

The elements of u^* are correlated with each other, because each element of u^* is a function of u_1, u_2, \dots, u_n .

The variance of u^* is:

$$V(u^*) = V(Z'u) = \sigma^2 Z' \Omega Z.$$

Go back to GMM:

$$\begin{aligned} & (y - X\beta)' Z (Z' \Omega Z)^{-1} Z' (y - X\beta) \\ &= y' Z (Z' \Omega Z)^{-1} Z' y - \beta' X' Z (Z' \Omega Z)^{-1} Z' y - y' Z (Z' \Omega Z)^{-1} Z' X \beta + \beta' X' Z (Z' \Omega Z)^{-1} Z' X \beta \\ &= y' Z W Z' y - 2y' Z (Z' \Omega Z)^{-1} Z' X \beta + \beta' X' Z (Z' \Omega Z)^{-1} Z' X \beta. \end{aligned}$$

Note that $\beta' X' Z (Z' \Omega Z)^{-1} Z' y = y' Z (Z' \Omega Z)^{-1} Z' X \beta$ because both sides are scalars.

Remember that $\frac{\partial Ax}{x} = A'$ and $\frac{\partial x' Ax}{x} = (A + A')x$.

Then, we obtain the following derivation:

$$\begin{aligned}
& \frac{\partial(y - X\beta)'Z(Z'\Omega Z)^{-1}Z'(y - X\beta)}{\partial\beta} \\
&= -2(y'Z(Z'\Omega Z)^{-1}Z'X)' + (X'Z(Z'\Omega Z)^{-1}Z'X + (X'Z(Z'\Omega Z)^{-1}Z'X)')\beta \\
&= -2X'Z(Z'\Omega Z)^{-1}Z'y + 2X'Z(Z'\Omega Z)^{-1}Z'X\beta = 0
\end{aligned}$$

The solution of β is denoted by β_{GMM} , which is:

$$\beta_{GMM} = (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'y.$$

The mean of β_{GMM} is asymptotically obtained.

$$\begin{aligned}
\beta_{GMM} &= (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'(X\beta + u) \\
&= \beta + (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u \\
&= \beta + \left(\left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{n}Z'u \right)
\end{aligned}$$

We assume that

$$\frac{1}{n}X'Z \longrightarrow M_{xz} \quad \text{and} \quad \frac{1}{n}Z'\Omega Z \longrightarrow M_{z\Omega z},$$

which are $k \times r$ and $r \times r$ matrices.

From the assumption of $\frac{1}{n}Z'u \longrightarrow 0$, we have the following result:

$$\beta_{GMM} \longrightarrow \beta + (M_{xz}M_{z\Omega z}^{-1}M'_{xz})^{-1}M_{xz}M_{z\Omega z}^{-1} \times 0 = \beta.$$

Thus, β_{GMM} is a consistent estimator of β (i.e., asymptotically unbiased estimator).

The variance of β_{GMM} is asymptotically obtained as follows:

$$\begin{aligned}
 V(\beta_{GMM}) &= E\left((\beta_{GMM} - E(\beta_{GMM}))(\beta_{GMM} - E(\beta_{GMM}))'\right) \approx E\left((\beta_{GMM} - \beta)(\beta_{GMM} - \beta)'\right) \\
 &= E\left((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'u)'\right) \\
 &= E\left((X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'uu'Z(Z'\Omega Z)^{-1}Z'X(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}\right) \\
 &\approx (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}X'Z(Z'\Omega Z)^{-1}Z'E(uu')Z(Z'\Omega Z)^{-1}Z'X(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1} \\
 &= \sigma^2(X'Z(Z'\Omega Z)^{-1}Z'X)^{-1}.
 \end{aligned}$$

Note that $\beta_{GMM} \rightarrow \beta$ implies $E(\beta_{GMM}) \rightarrow \beta$ in the 1st line.

\approx in the 4th line indicates that Z and X are treated as exogenous variables although they are stochastic.

We assume that $E(uu') = \sigma^2\Omega$ from the 4th line to the 5th line.

- We derive the asymptotic distribution of β_{GMM} .

