Example 3: $x_1, x_2, \cdots, x_n$ are mutually independently and identically distributed as $N(\mu, \sigma^2)$, where μ and σ^2 are unknown.

$$f_{x|\theta}(x|\theta) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\Big(-\frac{1}{2\sigma^2}(x_i - \mu)^2\Big)$$
$$= (2\pi\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2}(s^2 + n(\overline{x} - \mu)^2)\Big),$$

where $\overline{x} = (1/n)\sum_{i=1}^{n} x_i$ and $s^2 = \sum_{i=1}^{n}(x_i - \overline{x})^2$.

The prior density is:

$$f_\theta(\theta) = k(a, b, w)\sigma^{b+3} \exp\left(-\frac{1}{2\sigma^2}\Big(a + \frac{(\mu - m)^2}{w}\Big)\right),$$

where $k(a, b, w) = \dfrac{a^{b/2} 2^{-(b+1)/2} (\pi w)^{-1/2}}{\Gamma(\frac{1}{2}b)}$ is a constant.

The posterior density is:

$$f_{\theta|x}(\theta|x) = k(a_1, b_1, w_1)\sigma^{-(b_1+3)} \exp\left(-\frac{1}{2\sigma^2}\Big(a_1 + \frac{(\mu - m_1)^2}{w_1}\Big)\right),$$

where $w_1 = \dfrac{w}{1+nw}$, $m_1 = \dfrac{m+nw\overline{x}}{1+nw}$, $b_1 = b+n$, $a_1 = a + s^2 + \dfrac{n(\overline{x}-m)^2}{1+nw}$.

Inference on μ: The posterior density of μ is:

$$f(\mu|x) = \int_0^\infty f(\theta|x)\mathrm{d}\sigma^2 = k_\mu(t_1, b_1)\left(1 + \frac{(\mu - m_1)^2}{b_1 t_1}\right)^{-(b_1+1)/2},$$

where $t_1 = \dfrac{w_1 a_1}{b_1}$ and $k_\mu(t_1, b_1) = \dfrac{1}{\sqrt{t_1 k_1}\,B(\frac{1}{2}, \frac{1}{2}b_1)}$.

Thus, $\dfrac{\mu - m_1}{\sqrt{t_1}}$ has a t distribution with b_1 degrees of freedom.

Inference of σ^2: The posterior density of σ^2 is:

$$f(\sigma^2|x) = \int_{-\infty}^\infty f(\theta|x)\mathrm{d}\mu = k_{\sigma^2}(a_1, b_1)\sigma^{-(b_1+2)}\exp\Big(-\frac{a_1}{2\sigma^2}\Big),$$

where $k_{\sigma^2}(a_1, b_1) = \dfrac{(\frac{1}{2}a_1)^{b_1/2}}{\Gamma(\frac{1}{2}b_1)}$.

Thus, $\dfrac{a_1}{\sigma^2}$ is chi-squared with b_1 degrees of freedom.

2.2 Inference

Posterior Distribution (事後分布): $f_{\theta|y}(\theta|y)$

2.2.1 Point Estimate

Posterior Mean (事後平均):

$$\bar{\theta} = \int_{-\infty}^{\infty} \theta f_{\theta|y}(\theta|y) \mathrm{d}\theta.$$

Posterior Mode (事後モード):

$$\hat{\theta} = \mathrm{argmax}_{\theta}\ f_{\theta|x}(\theta|y).$$

Posterior Median (事後メディアン):

$$\tilde{\theta} \ \text{ such that } \ \int_{-\infty}^{\tilde{\theta}} f_{\theta|y}(\theta|y) \mathrm{d}\theta = 0.5.$$

2.2.2 Interval Estimate

$$\int_R f_{\theta|y}(\theta|y)\mathrm{d}\theta = 1 - \alpha,$$

where R is called confidence interval.

Bayesian confidence interval (ベイズ信頼区間) or credible interval (信用区間):

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha.$$

θ_L and θ_U lead to lower and upper bounds.

(θ_L, θ_U) is called Bayesian confidence interval or credible interval.

