8.4 Cointegration (共和分)

1. For a scalar y_t, when $\Delta y_t = y_t - y_{t-1}$ is a white noise (i.e., iid), we write $\Delta y_t \sim I(0)$ or $y_t \sim I(1)$.

   $Delta y_t = (1-L)y_t$, $\Delta^d y_t = (1-L)^d y_t$.

   $I(0)$ indicates a stationary process.

   $\Delta^d y_t \sim I(0)$ indicates that y_t has d unit roots.

   $\Delta^d y_t \sim I(0)$ indicates $y_t \sim I(d)$.

2. **Definition of Cointegration:**

   Suppose that each series in a $g \times 1$ vector y_t is $I(1)$, i.e., each series has unit root, and that a linear combination of each series (i.e, $a' y_t$ for a nonzero vector a) is $I(0)$, i.e., stationary.

   Then, we say that y_t has a cointegration.

   a is called the cointegrating vector.

3. **Example:**

   Suppose that $y_t = (y_{1,t},\ y_{2,t})'$ is the following vector autoregressive process:

$$y_{1,t} = \phi_1 y_{2,t} + \epsilon_{1,t},$$

$$y_{2,t} = y_{2,t-1} + \epsilon_{2,t}.$$

Then,

$$\begin{aligned}
\Delta y_{1,t} &= \phi_1 \epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \qquad \text{(MA(1) process)}, \\
\Delta y_{2,t} &= \epsilon_{2,t},
\end{aligned}$$

where both $y_{1,t}$ and $y_{2,t}$ are $I(1)$ processes.

The linear combination $y_{1,t} - \phi_1 y_{2,t}$ is $I(0)$.

In this case, we say that $y_t = (y_{1,t},\ y_{2,t})'$ is cointegrated with $a = (1,\ -\phi_1)$.

$a = (1,\ -\phi_1)$ is called the cointegrating vector, which is not unique.

Therefore, the first element of a is set to be one.

8.5 Spurious Regression (見せかけ回帰)

1. Suppose that $y_t \sim I(1)$ and $x_t \sim I(1)$.

   For the regression model $y_t = x_t\beta + u_t$, OLS does not work well if we do not have the β which satisfies $u_t \sim I(0)$.

$\Longrightarrow$ **Spurious regression (見せかけ回帰)**

2. Suppose that $y_t \sim I(1)$, y_t is a $g \times 1$ vector and $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$.

$y_{2,t}$ is a $k \times 1$ vector, where $k = g - 1$.

Consider the following regression model:

$$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t, \qquad t = 1, 2, \cdots, T.$$

OLSE is given by:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}.$$

Next, consider testing the null hypothesis $H_0: \ R\gamma = r$, where R is a $m \times k$ matrix ($m \leq k$) and r is a $m \times 1$ vector.

The F statistic, denoted by F_T, is given by:

$$F_T = \frac{1}{m}(R\hat{\gamma} - r)' \left(s_T^2 \,(0 \quad R) \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R' \end{pmatrix} \right)^{-1} (R\hat{\gamma} - r),$$

where

$$s_T^2 = \frac{1}{T - g} \sum_{t=1}^{T} (y_{1,t} - \hat{\alpha} - \hat{\gamma}' y_{2,t})^2.$$

When we have the γ such that $y_{1,t} - \gamma y_{2,t}$ is stationary, OLSE of γ, i.e., $\hat{\gamma}$, is not statistically equal to zero.

When the sample size T is large enough, H_0 is rejected by the F test.

3. Phillips, P.C.B. (1986) "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, Vol.33, pp.95 – 131.

Consider a $g \times 1$ vector y_t whose first difference is described by:

$$\Delta y_t = \Psi(L)\epsilon_t = \sum_{s=0}^{\infty} \Psi_s \epsilon_{t-s},$$

for ϵ_t an i.i.d. $g \times 1$ vector with mean zero , variance $\mathrm{E}(\epsilon_t \epsilon_t') = PP'$, and finite fourth moments and where $\{s\Psi_s\}_{s=0}^{\infty}$ is absolutely summable.

Let $k = g - 1$ and $\Lambda = \Psi(1)P$.

Partition y_t as $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$ and $\Lambda\Lambda'$ as $\Lambda\Lambda' = \begin{pmatrix} \Sigma_{11} & \Sigma_{21}' \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$, where $y_{1,t}$ and Σ_{11} are scalars, $y_{2,t}$ and Σ_{21} are $k \times 1$ vectors, and Σ_{22} is a $k \times k$ matrix.

Suppose that $\Lambda\Lambda'$ is nonsingular,and define $\sigma_1^{*2} = \Sigma_{11} - \Sigma_{21}'\Sigma_{22}^{-1}\Sigma_{21}$.