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{z\Omega z}).$$

Accordingly, β_{GMM} is asymptotically distributed as:

$$\begin{aligned} \sqrt{n}(\beta_{GMM} - \beta) &= \left(\left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{n}Z'X \right) \right)^{-1} \left(\frac{1}{n}X'Z \right) \left(\frac{1}{n}Z'\Omega Z \right)^{-1} \left(\frac{1}{\sqrt{n}}Z'u \right) \\ &\longrightarrow N(0, \sigma^2 (M_{xz} M_{z\Omega z}^{-1} M'_{xz})^{-1}). \end{aligned}$$

Practically, we use: $\beta_{GMM} \sim N(\beta, s^2 (X'Z(Z'\Omega Z)^{-1}Z'X)^{-1})$,

where $s^2 = \frac{1}{n-k} (y - X\beta_{GMM})' \Omega^{-1} (y - X\beta_{GMM})$.

We may use n instead of $n - k$.

Identically and Independently Distributed Errors:

- If u_1, u_2, \dots, u_n are mutually independent and u_i is distributed with mean zero and variance σ^2 , the mean and variance of u^* are given by:

$$E(u^*) = 0 \quad \text{and} \quad V(u^*) = E(u^*u^{*\prime}) = \sigma^2 Z'Z.$$

Using GLS, GMM is obtained as:

$$\beta_{GMM} = (X^{*\prime}(Z'Z)^{-1}X^*)^{-1}X^{*\prime}(Z'Z)^{-1}y^* = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y.$$

- We derive the asymptotic distribution of β_{GMM} .

From the central limit theorem,

$$\frac{1}{\sqrt{n}}Z'u \longrightarrow N(0, \sigma^2 M_{zz}).$$

Accordingly, β_{GMM} is distributed as:

$$\begin{aligned}\sqrt{n}(\beta_{GMM} - \beta) &= \left(\left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{n} Z'X \right) \right)^{-1} \left(\frac{1}{n} X'Z \right) \left(\frac{1}{n} Z'Z \right)^{-1} \left(\frac{1}{\sqrt{n}} Z'u \right) \\ &\rightarrow N\left(0, \sigma^2 (M_{xz} M_{zz}^{-1} M'_{xz})^{-1}\right).\end{aligned}$$

Practically, for large n we use the following distribution:

$$\beta_{GMM} \sim N\left(\beta, s^2 (X'Z(Z'Z)^{-1}Z'X)^{-1}\right),$$

where $s^2 = \frac{1}{n-k} (y - X\beta_{GMM})'(y - X\beta_{GMM})$.

- The above GMM is equivalent to 2SLS.

$X: n \times k, \quad Z: n \times r, \quad r > k.$

Assume:

$$\frac{1}{n}X'u = \frac{1}{n} \sum_{i=1}^n x'_i u_i \longrightarrow E(x'u) \neq 0,$$

$$\frac{1}{n}Z'u = \frac{1}{n} \sum_{i=1}^n z'_i u_i \longrightarrow E(z'u) = 0.$$

Regress X on Z , i.e., $X = Z\Gamma + V$ by OLS, where Γ is a $r \times k$ unknown parameter matrix and V is an error term,

Denote the predicted value of X by $\hat{X} = Z\hat{\Gamma} = Z(Z'Z)^{-1}Z'X$, where $\hat{\Gamma} = (Z'Z)^{-1}Z'X$.

Review — IV estimator: Consider the regression model is:

$$y = X\beta + u,$$

Assumption: $E(X'u) \neq 0$ and $E(Z'u) = 0$.

The $n \times k$ matrix Z is called the instrumental variable (IV).

The IV estimator is given by:

$$\beta_{IV} = (Z'X)^{-1}Z'y,$$

- Note that 2SLS is equivalent to IV in the case of $Z = \hat{X}$, where this Z is different from the previous Z .

This Z is a $n \times k$ matrix, while the previous Z is a $n \times r$ matrix.

Z in the IV estimator is replaced by \hat{X} .

Then,

$$\beta_{2SLS} = (\hat{X}'X)^{-1}\hat{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y = \beta_{GMM}.$$

GMM is interpreted as the GLS applied to MM.