[Review] Trace (トレース):

1. A: $n \times n$, $\quad \mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$, where a_{ij} denotes an element in the ith row and the jth column of a matrix A.

2. a: scalar (1×1), $\quad \mathrm{tr}(a) = a$

3. A: $n \times k$, B: $k \times n$, $\quad \mathrm{tr}(AB) = \mathrm{tr}(BA)$

4. $\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$

5. When X is a square matrix of random variables, $\mathrm{E}(\mathrm{tr}(AX)) = \mathrm{tr}(A\mathrm{E}(X))$

End of Review

s^2 is taken as follows:

$$s^2 = \frac{1}{n-k}\sum_{i=1}^{n} e_i^2 = \frac{1}{n-k}e'e = \frac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta}),$$

which leads to an unbiased estimator of σ^2.

Proof:

Substitute $y = X\beta + u$ and $\hat{\beta} = \beta + (X'X)^{-1}X'u$ into $e = y - X\hat{\beta}$.

$$\begin{aligned} e &= y - X\hat{\beta} = X\beta + u - X(\beta + (X'X)^{-1}X'u) \\ &= u - X(X'X)^{-1}X'u = (I_n - X(X'X)^{-1}X')u \end{aligned}$$

$I_n - X(X'X)^{-1}X'$ is idempotent and symmetric, because we have:

$$(I_n - X(X'X)^{-1}X')(I_n - X(X'X)^{-1}X') = I_n - X(X'X)^{-1}X',$$
$$(I_n - X(X'X)^{-1}X')' = I_n - X(X'X)^{-1}X'.$$

s^2 is rewritten as follows:

$$s^2 = \frac{1}{n-k}e'e = \frac{1}{n-k}((I_n - X(X'X)^{-1}X')u)'(I_n - X(X'X)^{-1}X')u$$

$$= \frac{1}{n-k} u'(I_n - X(X'X)^{-1}X')'(I_n - X(X'X)^{-1}X')u$$
$$= \frac{1}{n-k} u'(I_n - X(X'X)^{-1}X')u$$

Take the expectation of $u'(I_n - X(X'X)^{-1}X')u$ and note that $\mathrm{tr}(a) = a$ for a scalar a.

$$\begin{aligned}
\mathrm{E}(s^2) &= \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\Big(u'(I_n - X(X'X)^{-1}X')u\Big)\Big) = \frac{1}{n-k}\mathrm{E}\Big(\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')uu'\Big)\Big) \\
&= \frac{1}{n-k}\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')\mathrm{E}(uu')\Big) = \frac{1}{n-k}\sigma^2\mathrm{tr}\Big((I_n - X(X'X)^{-1}X')I_n\Big) \\
&= \frac{1}{n-k}\sigma^2\mathrm{tr}(I_n - X(X'X)^{-1}X') = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(X(X'X)^{-1}X')) \\
&= \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}((X'X)^{-1}X'X)) = \frac{1}{n-k}\sigma^2(\mathrm{tr}(I_n) - \mathrm{tr}(I_k)) \\
&= \frac{1}{n-k}\sigma^2(n-k) = \sigma^2
\end{aligned}$$

$\longrightarrow$ s^2 is an unbiased estimator of σ^2.

Note that we do not need normality assumption for unbiasedness of s^2.

[Review]

- $X'X \sim \chi^2(n)$ for $X \sim N(0, I_n)$.

- $(X - \mu)'\Sigma^{-1}(X - \mu) \sim \chi^2(n)$ for $X \sim N(\mu, \Sigma)$.

- $\dfrac{X'X}{\sigma^2} \sim \chi^2(n)$ for $X \sim N(0, \sigma^2 I_n)$.

- $\dfrac{X'AX}{\sigma^2} \sim \chi^2(G)$, where $X \sim N(0, \sigma^2 I_n)$ and A is a symmetric idempotent $n \times n$ matrix of rank $G \leq n$.

Remember that $G = \text{Rank}(A) = \text{tr}(A)$ when A is symmetric and idempotent.

