7 Unit Root (Again)

Consider estimating the AR(1) model:

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \qquad \epsilon_t \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \qquad y_0 = 0, \qquad t = 1, \cdots, T$$

y_t ios rewritten as:

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \epsilon_t \\ &= \phi_1(\phi_1 y_{t-2} + \epsilon_{t-1}) + \epsilon_t = \phi_1^2 y_{t-2} + \epsilon_t + \phi_1 \epsilon_{t-1} \\ &\vdots \\ &= \phi_1^n y_0 + \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \ \cdots \ + \phi_1^{n-1} \epsilon_1 \end{aligned}$$

$\phi_1^n y_0 \longrightarrow 0$ as $n \longrightarrow \infty$.

$$y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \ \cdots$$

$$
\begin{aligned}
\mathrm{E}(y_t) &= \mathrm{E}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \ \cdots) = 0 \\
\mathrm{V}(y_t) &= \mathrm{V}(\epsilon_t + \phi_1 \epsilon_{t-1} + \phi_1^2 \epsilon_{t-2} + \ \cdots) \\
&= \mathrm{V}(\epsilon_t) + \phi_1^2 \mathrm{V}(\epsilon_{t-1}) + \phi_1^4 \mathrm{V}(\epsilon_{t-2}) + \ \cdots \\
&= \sigma_\epsilon^2 (1 + \phi_1^2 + \phi_1^4 + \ \cdots) = \frac{\sigma_\epsilon^2}{1 - \phi_1^2} = \gamma(0)
\end{aligned}
$$

where $\gamma(\tau) = \mathrm{Cov}(y_t,\ y_{t-\tau}) = \mathrm{E}(y_t y_{t-\tau}) \quad \longrightarrow$ **Autocovariance function**

● **The Case of $|\phi_1| < 1$:**

Then, OLSE of ϕ_1 is:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^T y_{t-1} y_t}{\sum_{t=1}^T y_{t-1}^2}.$$

In the case of $|\phi_1| < 1$,

$$\hat{\phi}_1 = \phi_1 + \frac{\frac{1}{T}\sum_{t=1}^T y_{t-1}\epsilon_t}{\frac{1}{T}\sum_{t=1}^T y_{t-1}^2} \longrightarrow \phi_1 + \frac{\mathrm{E}(y_{t-1}\epsilon_t)}{\mathrm{E}(y_{t-1}^2)} = \phi_1.$$

Note as follows:

$$\frac{1}{T}\sum_{t=1}^T y_{t-1}\epsilon_t \longrightarrow \mathrm{E}(y_{t-1}\epsilon_t) = 0.$$

By the central limit theorem,

$$\frac{\overline{y\epsilon} - \mathrm{E}(\overline{y\epsilon})}{\sqrt{\mathrm{V}(\overline{y\epsilon})}} \longrightarrow N(0,1)$$

where

$$\overline{y\epsilon} = \frac{1}{T}\sum_{t=1}^T y_{t-1}\epsilon_t.$$

$$\begin{aligned}
\mathrm{E}(\overline{y\epsilon}) &= 0, \\
\mathrm{V}(\overline{y\epsilon}) &= \mathrm{V}\big(\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t\big) = \mathrm{E}\Big(\big(\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\epsilon_t\big)^2\Big) \\
&= \frac{1}{T^2}\mathrm{E}\Big(\sum_{t=1}^{T}\sum_{s=1}^{T} y_{t-1}y_{s-1}\epsilon_t\epsilon_s\Big) = \frac{1}{T^2}\mathrm{E}\Big(\sum_{t=1}^{T} y_{t-1}^2\epsilon_t^2\Big) = \frac{1}{T}\sigma^2\gamma(0).
\end{aligned}$$

Therefore,

$$\frac{\overline{y\epsilon}}{\sqrt{\sigma^2\gamma(0)/T}} = \frac{1}{\sigma_\epsilon\sqrt{\gamma(0)}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0,1),$$

which is rewritten as:

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow N(0,\sigma_\epsilon^2\gamma(0)).$$

Using $\dfrac{1}{T}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \mathrm{E}(y_{t-1}^2) = \gamma(0)$, we have the following asymptotic distribution:

$$\sqrt{T}(\hat{\phi}_1 - \phi_1) = \frac{\dfrac{1}{\sqrt{T}}\displaystyle\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\dfrac{1}{T}\displaystyle\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow N\left(0, \frac{\sigma_\epsilon^2}{\gamma(0)}\right) = N\left(0, 1 - \phi_1^2\right).$$

Note that $\gamma(0) = \dfrac{\sigma_\epsilon^2}{1 - \phi_1^2}$.

In the case of $\phi_1 = 1$, as expected, we have:

$$\sqrt{T}(\hat{\phi}_1 - 1) \longrightarrow 0.$$

That is, $\hat{\phi}_1$ has the distribution which converges in probability to $\phi_1 = 1$ (i.e., degenerated distribution).

