2025/05/20

Econometrics I's Homework

Deadline: June 24, 2025, AM10:20

- The answer should be written in English or Japanese.
- Your name and student ID number should be included in your answer sheet.
- Give me your answer sheet by hand in the classroom.

$\boxed{1}$ Consider the following single regression model:

$$y_i = \alpha + \beta X_i + u_i, \qquad i = 1, 2, \cdots, n,$$

where y_i and X_i denote dependent and independent variables, respectively. n is the sample size. u_1, u_2, $\cdots$, u_n are mutually independently distributed with mean zero and variance σ^2. α and β are unknown parameters to be estimated.

(1) Derive the ordinary least squares estimators of α and β, which should be denoted by $\hat{\alpha}$ and $\hat{\beta}$.

(2) Obtain mean and variance of $\hat{\beta}$.

(3) Obtain mean and variance of $\hat{\alpha}$.

(4) Prove that $\hat{\beta}$ is a linear estimator of β.

(5) Prove that $\hat{\beta}$ is a linear unbiased estimator of β.

(6) Prove that $\hat{\beta}$ has minimum variance within a class of linear unbiased estimators

(7) Prove that $\hat{\beta}$ is a consistent estimator of β.

(8) Derive an asymptotic distribution of $\sqrt{n}(\hat{\beta} - \beta)$.

(9) Suppose $u_i \sim N(0,\ \sigma^2)$ for all i as an extra assumption. Derive an exact distribution of $\hat{\beta}$, using the moment-generating function.

$\boxed{2}$ Consider the following multiple regression model:

$$y = X\beta + u$$

where y, X, β and u denote $n \times 1$, $n \times k$, $k \times 1$ and $n \times 1$ matrices. k and n are the numbers of explanatory variables and the sample size. $u_1,\ u_2,\ \cdots,\ u_n$ are mutually independently and **normally** distributed with mean zero and variance σ^2, i.e., $u \sim N(0, \sigma^2 I_n)$ for $u = (u_1,\ u_2,\ \cdots,\ u_n)'$. β is a vector of unknown parameters to be estimated. Let $\hat{\beta}$ be the ordinary least squares estimator of β.

(1) Derive $\hat{\beta}$.

(2) Derive mean and variance of $\hat{\beta}$.

(3) Derive a distribution of $\hat{\beta}$, using the moment-generating function.

(4) Show that $s^2 = \dfrac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta})$ is an unbiased estimator of σ^2.

(5) Show that $\dfrac{(n-k)s^2}{\sigma^2}$ is distributed as a χ^2 random variavble with $n-k$ degrees of freedom.

(6) Show that $\hat{\beta}$ is a best linear unbiased estimator.

(7) $\hat{\beta}$ is normally distributed with mean β and variance $\sigma^2(X'X)^{-1}$.

Then, show that $\dfrac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)}{\sigma^2} \sim \chi^2(k)$.

Especially, why is the degree of freedom equal to k?

(8) Show that $\hat{\beta}$ is independent of $s^2 = \dfrac{1}{n-k}(y - X\hat{\beta})'(y - X\hat{\beta})$.

(9) Show that $\dfrac{(\hat{\beta}-\beta)'X'X(\hat{\beta}-\beta)/k}{(y - X\hat{\beta})'(y - X\hat{\beta})/(n-k)} \sim F(k, n-k)$.

(10) Show that $\displaystyle\sum_{i=1}^{n}(y_i - \overline{y})^2 = y'(I_n - \frac{1}{n}ii')y$, where $y = (y_1, y_2, \cdots, y_n)'$ and $i = (1, 1, \cdots, 1)'$.

(11) Show that $I_n - \dfrac{1}{n}ii'$ is symmetric and idempotent.

$\boxed{3}$ (Statistics Questions) n random variables $X_1, X_2, \cdots, X_n$ are assumed to be mutually independently, identically and normally distributed with mean μ and variance σ^2. We want to prove $\dfrac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$, which is shown in standard statistics textbooks, where $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{j=1}^n X_j$.

Define $X = (X_1, X_2, \cdots, X_n)'$ and $i = (1, 1, \cdots, 1)'$, which are $n \times 1$ vectors. Then, we can rewrite as follows:

$$X \sim N(\mu i, \sigma^2 I_n).$$

Answer the following questions.

(1) What is the distribution of $\dfrac{(X - \mu i)'(X - \mu i)}{\sigma^2}$?

(2) Show that

$$(X - \mu i)'(I_n - \frac{1}{n} i i')(X - \mu i) = \sum_{j=1}^n (X_j - \overline{X})^2.$$

(3) Show that $\dfrac{(X - \mu i)'(I_n - \frac{1}{n} i i')(X - \mu i)}{\sigma^2} \sim \chi^2(n-1)$. (That is, $\dfrac{\sum_{j=1}^n (X_j - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$ is obtained.)