean zero and variance r and it is called the **Brownian motion**.

Moreover, we consider the following:

$$\frac{1}{\sqrt{T}\sigma}y_{t'} = \frac{1}{\sqrt{T}\sigma}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{t'}) \longrightarrow N(0, r') \equiv W(r').$$

For $t' > t$, we have the following:

$$\begin{aligned}\frac{1}{\sqrt{T}\sigma}y_{t'} &= \frac{1}{\sqrt{T}\sigma}(\epsilon_1 + \cdots + \epsilon_t + \epsilon_{t+1} + \cdots \epsilon_{t'}) \longrightarrow N(0, r') \equiv W(r') \\ &= \frac{1}{\sqrt{T}\sigma}y_t + \frac{1}{\sqrt{T}\sigma}(\epsilon_{t+1} + \cdots \epsilon_{t'}).\end{aligned}$$

Therefore, we have

$$\frac{1}{\sqrt{T}\sigma}y_{t'} - \frac{1}{\sqrt{T}\sigma}y_t = \frac{1}{\sqrt{T}\sigma}(\epsilon_{t+1} + \cdots \epsilon_{t'}) \longrightarrow N(0, r' - r) \equiv W(r') - W(r).$$

That is, $W(r)$ is independent of $W(r') - W(r)$ for $r' > r$.

Moreover, note as follows:

$$\frac{1}{T\sqrt{T}\sigma}\sum_{t=1}^{T} y_t = \frac{1}{T}\sum_{t=1}^{T}\Big(\frac{y_t}{\sqrt{T}\sigma}\Big) \longrightarrow \int_0^1 W(r)\mathrm{d}r$$

where $\dfrac{1}{T}$ and $\displaystyle\sum_{t=1}^{T}$ are replaced by dr and $\displaystyle\int_0^1$ as T goes to infinity.

We devide the time interval $(0,\ 1)$ into T time intervals $\Big(\dfrac{t}{T},\ \dfrac{t+1}{T}\Big)$.

That is, time interval $(1,\ T)$ is transformed into $(0, 1)$.

(*) We know that $\dfrac{y_t}{\sqrt{T}\sigma} \longrightarrow W(r)$ as $\dfrac{t}{T} \longrightarrow r$.

Summary: **<u>Properties of $W(r)$ for $0 < r < 1$</u>:**

1. $W(r) \equiv N(0, r)$ $\Longrightarrow$ $W(r)$ is a random variable.

2. $W(1) \equiv N(0, 1)$

3. $W(1)^2 \equiv \chi^2(1)$ $\Longrightarrow$ Remember that $Z^2 \sim \chi^2(1)$ when $Z \sim N(0, 1)$.

4. $W(r)$ is independent of $W(r') - W(r)$ for $r < r'$.

5. $W(r_4) - W(r_3)$ is independent of $W(r_2) - W(r_1)$ for $0 \leq r_1 < r_2 < r_3 < r_4 \leq 1$.

   $\Longrightarrow$ The interval between r_4 and r_3 is not overlapped with the interval between r_2 and r_1.

- **True Model $y_t = y_{t-1} + \epsilon_t$ vs Estimated Model $y_t = \phi y_{t-1} + \epsilon_t$:** Under $\phi = 1$, we estimate ϕ in the regression model:

$$y_t = \phi y_{t-1} + \epsilon_t$$

OLS of ϕ is:

$$\hat{\phi} = \frac{\sum_{t=1}^{T} y_{t-1} y_t}{\sum_{t=1}^{T} y_{t-1}^2} = \phi + \frac{\sum_{t=1}^{T} y_{t-1} \epsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}$$

As mentioned aove, the numerator is related to:

$$\frac{1}{\sigma^2 T} \sum_{t=1}^{T} y_{t-1} \epsilon_t = \frac{1}{2} \left(\frac{y_T}{\sigma \sqrt{T}} \right)^2 - \frac{1}{2\sigma^2} \frac{1}{T} \sum_{t=1}^{T} \epsilon_t^2 \longrightarrow \frac{1}{2} W(1)^2 - \frac{1}{2}$$

which is rewritten by using the Brownian motion $W(1)$.