[End of Review]

Under normality assumption for u, the distribution of s^2 is:

$$\frac{(n-k)s^2}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(\mathrm{tr}(I_n - X(X'X)^{-1}X'))$$

Note that $\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k$, because

$$\mathrm{tr}(I_n) = n$$

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

Asymptotic Normality (without normality assumption on u): Using the central limit theorem, without normality assumption we can show that as $n \longrightarrow \infty$, under the condition of $\frac{1}{n}X'X \longrightarrow M$ we have the following result:

$$\frac{\hat{\beta}_j - \beta_j}{s\sqrt{a_{jj}}} \longrightarrow N(0,1),$$

where M denotes a $k \times k$ constant matrix.

Thus, we can construct the confidence interval and the testing procedure, using the t distribution under the normality assumption or the normal distribution without the normality assumption.

3 Properties of OLSE

1. Properties of $\hat{\beta}$: **BLUE (best linear unbiased estimator，最良線形不偏推定量)**, i.e., minimum variance within the class of linear unbiased estimators **(Gauss-Markov theorem，ガウス・マルコフの定理)**

   Proof:

Consider another linear unbiased estimator, which is denoted by $\tilde{\beta} = Cy$.

$$\tilde{\beta} = Cy = C(X\beta + u) = CX\beta + Cu,$$

where C is a $k \times n$ matrix.

Taking the expectation of $\tilde{\beta}$, we obtain:

$$\mathrm{E}(\tilde{\beta}) = CX\beta + C\mathrm{E}(u) = CX\beta$$

Because we have assumed that $\tilde{\beta} = Cy$ is unbiased, $\mathrm{E}(\tilde{\beta}) = \beta$ holds.

That is, we need the condition: $CX = I_k$.

Next, we obtain the variance of $\tilde{\beta} = Cy$.

$$\tilde{\beta} = C(X\beta + u) = \beta + Cu.$$

Therefore, we have:

$$\mathrm{V}(\tilde{\beta}) = \mathrm{E}((\tilde{\beta} - \beta)(\tilde{\beta} - \beta)') = \mathrm{E}(Cuu'C') = \sigma^2 CC'$$

Defining $C = D + (X'X)^{-1}X'$, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\mathrm{V}(\tilde{\beta}) = \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')'.$$

Moreover, because $\hat{\beta}$ is unbiased, we have the following:

$$CX = I_k = (D + (X'X)^{-1}X')X = DX + I_k.$$

Therefore, we have the following condition:

$$DX = 0.$$

Accordingly, $\mathrm{V}(\tilde{\beta})$ is rewritten as:

$$\begin{aligned}\mathrm{V}(\tilde{\beta}) &= \sigma^2 CC' = \sigma^2 (D + (X'X)^{-1}X')(D + (X'X)^{-1}X')' \\ &= \sigma^2 (X'X)^{-1} + \sigma^2 DD' = \mathrm{V}(\hat{\beta}) + \sigma^2 DD'\end{aligned}$$

Thus, for $D \neq 0$, $\mathrm{V}(\tilde{\beta}) - \mathrm{V}(\hat{\beta})$ is a positive definite matrix.

$\Longrightarrow \mathrm{V}(\tilde{\beta}_i) - \mathrm{V}(\hat{\beta}_i) > 0$

$\Longrightarrow \hat{\beta}$ is a minimum variance (i.e., best) linear unbiased estimator of β.

Note as follows:

$\Longrightarrow$ A is positive definite when $d'Ad > 0$ except $d = 0$.

$\Longrightarrow$ The ith diagonal element of A, i.e., a_{ii}, is positive (choose d such that the ith element of d is one and the other elements are zeros).

[Review] F **Distribution:**
Suppose that $U \sim \chi(n)$, $V \sim \chi(m)$, and U is independent of V.
Then, $\dfrac{U/n}{V/m} \sim F(n, m)$.
[End of Review]

***F* Distribution ($H_0 : \beta = 0$):** Final Result in this Section:

$$\frac{(\hat\beta - \beta)'X'X(\hat\beta - \beta)/k}{e'e/(n-k)} \sim F(k, n-k).$$

Consider the numerator and the denominator, separately.

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat\beta \sim N(\beta, \sigma^2(X'X)^{-1})$.

   Therefore, $\dfrac{(\hat\beta - \beta)'X'X(\hat\beta - \beta)}{\sigma^2} \sim \chi^2(k)$.

