3.2 Panel Model Basic

Model:

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T$$

where i indicates individual and t denotes time.

There are n observations for each t.

u_{it} indicates the error term, assuming that $\mathrm{E}(u_{it}) = 0$, $\mathrm{V}(u_{it}) = \sigma_u^2$ and $\mathrm{Cov}(u_{it}, u_{js}) = 0$ for $i \neq j$ and $t \neq s$.

v_i denotes the individual effect, which is fixed or random.

3.2.1 Fixed Effect Model (固定効果モデル)

In the case where v_i is fixed, the case of $v_i = z_i\alpha$ is included.

$$y_{it} = X_{it}\beta + v_i + u_{it}, \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

$$\overline{y}_i = \overline{X}_i\beta + v_i + \overline{u}_i, \qquad i = 1, 2, \cdots, n,$$

where $\overline{y}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} y_{it}$, $\overline{X}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} X_{it}$, and $\overline{u}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} u_{it}$.

$$(y_{it} - \overline{y}_i) = (X_{it} - \overline{X}_i)\beta + (u_{it} - \overline{u}_i), \qquad i = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, T,$$

Taking an example of y, the left-hand side of the above equation is rewritten as:

$$y_{it} - \overline{y}_i = y_{it} - \frac{1}{T} 1_T' y_i,$$

where $1_T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, which is a $T \times 1$ vector, and $y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$.

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = I_T y_i - 1_T \overline{y}_i = I_T y_i - \frac{1}{T} 1_T 1_T' y_i = (I_T - \frac{1}{T} 1_T 1_T') y_i$$

Thus,

$$\begin{pmatrix} y_{i1} - \overline{y}_i \\ y_{i2} - \overline{y}_i \\ \vdots \\ y_{iT} - \overline{y}_i \end{pmatrix} = \begin{pmatrix} X_{i1} - \overline{X}_i \\ X_{i2} - \overline{X}_i \\ \vdots \\ X_{iT} - \overline{X}_i \end{pmatrix} \beta + \begin{pmatrix} u_{i1} - \overline{u}_i \\ u_{i2} - \overline{u}_i \\ \vdots \\ u_{iT} - \overline{u}_i \end{pmatrix}, \qquad i = 1, 2, \cdots, n,$$

which is re-written as:

$$(I_T - \frac{1}{T}1_T1_T')y_i = (I_T - \frac{1}{T}1_T1_T')X_i\beta + (I_T - \frac{1}{T}1_T1_T')u_i, \qquad i = 1, 2, \cdots, n,$$

i.e.,

$$D_Ty_i = D_TX_i\beta + D_Tu_i, \qquad i = 1, 2, \cdots, n,$$

where $D_T = (I_T - \frac{1}{T}1_T1_T')$, which is a $T \times T$ matrix.
Note that $D_TD_T' = D_T$, i.e., D_T is a symmetric and idempotent matrix.
Using the matrix form for $i = 1, 2, \cdots, n$, we have:

$$\begin{pmatrix} D_Ty_1 \\ D_Ty_2 \\ \vdots \\ D_Ty_n \end{pmatrix} = \begin{pmatrix} D_TX_1 \\ D_TX_2 \\ \vdots \\ D_TX_n \end{pmatrix}\beta + \begin{pmatrix} D_Tu_1 \\ D_Tu_2 \\ \vdots \\ D_Tu_n \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} y = \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} X\beta + \begin{pmatrix} D_T & 0 & \cdots & 0 \\ 0 & D_T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D_T \end{pmatrix} u,$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $X \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, and $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, which are $Tn \times 1$, $Tn \times k$ and $Tn \times 1$ matrices, respectively

Using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Kronecker Product — Review:

1. A: $n \times m$, B: $T \times k$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}, \text{ which is a } nT \times mk \text{ matrix.}$$

2. A: $n \times n$, B: $m \times m$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}, \qquad |A \otimes B| = |A|^m |B|^n,$$

$$(A \otimes B)' = A' \otimes B', \qquad \mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B).$$

3. For A, B, C and D such that the products are defined,

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

End of Review

Going back to the previous slide, using the Kronecker product, we obtain the following expression:

$$(I_n \otimes D_T)y = (I_n \otimes D_T)X\beta + (I_n \otimes D_T)u,$$

where $(I_n \otimes D_T)$, y, X, and u are $nT \times nT$, $nT \times 1$, $nT \times k$, and $nT \times 1$, respectively.

Apply OLS to the above regression model.

$$\begin{aligned}\hat{\beta} &= \Big(((I_n \otimes D_T)X)'(I_n \otimes D_T)X\Big)^{-1}((I_n \otimes D_T)X)'(I_n \otimes D_T)y \\ &= \Big(X'(I_n \otimes D_T'D_T)X\Big)^{-1}X'(I_n \otimes D_T'D_T)y \\ &= \Big(X'(I_n \otimes D_T)X\Big)^{-1}X'(I_n \otimes D_T)y.\end{aligned}$$

Note that the inverse matrix of D_T is not available, because the rank of D_T is $T-1$, not T (full rank).

The rank of a symmetric and idempotent matrix is equal to its trace.

The fixed effect v_i is estimated as:

$$\hat{v}_i = \bar{y}_i - \overline{X}_i\hat{\beta}.$$

Possibly, we can estimate the following regression:

$$\hat{v}_i = Z_i\alpha + \epsilon_i,$$

where it is assumed that the individual-specific effect depends on Z_i.

The estimator of σ_u^2 is given by:

$$\hat{\sigma}_u^2 = \frac{1}{nT - k - n} \sum_{i=1}^{n} \sum_{t=1}^{T} (y_{it} - X_{it}\hat{\beta} - \hat{v}_i)^2.$$

[Remark]

More than ten years ago, “fixed” indicates that v_i is nonstochastic.

Recently, however, “fixed” does not mean anything.

“fixed” indicates that OLS is applied and that v_i may be correlated with X_{it}.

Possibly, $\mathrm{E}(v_i|X) = \alpha_i(X)$, where $\alpha_i(X)$ is a function of X_{it} for $i = 1, 2, \cdots, n$ and $t = 1, 2, \cdots, T$, and it is normalized to $\sum_{i=1}^{n} \alpha_i(X) = 0$.