Is this true?

● **The Case of $\phi_1 = 1$:** $\Longrightarrow$ Random Walk Process

$y_t = y_{t-1} + \epsilon_t$ with $y_0 = 0$ is written as:

$$y_t = \epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots + \epsilon_1.$$

Therefore, we can obtain:

$$y_t \sim N(0, \sigma_\epsilon^2 t).$$

The variance of y_t depends on time t. $\Longrightarrow$ y_t is nonstationary.

Remember that $\hat{\phi}_1 = \phi_1 + \dfrac{\sum y_{t-1}\epsilon_t}{\sum y_{t-1}^2}$.

1. First, consider the numerator $\sum y_{t-1}\epsilon_t$.

   We have $y_t^2 = (y_{t-1} + \epsilon_t)^2 = y_{t-1}^2 + 2y_{t-1}\epsilon_t + \epsilon_t^2$.

   Therefore, we obtain:

$$y_{t-1}\epsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \epsilon_t^2).$$

Taking into account $y_0 = 0$, we have:

$$\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}y_T^2 - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t^2.$$

Divided by $\sigma_\epsilon^2 T$ on both sides, we have the following:

$$\frac{1}{\sigma_\epsilon^2 T}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma_\epsilon\sqrt{T}}\right)^2 - \frac{1}{2\sigma_\epsilon^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2.$$

From $y_t \sim N(0, \sigma_\epsilon^2 t)$, we obtain the following result:

$$\left(\frac{y_T}{\sigma_\epsilon\sqrt{T}}\right)^2 \sim \chi^2(1).$$

Moreover, the second term is derived from:

$$\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \sigma_\epsilon^2.$$

Therefore,

$$\frac{1}{\sigma_\epsilon^2 T}\sum_{t=1}^{T} y_{t-1}\epsilon_t = \frac{1}{2}\left(\frac{y_T}{\sigma\sqrt{T}}\right)^2 - \frac{1}{2\sigma_\epsilon^2}\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \longrightarrow \frac{1}{2}(\chi^2(1)-1).$$

Next, consider $\sum y_{t-1}^2$.

$$\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) = \sum_{t=1}^{T}\mathrm{E}(y_{t-1}^2) = \sum_{t=1}^{T}\sigma_\epsilon^2(t-1) = \sigma_\epsilon^2\frac{T(T-1)}{2}.$$

Thus, we obtain the following result:

$$\frac{1}{T^2}\mathrm{E}\left(\sum_{t=1}^{T} y_{t-1}^2\right) \longrightarrow \text{a fixed value.}$$

Therefore,

$$\frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \text{a distribution.}$$

Summarizing the results up to now, $T(\hat{\phi}_1 - \phi_1)$, not $\sqrt{T}(\hat{\phi}_1 - \phi_1)$, has limiting distribution in the case of $\phi_1 = 1$.

$$T(\hat{\phi}_1 - \phi_1) = \frac{(1/T)\sum y_{t-1}\epsilon_t}{(1/T^2)\sum y_{t-1}^2} \longrightarrow \text{ a distribution.}$$

$$y_t = \phi_1 y_{t-1} + \epsilon_t.$$

Test $H_0:\ \phi_1 = 1$ against $H_1:\ \phi_1 < 1$.

Equivalently,

$$\Delta y_t = \rho y_{t-1} + \epsilon_t.$$

Test $H_0:\ \rho = 0$ against $H_1:\ \rho < 0$.

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.49	−2.06	−1.71	−1.32	1.32	1.71	2.06	2.49
50	−2.40	−2.01	−1.68	−1.30	1.30	1.68	2.01	2.40
100	−2.36	−1.98	−1.66	−1.29	1.29	1.66	1.98	2.36
250	−2.34	−1.97	−1.65	−1.28	1.28	1.65	1.97	2.34
500	−2.33	−1.96	−1.65	−1.28	1.28	1.65	1.96	2.33
∞	−2.33	−1.96	−1.64	−1.28	1.28	1.64	1.96	2.33

(a) $\boldsymbol{H_0 : \ y_t = y_{t-1} + \epsilon_t}$

$\boldsymbol{H_1 : \ y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{\phi_1 < 1}$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

To test $H_0 : \ \rho = 0$ against $H_1 : \ \rho < 0$, estimate $\Delta y_t = \rho y_{t-1} + \epsilon_t$ and compare the t-value of ρ with the above table.

Basic Concepts of Random Walk Process (More Formally):

1. Model: $y_t = y_{t-1} + \epsilon_t, \qquad y_0 = 0, \qquad \epsilon_t \sim N(0, 1).$

   Then,

   $$y_t = \epsilon_t + \epsilon_{t-1} + \cdots + \epsilon_1.$$

   Therefore,

   $$y_t \sim N(0, t).$$

   $\Longrightarrow$ Nonstationary Process (i.e., variance depends on time t.)