2. **Proof:**

   Using $\hat\beta - \beta = (X'X)^{-1}X'u$, we obtain:

$$\begin{aligned}(\hat\beta - \beta)'X'X(\hat\beta - \beta) &= ((X'X)^{-1}X'u)'X'X(X'X)^{-1}X'u \\ &= u'X(X'X)^{-1}X'X(X'X)^{-1}X'u = u'X(X'X)^{-1}X'u\end{aligned}$$

   Note that $X(X'X)^{-1}X'$ is symmetric and idempotent, i.e., $A'A = A$.

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(X(X'X)^{-1}X')\Big)$$

The degree of freedom is given by:

$$\mathrm{tr}(X(X'X)^{-1}X') = \mathrm{tr}((X'X)^{-1}X'X) = \mathrm{tr}(I_k) = k$$

Therefore, we obtain:

$$\frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k)$$

3. (*) Formula:

   Suppose that $X \sim N(0, I_k)$.

   If A is symmetric and idempotent, i.e., $A'A = A$, then $X'AX \sim \chi^2(\mathrm{tr}(A))$.

   Here, $X = \dfrac{1}{\sigma}u \sim N(0, I_n)$ from $u \sim N(0, \sigma^2 I_n)$, and $A = X(X'X)^{-1}X'$.

4. **Sum of Residuals:** e is rewritten as:

$$e = (I_n - X(X'X)^{-1}X')u.$$

Therefore, the sum of residuals is given by:

$$e'e = u'(I_n - X(X'X)^{-1}X')u.$$

Note that $I_n - X(X'X)^{-1}X'$ is symmetric and idempotent.

We obtain the following result:

$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2\Big(\mathrm{tr}(I_n - X(X'X)^{-1}X')\Big),$$

where the trace is:

$$\mathrm{tr}(I_n - X(X'X)^{-1}X') = n - k.$$

Therefore, we have the following result:

$$\frac{e'e}{\sigma^2} = \frac{(n-k)s^2}{\sigma^2} \sim \chi^2(n-k),$$

where

$$s^2 = \frac{1}{n-k} e'e.$$

5. We show that $\hat{\beta}$ is independent of e.

Proof:

Because $u \sim N(0, \sigma^2 I_n)$, we show that $\mathrm{Cov}(e, \hat{\beta}) = 0$.

$$\begin{aligned}
\mathrm{Cov}(e, \hat{\beta}) &= \mathrm{E}(e(\hat{\beta} - \beta)') = \mathrm{E}\Big((I_n - X(X'X)^{-1}X')u((X'X)^{-1}X'u)'\Big) \\
&= \mathrm{E}\Big((I_n - X(X'X)^{-1}X')uu'X(X'X)^{-1}\Big) = (I_n - X(X'X)^{-1}X')\mathrm{E}(uu')X(X'X)^{-1} \\
&= (I_n - X(X'X)^{-1}X')(\sigma^2 I_n)X(X'X)^{-1} = \sigma^2(I_n - X(X'X)^{-1}X')X(X'X)^{-1} \\
&= \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}X'X(X'X)^{-1}) = \sigma^2(X(X'X)^{-1} - X(X'X)^{-1}) = 0.
\end{aligned}$$

$\hat{\beta}$ is independent of e, because of normality assumption on u

[Review]

- Suppose that X is independent of Y. Then, $\text{Cov}(X, Y) = 0$. However, $\text{Cov}(X, Y) = 0$ does not mean in general that X is independent of Y.

- In the case where X and Y are normal, $\text{Cov}(X, Y) = 0$ indicates that X is independent of Y.

[End of Review]

[Review] Formulas — F Distribution:

- $\dfrac{U/n}{V/m} \sim F(n,m)$ when U $sim\chi^2(n)$, $V \sim \chi^2(m)$, and U is independent of V.

- When $X \sim N(0, I_n)$, A and B are $n \times n$ symmetric idempotent matrices, $\text{Rank}(A) = \text{tr}(A) = G$, $\text{Rank}(B) = \text{tr}(B) = K$ and $AB = 0$, then $\dfrac{X'AX/G}{X'BX/K} \sim F(G, K)$.

Note that the covariance of AX and BX is zero, which implies that AX is independent of BX under normality of X.

[End of Review]

6. Therefore, we obtain the following distribution:

$$\frac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)}{\sigma^2} = \frac{u'X(X'X)^{-1}X'u}{\sigma^2} \sim \chi^2(k),$$
$$\frac{e'e}{\sigma^2} = \frac{u'(I_n - X(X'X)^{-1}X')u}{\sigma^2} \sim \chi^2(n-k)$$

$\hat{\beta}$ is independent of e, because $X(X'X)^{-1}X'(I_n - X(X'X)^{-1}X') = 0$.