Let L_{22} denote the Cholesky factor of Σ_{22}^{-1}, i.e., L_{22} is the lower triangular matrix satisfying $\Sigma_{22}^{-1} = L_{22}L_{22}'$.

Then, (a) – (c) hold.

(a) OLSEs of α and γ in the regression model $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$, denoted by $\hat{\alpha}_T$ and $\hat{\gamma}_T$, are characterized by:

$$\begin{pmatrix} T^{-1/2}\hat{\alpha}_T \\ \hat{\gamma}_T - \Sigma_{22}^{-1}\Sigma_{21} \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1^* h_1 \\ \sigma_1^* L_{22} h_2 \end{pmatrix},$$

where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & \int_0^1 W_2^*(r)'\mathrm{d}r \\ \int_0^1 W_2^*(r)\mathrm{d}r & \int_0^1 W_2^*(r)W_2^*(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} \int_0^1 W_1^*(r)\mathrm{d}r \\ \int_0^1 W_2^*(r)W_1^*(r)\mathrm{d}r \end{pmatrix}.$

$W_1^*(r)$ and $W_2^*(r)$ denote scalar and g-dimensional standard Brownian motions, and $W_1^*(r)$ is independent of $W_2^*(r)$.

(b) The sum of squared residuals, denoted by $\mathrm{RSS}_T = \sum_{t=1}^{T} \hat{u}_t^2$, satisfies

$$T^{-2}\mathrm{RSS}_T \longrightarrow \sigma_1^{*2} H,$$

where $H = \int_0^1 (W_1^*(r))^2 \mathrm{d}r - \left(\begin{pmatrix} \int_0^1 W_1^*(r)\mathrm{d}r \\ \int_0^1 W_2^*(r)W_1^*(r)\mathrm{d}r \end{pmatrix}' \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right)^{-1}.$

(c) The F_T test satisfies:

$$
\begin{aligned}
T^{-1}F_T \;\longrightarrow\; & \frac{1}{m}(\sigma_1^* R^* h_2 - r^*)' \\
& \times\left(\sigma_1^{*2} H \begin{pmatrix} 0 & R^* \end{pmatrix} \begin{pmatrix} 1 & \int_0^1 W_2^*(r)'\mathrm{d}r \\ \int_0^1 W_2^*(r)\mathrm{d}r & \int_0^1 W_2^*(r)W_2^*(r)'\mathrm{d}r \end{pmatrix}^{-1} \begin{pmatrix} 0 & R^* \end{pmatrix}' \right)^{-1} \\
& \times(\sigma_1^* R^* h_2 - r^*),
\end{aligned}
$$

where $R^* = RL_{22}$ and $r^* = r - R\Sigma_{22}^{-1}\Sigma_{21}$.

Summary: Spurious regression (見せかけの回帰)

Consider the regression model: $y_{1.t} = \alpha + y_{2,t}\gamma + u_t$ for $t = 1, 2, \cdots, T$

and $y_t \sim I(1)$ for $y_t = (y_{1,t}, y_{2,t})'$.

(a) indicates that OLSE $\hat{\gamma}_T$ is not consistent.

(b) indicates that $s_T^2 = \dfrac{1}{T-g}\displaystyle\sum_{t=1}^{T}\hat{u}_t^2$ diverges.

(c) indicates that F_T diverges.

$\Longrightarrow$ It seems that the coefficients are statistically significant, based on the conventional t statistics.

4. **Resolution for Spurious Regression:**

Suppose that $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ is a spurious regression.

(1) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + \phi y_{1,t-1} + \delta y_{2,t-1} + u_t$.

Then, $\hat{\gamma}_T$ is $\sqrt{T}$-consistent, and the t test statistic goes to the standard normal distribution under $H_0 : \gamma = 0$.

(2) Estimate $\Delta y_{1,t} = \alpha + \gamma' \Delta y_{2,t} + u_t$. Then, $\hat{\alpha}_T$ and $\hat{\beta}_T$ are $\sqrt{T}$-consistent, and the t test and F test make sense.

(3) Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by the Cochrane-Orcutt method, assuming that u_t is the first-order serially correlated error.

Usually, choose (2).

However, there are two exceptions.

(i) The true value of ϕ is not one, i.e., less than one.

(ii) $y_{1,t}$ and $y_{2,t}$ are the cointegrated processes.

In these two cases, taking the first difference leads to the misspecified regression.

5. **Cointegrating Vector:**

Suppose that each element of y_t is $I(1)$ and that $a'y_t$ is $I(0)$.

a is called a **cointegrating vector (共和分ベクトル)**, which is not unique.