Highest posterior density interval (最高事後密度区間):

$$f_{\theta|y}(\theta_0|y) \geq f_{\theta|y}(\theta_1|y), \qquad \text{for } \theta_0 \in R \text{ and } \theta_1 \notin R.$$

2.2.3 Marginal Likelihood (周辺尤度)

Marginal Likelihood $\Longrightarrow$ Fitness of the Model:

$$f_y(y) = \int f_{y|\theta}(y|\theta) f_\theta(\theta) \mathrm{d}\theta,$$

which corresponds to the denominator in the posterior distribution.

2.3 Example: Linear Regression

Regression Model:

$$y = X\beta + u, \qquad u \sim N(0, \sigma^2 I_n),$$

where y and u are $n \times 1$ vectors, X is an $n \times k$ matrix and β is a $k \times 1$ vector.

Likelihood Function: $\theta = (\beta, \sigma^2)$

$$f_{y|\theta}(y|\theta) = (2\pi\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\Big)$$

Prior Distributions:

$$f_\theta(\beta, \sigma^2) = f_{\beta|\sigma^2}(\beta|\sigma^2) f_{\sigma^2}(\sigma^2),$$

where

$$f_{\beta|\sigma^2}(\beta|\sigma^2) = N(\beta_0, \sigma^2 A^{-1}) = (2\pi\sigma^2)^{-k/2}|A|^{1/2} \exp\Big(-\frac{1}{2\sigma^2}(\beta - \beta_0)' A(\beta - \beta_0)\Big),$$

$$f_{\sigma^2}(\sigma^2) = IG\Big(\frac{\nu_0}{2}, \frac{\lambda_0}{2}\Big) = \frac{(\lambda_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)} (\sigma^2)^{-\nu_0/2-1} \exp\Big(-\frac{\lambda_0}{2\sigma^2}\Big).$$

β_0, A, ν_0 and λ_0 are called the hyper-parameters.

Note that $Y \sim IG(a, b)$ for $X \sim G(a, b)$ and $Y = \dfrac{1}{X}$.

The posterior distribution of β and σ^2 is:

$$\begin{aligned} f_{\theta|y}(\beta, \sigma^2|y) &\propto f_{y|\theta}(y|\beta, \sigma^2) f_{\beta|\sigma^2}(\beta|\sigma^2) f_{\sigma^2}(\sigma^2) \\ &= (2\pi\sigma^2)^{-n/2} \exp\Big(-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\Big) \end{aligned}$$

$$
\begin{aligned}
&\times(2\pi\sigma^2)^{-k/2}|A|^{1/2}\exp\Big(-\frac{1}{2\sigma^2}(\beta-\beta_0)'A(\beta-\beta_0)\Big)\\
&\times\frac{(\lambda_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)}(\sigma^2)^{-\nu_0/2-1}\exp\Big(-\frac{\lambda_0}{2\sigma^2}\Big)\\
\propto\ &(\sigma^2)^{-(n+k+\nu_0)/2-1}\exp\Big(-\frac{(y-X\beta)'(y-X\beta)+(\beta-\beta_0)'A(\beta-\beta_0)+\lambda_0}{2\sigma^2}\Big)\\
\propto\ &|\sigma^2\hat{A}|^{-1/2}\exp\Big(-\frac{(\beta-\hat{\beta})'\hat{A}^{-1}(\beta-\hat{\beta})}{2\sigma^2}\Big)\ \times\ (\sigma^2)^{-\hat{\nu}/2-1}\exp\Big(-\frac{\hat{\lambda}}{2\sigma^2}\Big)\\
\propto\ &f_{\beta|\sigma^2,y}(\beta|\sigma^2,y)\ \times\ f_{\sigma^2|y}(\sigma^2|y)=N(\hat{\beta},\sigma^2\hat{A})\ \times\ IG(\frac{\hat{\nu}}{2},\frac{\hat{\lambda}}{2})
\end{aligned}
$$

where

$$
\hat{\beta}=(X'X+A)^{-1}(X'X\hat{\beta}_{OLS}+A\beta_0),\qquad \hat{\beta}_{OLS}=(X'X)^{-1}X'y,
$$

$$
\hat{A}=(X'X+A)^{-1},\qquad \hat{\nu}=\nu_0+n,
$$

$$
\hat{\lambda}=\lambda_0+(y-X\hat{\beta})'(y-X\hat{\beta})+(\beta_0-\hat{\beta}_{OLS})'((X'X)^{-1}+A^{-1})^{-1}(\beta_0-\hat{\beta}_{OLS}).
$$