Accordingly, we can derive:

$$\frac{\dfrac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)}{\sigma^2}\Big/k}{\dfrac{e'e}{\sigma^2}\Big/(n-k)} = \frac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)/k}{s^2} \sim F(k, n-k)$$

Under the null hypothesis $H_0:\ \beta = 0$, $\dfrac{\hat{\beta}'X'X\hat{\beta}/k}{s^2} \sim F(k, n-k)$.

Given data, $\dfrac{\hat{\beta}'X'X\hat{\beta}/k}{s^2}$ is compared with $F(k, n-k)$.

If $\dfrac{\hat{\beta}'X'X\hat{\beta}/k}{s^2}$ is in the tail of the F distribution, the null hypothesis is rejected.

Coefficient of Determination (決定係数), R^2:

1. Definition of the Coefficient of Determination, R^2: $\quad R^2 = 1 - \dfrac{\sum_{i=1}^n e_i^2}{\sum_{i=1}^n (y_i - \overline{y})^2}$

2. Numerator: $\quad \displaystyle\sum_{i=1}^n e_i^2 = e'e$

3. Denominator: $\quad \displaystyle\sum_{i=1}^n (y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')'(I_n - \frac{1}{n}ii')y = y'(I_n - \frac{1}{n}ii')y$

   (*) Remark

$$\begin{pmatrix} y_1 - \overline{y} \\ y_2 - \overline{y} \\ \vdots \\ y_n - \overline{y} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{pmatrix} = y - \frac{1}{n}ii'y = (I_n - \frac{1}{n}ii')y,$$

   where $i = (1, 1, \cdots, 1)'$.

4. In a matrix form, we can rewrite as: $\quad R^2 = 1 - \dfrac{e'e}{y'(I_n - \frac{1}{n}ii')y}$

F Distribution and Coefficient of Determination:

$\Longrightarrow$ This will be discussed later.

Testing Linear Restrictions (F Distribution):

1. If $u \sim N(0, \sigma^2 I_n)$, then $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$.

   Consider testing the hypothesis $H_0 : R\beta = r$.

   $R: \ G \times k, \qquad \text{rank}(R) = G \leq k.$

   $R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1} R').$

   Therefore, $\dfrac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(G).$

   Note that $R\beta = r$.

   (a) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the mean of $R\hat{\beta}$ is:

   $$\mathrm{E}(R\hat{\beta}) = R\mathrm{E}(\hat{\beta}) = R\beta.$$

   (b) When $\hat{\beta} \sim N(\beta, \sigma^2 (X'X)^{-1})$, the variance of $R\hat{\beta}$ is:

   $$\begin{aligned}\mathrm{V}(R\hat{\beta}) &= \mathrm{E}((R\hat{\beta} - R\beta)(R\hat{\beta} - R\beta)') = \mathrm{E}(R(\hat{\beta} - \beta)(\hat{\beta} - \beta)'R') \\ &= R\mathrm{E}((\hat{\beta} - \beta)(\hat{\beta} - \beta)')R' = R\mathrm{V}(\hat{\beta})R' = \sigma^2 R(X'X)^{-1}R'.\end{aligned}$$

2. We know that $\dfrac{(n-k)s^2}{\sigma^2} = \dfrac{e'e}{\sigma^2} = \dfrac{(y - X\hat{\beta})'(y - X\hat{\beta})}{\sigma^2} \sim \chi^2(n-k)$.

3. Under normality assumption on u, $\hat{\beta}$ is independent of e.

4. Therefore, we have the following distribution:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(G, n-k)$$

5. **Some Examples:**

(a) t Test:

The case of $G = 1$, $r = 0$ and $R = (0, \cdots, 1, \cdots, 0)$ (the ith element of R is one and the other elements are zero):

The test of $H_0: \beta_i = 0$ is given by:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/G}{s^2} = \frac{\hat{\beta}_i^2}{s^2 a_{ii}} \sim F(1, n-k),$$

where $s^2 = e'e/(n-k)$, $R\hat{\beta} = \hat{\beta}_i$ and

$$a_{ii} = R(X'X)^{-1}R' = \text{ the } i \text{ row and } i\text{th column of } (X'X)^{-1}.$$

*) Recall that $Y \sim F(1, m)$ when $X \sim t(m)$ and $Y = X^2$.