Set $z_t = a'y_t$, where z_t is scalar, and a and y_t are $g \times 1$ vectors.

For $z_t \sim I(0)$ (i.e., stationary),

$$T^{-1} \sum_{t=1}^{T} z_t^2 = T^{-1} \sum_{t=1}^{T} (a'y_t)^2 \longrightarrow \mathrm{E}(z_t^2).$$

For $z_t \sim I(1)$ (i.e., nonstationary, i.e., a is not a cointegrating vector),

$$T^{-2} \sum_{t=1}^{T} (a'y_t)^2 \longrightarrow \lambda^2 \int_0^1 (W(r))^2 \,\mathrm{d}r,$$

where $W(r)$ denotes a standard Brownian motion and λ^2 indicates variance of $(1-L)z_t$.

If a is not a cointegrating vector, $T^{-1} \sum_{t=1}^{T} z_t^2$ diverges.

$\Longrightarrow$ We can obtain a consistent estimate of a cointegrating vector by minimizing $\sum_{t=1}^{T} z_t^2$ with respect to a, where a normalization condition on a has to be imposed.

The estimator of the a including the normalization condition is super-consistent (T-consistent).

● Stock, J.H. (1987) "Asymptotic Properties of Least Squares Estimators of Cointegrating Vectors," *Econometrica*, Vol.55, pp.1035 – 1056.

Proposition:

Let $y_{1,t}$ be a scalar, $y_{2,t}$ be a $k \times 1$ vector, and $(y_{1,t}, y'_{2,t})'$ be a $g \times 1$ vector, where $g = k + 1$.

Consider the following model:

$$\begin{aligned} y_{1,t} &= \alpha + \gamma' y_{2,t} + z_t^*, \\ \Delta y_{2,t} &= u_{2,t}, \end{aligned} \qquad \begin{pmatrix} z_t^* \\ u_{2,t} \end{pmatrix} = \Psi^*(L)\epsilon_t,$$

ϵ_t is a $g \times 1$ i.i.d. vector with $\mathrm{E}(\epsilon_t) = 0$ and $\mathrm{E}(\epsilon_t \epsilon_t') = PP'$.

OLSE is given by: $\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y_{2,t}' \\ \sum y_{2,t} & \sum y_{2,t} y_{2,t}' \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}$.

Define λ_1^*, which is a $g \times 1$ vector, and Λ_2^*, which is a $k \times g$ matrix, as follows:

$$\Psi^*(1)\, P = \begin{pmatrix} \lambda_1^{*\prime} \\ \Lambda_2^* \end{pmatrix}.$$

Then, we have the following results:

$$\begin{pmatrix} T^{1/2}(\hat{\alpha} - \alpha) \\ T(\hat{\gamma} - \gamma) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \left(\Lambda_2^* \int W(r)\mathrm{d}r\right)' \\ \Lambda_2^* \int W(r)\mathrm{d}r & \Lambda_2^* \left(\int (W(r))\,(W(r))'\, \mathrm{d}r \right) \Lambda_2^{*\prime} \end{pmatrix}^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^{*\prime} W(1) \\ \Lambda_2^* \left(\int W(r)\,(\mathrm{d}W(r))' \right) \lambda_1^* + \sum_{\tau=0}^{\infty} \mathrm{E}(u_{2,t} z_{t+\tau}^*) \end{pmatrix}$.

$W(r)$ denotes a g-dimensional standard Brownian motion.

1) OLSE of the cointegrating vector is consistent even though u_t is serially correlated.

2) The consistency of OLSE implies that $T^{-1} \sum \hat{u}_t^2 \longrightarrow \sigma^2$.

3) Because $T^{-1} \sum (y_{1,t} - \overline{y}_1)^2$ goes to infinity, a coefficient of determination, R^2, goes to one.

8.6 Testing Cointegration

8.6.1 Engle-Granger Test

$y_t \sim I(1)$

$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$

- $u_t \sim I(0) \implies$ Cointegration
- $u_t \sim I(1) \implies$ Spurious Regression

Estimate $y_{1,t} = \alpha + \gamma' y_{2,t} + u_t$ by OLS, and obtain $\hat{u}_t$.

Estimate $\hat{u}_t = \rho \hat{u}_{t-1} + \delta_1 \Delta \hat{u}_{t-1} + \delta_2 \Delta \hat{u}_{t-2} + \cdots + \delta_{p-1} \Delta \hat{u}_{t-p+1} + e_t$ by OLS.

ADF Test:

- $H_0 : \ \rho = 1$ (Sprious Regression)
- $H_1 : \ \rho < 1$ (Cointegration)

$\implies$ **Engle-Granger Test**

For example, see Engle and Granger (1987), Phillips and Ouliaris (1990) and Hansen (1992).