The marginal posterior distribution of β is:

$$f_{\beta|y}(\beta|y) = \int f_{\theta|y}(\beta, \sigma^2|y)\mathrm{d}\sigma^2 = \int f_{\beta|\sigma^2,y}(\beta|\sigma^2, y) f_{\sigma^2|y}(\sigma^2|y)\mathrm{d}\sigma^2$$

$$\propto \left(1 + \frac{1}{\hat{\nu}}(\beta - \hat{\beta})'\Big(\frac{\hat{\lambda}}{\hat{\nu}}\hat{A}\Big)^{-1}(\beta - \hat{\beta})\right)^{-(\hat{\nu}+k)/2},$$

which is a k-dimensional t distribution with parameters $\hat{\beta}$, $\dfrac{\hat{\lambda}}{\hat{\nu}}\hat{A}$ and $\hat{\nu}$.

Note that the k-dimensional t distribution with parameters μ, Σ and ν is given by:

$$f(x) = \frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\frac{\nu}{2})(\nu\pi)^{k/2}}|\Sigma|^{-1/2}\Big(1 + \frac{1}{\nu}(x - \mu)'\Sigma^{-1}(x - \mu)\Big)^{-(\nu+k)/2}.$$

The marginal likelihood is:

$$f_y(y) = \frac{f_{y|\theta}(y|\theta) f_\theta(\theta)}{f_{\theta|y}(\theta|y)} = \frac{|\hat{A}|^{1/2}|A|^{1/2}(\lambda_0/2)^{\nu_0/2}\Gamma(\hat{\nu}/2)}{\pi^{n/2}\Gamma(\nu_0/2)(\hat{\lambda}/2)^{\hat{\nu}/2}},$$

which is utilized for model selection.

In general, how do we evaluate $f_{\theta|y}(\theta|y)$, $\mathrm{E}(\theta|y)$, $f_y(y)$ and so on?

2.4 On Prior Distribution

2.4.1 Non-informative Prior

$$f_\theta(\theta) = \text{ const.}$$

In this case, the posterior distribution is:

$$f_{\theta|y}(\theta|y) \propto f_{y|\theta}(y|\theta),$$

which is proportional to the likelihood function.

However, we have the case where the integration of prior diverges, i.e.,

$$\int f_\theta(\theta)\mathrm{d}\theta = \infty.$$

In this case, $f_\theta(\theta)$ is called an improper prior.

2.4.2 Jeffreys' Prior

$$f_\theta(\theta) \propto |J(\theta)|^{1/2},$$

where

$$J(\theta) = -\int \frac{\partial^2 \log f_{y|\theta}(y|\theta)}{\partial\theta\partial\theta'} f_{y|\theta}(y|\theta)\mathrm{d}y = -\mathrm{E}\Big(\frac{\partial^2 \log f_{y|\theta}(y|\theta)}{\partial\theta\partial\theta'}\Big),$$

which is Fisher's information matrix.

2.5 Evaluation of Expectation

Posterior distribution $f_{\theta|y}(\theta|y)$

$$\mathrm{E}(\theta|y) = \int \theta f_{\theta|y}(\theta|y)\mathrm{d}\theta = \frac{\int \theta f_{y|\theta}(y|\theta) f_\theta(\theta)\mathrm{d}\theta}{\int f_{y|\theta}(y|\theta) f_\theta(\theta)\mathrm{d}\theta}.$$

In the case where it is not easy to evaluate $\mathrm{E}(\theta|y)$, how do we do?

Bayesian Method = Evaluation of Integration (Too much to say?)

- Numerical Integration
- Monte Carlo Integration
- Random Number Generation from $f_{\theta|y}(\theta|y)$