Therefore, the test of $H_0: \beta_i = 0$ is given by:

$$\frac{\hat{\beta}_i}{s\sqrt{a_{ii}}} \sim t(n-k).$$

(b) Test of structural change (Part 1):

$$y_i = \begin{cases} x_i\beta_1 + u_i, & i = 1, 2, \cdots, m \\ x_i\beta_2 + u_i, & i = m+1, m+2, \cdots, n \end{cases}$$

Assume that $u_i \sim N(0, \sigma^2)$.

In a matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \\ \vdots & \vdots \\ x_m & 0 \\ 0 & x_{m+1} \\ 0 & x_{m+2} \\ \vdots & \vdots \\ 0 & x_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}$$

Moreover, rewriting,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u$$

Again, rewriting,

$$Y = X\beta + u$$

The null hypothesis is $H_0 : \beta_1 = \beta_2$.

Apply the F test, using $R = (I_k \quad -I_k)$ and $r = 0$.

In this case, $G = \text{rank}(R) = k$ and β is a $2k \times 1$ vector.

The distribution is $F(k, n - 2k)$.

(c) The hypothesis in which sum of the 1st and 2nd coefficients is equal to one:

$R = (1, 1, 0, \cdots, 0)$, $r = 1$

In this case, $G = \text{rank}(R) = 1$

The distribution of the test statistic is $F(1, n - k)$.

(d) Testing seasonality:

In the case of **quarterly data (四半期データ)**, the regression model is:

$$y = \alpha + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_3 + X\beta_0 + u$$

$D_j = 1$ in the jth quarter and 0 otherwise, i.e., D_j, $j = 1, 2, 3$, are seasonal dummy variables.

Testing seasonality $\Longrightarrow H_0:\ \alpha_1 = \alpha_2 = \alpha_3 = 0$

$$\beta = \begin{pmatrix} \alpha \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In this case, $G = \text{rank}(R) = 3$, and β is a $k \times 1$ vector.

The distribution of the test statistic is $F(3, n-k)$.

(e) Cobb-Douglas Production Function:

Let Q_i, K_i and L_i be production, capital stock and labor.

We estimate the following production function:

$$\log(Q_i) = \beta_1 + \beta_2 \log(K_i) + \beta_3 \log(L_i) + u_i.$$

We test a linear homogeneous (一次同次) production function.

The null and alternative hypotheses are:

$$H_0:\ \beta_2 + \beta_3 = 1,$$
$$H_1:\ \beta_2 + \beta_3 \neq 1.$$

Then, set as follows:

$$R = (\,0 \quad 1 \quad 1\,), \qquad r = 1.$$

(f) Test of structural change (Part 2):

Test the structural change between time periods m and $m + 1$.

In the case where both the constant term and the slope are changed, the regression model is as follows:

$$y_i = \alpha + \beta x_i + \gamma d_i + \delta d_i x_i + u_i,$$

where

$$d_i = \begin{cases} 0, & \text{for } i = 1, 2, \cdots, m, \\ 1, & \text{for } i = m + 1, m + 2, \cdots, n. \end{cases}$$

We consider testing the structural change at time $m + 1$.

The null and alternative hypotheses are as follows:

$$H_0: \ \gamma = \delta = 0,$$
$$H_1: \ \gamma \neq 0, \text{ or, } \delta \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(g) Multiple regression model:

Consider the case of two explanatory variables:

$$y_i = \alpha + \beta x_i + \gamma z_i + u_i.$$

We want to test the hypothesis that neither x_i nor z_i depends on y_i.

In this case, the null and alternative hypotheses are as follows:

$$H_0: \beta = \gamma = 0,$$
$$H_1: \beta \neq 0, \text{ or, } \gamma \neq 0.$$

Then, set as follows:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Coefficient of Determination R^2 and F distribution:

- The regression model:

$$y_i = x_i\beta + u_i = \beta_1 + x_{2i}\beta_2 + u_i$$

where

$$x_i = (\,1 \quad x_{2i}\,), \qquad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

x_i : $1 \times k$, $\quad x_{2i}$: $1 \times (k-1)$, $\quad \beta$: $k \times 1$, $\quad \beta_2$: $(k-1) \times 1$

Define:

$$X_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{pmatrix}$$

Then,

$$y = X\beta + u = (i \quad X_2)\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u = i\beta_1 + X_2\beta_2 + u,$$

where the first column of X corresponds to a constant term, i.e.,

$$X = (\, i \quad X_2 \,), \qquad i = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

● Consider testing $H_0:\ \beta_2 = 0$.
The F distribution is set as follows:

$$R = (\, 0 \quad I_{k-1} \,), \qquad r = 0$$

where R is a $(k-1) \times k$ matrix and r is a $(k-1) \times 1$ vector.