   Let T be the sample size.

   $$\frac{y_t}{\sqrt{T}} \sim N(0, \frac{t}{T})$$

Let us define $r = \dfrac{t}{T}$.

Keeping $r = \dfrac{t}{T}$, as $T \longrightarrow \infty$,

$$\frac{y_t}{\sqrt{T}} \sim N(0,\ \frac{t}{T}) \longrightarrow N(0, r) = W(r)$$

The limit when $T \longrightarrow \infty$ is a **continuous time (連続時間)** process known as **standard Brownian motion** or **Wiener process**.

$W(r)$ is a normal random variable with mean zero and variance r and it is a continuous function in r.

2. **Examples:**

$W(0) = 0$

$W(1) = N(0,\ 1)$

$W(1)^2 = \chi^2(1)$

$$\sigma W(r) \sim N(0, \sigma^2 r)$$

$$W(r)^2 = \sqrt{r} W(1)^2 \sim r \times \chi^2(1)$$

3. Suppose that $y_t = y_{t-1} + \epsilon_t, \quad y_0 = 0, \quad \epsilon_t \sim N(0, \sigma^2)$.

$$y_t \sim N(0,\ \sigma^2 t), \text{ i.e.,}$$

$$\frac{y_t}{\sqrt{T}} \sim N(0,\ \sigma^2 \frac{t}{T}), \text{ i.e.,}$$

$$\frac{y_t}{\sigma\sqrt{T}} \sim N(0,\ \frac{t}{T}) \longrightarrow N(0, r) = W(r).$$

- $\displaystyle\sum_{t=1}^{T} \frac{1}{T} \frac{y_t}{\sigma\sqrt{T}} \longrightarrow \int_0^1 W(r)\,\mathrm{d}r.$

$$\Big(\frac{y_t}{\sigma\sqrt{T}}\Big)^2 \longrightarrow W(r)^2.$$

- $\displaystyle\sum_{t=1}^{T} \frac{1}{T} \Big(\frac{y_t}{\sigma\sqrt{T}}\Big)^2 \longrightarrow \int_0^1 W(r)^2\,\mathrm{d}r.$

(*) **Continuous Mapping Theorem (連続写像定理):**

if $x_T \longrightarrow x$ (convergence in distribution) and $g(\cdot)$ is a continuous function, then

$g(x_T) \longrightarrow g(x)$ (convergence in distribution).

Threfore, we have the follwoing result:

$$\frac{1}{T}\sum_{t=1}^{T}\Big(\frac{y_t}{\sqrt{T}}\Big)^2 = \frac{1}{T^2}\sum_{t=1}^{T} y_t^2 \longrightarrow \sigma^2 \int_0^1 W(r)^2 \,\mathrm{d}r.$$

Asymptotic Distribution of AR(1) Model:

1. $\boldsymbol{H_0 : \; y_t = y_{t-1} + \epsilon_t}$ **and** $\boldsymbol{H_1 : \; y_t = \phi_1 y_{t-1} + \epsilon_t}$ **for** $\boldsymbol{|\phi_1| < 1}$

   OLSE of ϕ_1, denoted by $\hat{\phi}_1$, is given by:

$$\hat{\phi}_1 = \frac{\sum_{t=1}^{T} y_{t-1}y_t}{\sum_{t=1}^{T} y_{t-1}^2} = \phi_1 + \frac{\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}$$

   Using $\phi_1 = 1$ and some formulas shown above, we obtain:

$$T(\hat{\phi}_1 - 1) = \frac{T^{-1}\sum_{t=1}^{T} y_{t-1}\epsilon_t}{T^{-2}\sum_{t=1}^{T} y_{t-1}^2} \longrightarrow \frac{\frac{1}{2}(W(1)^2 - 1)}{\int_0^1 W(r)^2 \mathrm{d}r}$$

   Remember that

$$T^{-1}\sum_{t=1}^{T} y_{t-1}\epsilon_t \longrightarrow \frac{1}{2}\sigma^2(W(1)^2 - 1)$$

   and

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \longrightarrow \sigma^2 \int_0^1 W(r)^2 \, \mathrm{d}r,$$

where $(W(1))^2 = \chi^2(1)$.

We say that $\hat{\phi}_1$ is **super-consistent (超一致性)** or ***T*-consistent**.

Remember that when $|\phi_1| < 1$ we have $\sqrt{T}(\hat{\phi}_1 - \phi_1) \longrightarrow N(0, 1 - \phi_1^2)$, and in this case we say that $\hat{\phi}_1$ is $\boldsymbol{\sqrt{T}}$**-consistent**.