The denominator is:

$$\frac{1}{\sigma^2 T^2} \sum_{t=1}^{T} y_{t-1}^2 \approx \frac{1}{\sigma^2 T^2} \sum_{t=1}^{T} y_t^2 = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{y_t}{\sigma \sqrt{T}} \right)^2 \longrightarrow \int_0^1 W(r)^2 \mathrm{d}r$$

where $\frac{1}{T} \longrightarrow \mathrm{d}r$ and $\frac{y_t}{\sigma\sqrt{T}} \longrightarrow W(r)$ for $\frac{t}{T} \longrightarrow r$.

Thus, under $\phi = 1$, $T(\hat{\phi} - \phi)$ is asymptotically distributed as follows:

$$T(\hat{\phi} - \phi) = T(\hat{\phi} - 1) = \frac{\dfrac{1}{\sigma^2 T}\displaystyle\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\dfrac{1}{\sigma^2 T^2}\displaystyle\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}\Big(W(1)^2 - 1\Big)}{\displaystyle\int_0^1 W(r)^2 \mathrm{d}r}$$

- t value:

In the regression model: $y_t - y_{t-1} \equiv \Delta y_t = \rho y_{t-1} + \epsilon_t$, OLSE of $\rho = \phi - 1$ is given by $\hat{\rho} = \hat{\phi} - 1$.

t value is $\frac{\hat{\rho}}{s_\rho} = \frac{\hat{\phi} - 1}{s_\phi}$, where s_ρ and s_ϕ denote the standard errors of $\hat{\rho}$ and $\hat{\phi}$.
Note that $s_\rho = s_\phi$ because of $\mathrm{V}(\hat{\rho}) = \mathrm{V}(\hat{\phi} - 1) = \mathrm{V}(\hat{\phi})$.

The standard error of $\hat{\phi}$, denoted by s_ϕ, is given by: $s_\phi^2 = \frac{s^2}{\sum_{t=1}^{T} y_{t-1}^2}$, where $s^2 = \frac{1}{T}\sum_{t=1}^{T}(y_t - \hat{\phi} y_{t-1})^2$, called the standard error of regression.

$T(\hat{\phi} - \phi) = T(\hat{\phi} - 1)$ converges in distribution.

$\hat{\phi}$ is a consistent estimatior of $\phi = 1$, i.e., $\hat{\phi} \longrightarrow \phi = 1$.

Threfore,

$$s^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - \hat{\phi} y_{t-1})^2 \longrightarrow \mathrm{E}((y_t - \phi y_{t-1})^2) = \mathrm{E}(\epsilon_t^2) = \sigma^2.$$

Moreover, as shown above,

$$\frac{1}{T^2 \sigma^2} \sum_{t=1}^{T} y_t^2 \longrightarrow \int_0^1 W(r)^2 \mathrm{d}r$$

Therefore, from $s_\phi^2 = \dfrac{1}{T^2 \sigma^2} \dfrac{s^2}{\dfrac{1}{T^2 \sigma^2} \sum_{t=1}^{T} y_{t-1}^2}$, we obtain

$$T^2 s_\phi^2 = \frac{1}{\sigma^2} \frac{s^2}{\dfrac{1}{T^2 \sigma^2} \sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \Big(\int_0^1 W(r)^2 \mathrm{d}r \Big)^{-1}.$$

Therefore, t value is given by:

$$\frac{\hat{\phi}-1}{s_\phi} = \frac{T(\hat{\phi}-1)}{\sqrt{T^2 s_\phi^2}} \longrightarrow \frac{\dfrac{\frac{1}{2}\left(W(1)^2-1\right)}{\int_0^1 W(r)^2 \mathrm{d}r}}{\left(\int_0^1 W(r)^2 \mathrm{d}r\right)^{-1/2}} = \frac{\frac{1}{2}\left(W(1)^2-1\right)}{\left(\int_0^1 W(r)^2 \mathrm{d}r\right)^{1/2}},$$

which is not a normal distribution.