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} \;\sim\; F(k-1, n-k)$$

We are going to show:

$$(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r) = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2,$$

where $M = I_n - \frac{1}{n} i i'$.

Note that M is symmetric and idempotent, i.e., $M'M = M$.

$$\begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix} = My$$

$R(X'X)^{-1}R'$ is given by:

$$\begin{aligned} R(X'X)^{-1}R' &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \left(\begin{pmatrix} i' \\ X_2' \end{pmatrix} \begin{pmatrix} i & X_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} \end{aligned}$$

[Review] The inverse of a partitioned matrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square nonsingular matrices.

$$A^{-1} = \begin{pmatrix} B_{11} & -B_{11}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}A_{12}A_{22}^{-1} \end{pmatrix},$$

where $B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}$, or alternatively,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22} \\ -B_{22}A_{21}A_{11}^{-1} & B_{22} \end{pmatrix},$$

where $B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$.
[End of Review]

Go back to the F distribution.

$$\begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'X_2 - X_2'i(i'i)^{-1}i'X_2)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'(I_n - \frac{1}{n}ii')X_2)^{-1} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix}$$

Therefore, we obtain:

$$\begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} i'i & i'X_2 \\ X_2'i & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I_{k-1} \end{pmatrix} \begin{pmatrix} \cdot & \cdots \\ \vdots & (X_2'MX_2)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ I_{k-1} \end{pmatrix} = (X_2'MX_2)^{-1}.$$

Thus, under $H_0: \ \beta_2 = 0$, we obtain the following result:

$$\frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/(k-1)}{e'e/(n-k)} = \frac{\hat{\beta}_2'X_2'MX_2\hat{\beta}_2/(k-1)}{e'e/(n-k)} \ \sim \ F(k-1, n-k).$$

● Coefficient of Determination R^2:

Define e as $e = y - X\hat{\beta}$. The coefficient of determinant, R^2, is

$$R^2 = 1 - \frac{e'e}{y'My},$$

where $M = I_n - \frac{1}{n}ii'$, I_n is a $n \times n$ identity matrix and i is a $n \times 1$ vector consisting of 1, i.e., $i = (1, 1, \cdots, 1)'$.

$$Me = My - MX\hat{\beta}.$$

When $X = (\, i \quad X_2 \,)$ and $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$,

$$Me = e,$$

because $i'e = 0$, and

$$MX = M(\, i \quad X_2 \,) = (\, Mi \quad MX_2 \,) = (\, 0 \quad MX_2 \,),$$

because $Mi = 0$.

$$MX\hat{\beta} = (\, 0 \quad MX_2 \,)\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = MX_2\hat{\beta}_2.$$

Thus,

$$My = MX\hat{\beta} + Me \qquad \Longrightarrow \qquad My = MX_2\hat{\beta}_2 + e.$$

$y'My$ is given by: $y'My = \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2 + e'e$, because $X_2'e = 0$ and $Me = e$.
The coefficient of determinant, R^2, is rewritten as:

$$R^2 = 1 - \frac{e'e}{y'My} \qquad \Longrightarrow \qquad e'e = (1 - R^2)y'My,$$

$$R^2 = \frac{y'My - e'e}{y'My} = \frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2}{y'My} \qquad \Longrightarrow \qquad \hat{\beta}_2' X_2' M X_2 \hat{\beta}_2 = R^2 y'My.$$

Therefore,

$$\frac{\hat{\beta}_2' X_2' M X_2 \hat{\beta}_2/(k-1)}{e'e/(n-k)} = \frac{R^2 y'My/(k-1)}{(1-R^2)y'My/(n-k)} = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F(k-1, n-k).$$

Thus, using R^2, the null hypothesis $H_0 : \beta_2 = 0$ is easily tested.