Conventional t test statistic is given by:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi},$$

where

$$s_\phi = \left(s^2 \Big/ \sum_{t=1}^{T} y_{t-1}^2\right)^{1/2} \quad \text{and} \quad s^2 = \frac{1}{T-1}\sum_{t=1}^{T}(y_t - \hat{\phi}_1 y_{t-1})^2.$$

Next, consider t statistic.

The t test statistic, denoted by t, is represented as follows:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi} = \frac{T(\hat{\phi}_1 - 1)}{T s_\phi}$$

The denominator is:

$$T s_\phi = \left(s^2 \Big/ \frac{1}{T^2}\sum_{t=1}^{T} y_{t-1}^2\right)^{1/2}$$

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$$\longrightarrow \left(\sigma^2 \Big/ \left(\sigma^2 \int_0^1 (W(r))^2 \, dr\right)\right)^{1/2} = \left(\int_0^1 (W(r))^2 \, dr\right)^{-1/2},$$

where $s^2 \longrightarrow \sigma^2$ is utilized.

Therefore, we have the following asymptotic distribution:

$$t = \frac{\hat{\phi}_1 - 1}{s_\phi} \longrightarrow \frac{\frac{1}{2}\left((W(1))^2 - 1\right)}{\int_0^1 (W(r))^2 \, dr} \Big/ \left(\int_0^1 (W(r))^2 \, dr\right)^{-1/2}$$

$$= \frac{\frac{1}{2}\left((W(1))^2 - 1\right)}{\left(\int_0^1 (W(r))^2 \, dr\right)^{1/2}}.$$

Therefore, the distribution of the t statistic shown above is different from the t distribution.

The distributions of the t statistic: $\dfrac{\hat{\phi}_1 - 1}{s_\phi}$

t Distribution

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.49	−2.06	−1.71	−1.32	1.32	1.71	2.06	2.49
50	−2.40	−2.01	−1.68	−1.30	1.30	1.68	2.01	2.40
100	−2.36	−1.98	−1.66	−1.29	1.29	1.66	1.98	2.36
250	−2.34	−1.97	−1.65	−1.28	1.28	1.65	1.97	2.34
500	−2.33	−1.96	−1.65	−1.28	1.28	1.65	1.96	2.33
∞	−2.33	−1.96	−1.64	−1.28	1.28	1.64	1.96	2.33

(a) $H_0 : y_t = y_{t-1} + \epsilon_t$

$H_1 : y_t = \phi_1 y_{t-1} + \epsilon_t$ for $\phi_1 < 1$ or $-1 < \phi_1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00

(b) $H_0 : y_t = y_{t-1} + \epsilon_t$

$H_1 : y_t = \alpha_0 + \phi_1 y_{t-1} + \epsilon_t$ for $\phi_1 < 1$ or $-1 < \phi_1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60

(d) $H_0 : y_t = \alpha_0 + y_{t-1} + \epsilon_t$

$H_1 : y_t = \alpha_0 + \alpha_1 t + \phi_1 y_{t-1} + \epsilon_t$ for $\phi_1 < 1$ or $-1 < \phi_1$

T	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

7.1 AR(p) model: Augmented Dickey-Fuller (ADF) Test

Consider the case where the error term is serially correlated.

Consider the following AR(p) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t, \qquad \epsilon_t \sim \text{iid}(0, \sigma^2),$$

which is rewritten as:

$$\phi(L)y_t = \epsilon_t.$$

When the above model has a unit root, we have $\phi(1) = 0$, i.e., $\phi_1 + \phi_2 + \cdots + \phi_p = 1$.
The above AR(p) model is written as:

$$y_t = \rho y_{t-1} + \delta_1 \Delta y_{t-1} + \delta_2 \Delta y_{t-2} + \cdots + + \delta_{p-1} \Delta y_{t-p+1} + \epsilon_t,$$

where $\rho = \phi_1 + \phi_2 + \cdots + \phi_p$ and $\delta_j = -(\phi_{j+1} + \phi_{j+2} + \cdots + \phi_p)$.

The null and alternative hypotheses are:

$$H_0 : \ \rho = 1 \ (\text{Unit root}),$$

$$H_1 : \ \rho < 1 \ (\text{Stationary}).$$

Use the *t* test, where we have the same asymptotic distributions.

We can utilize the same tables as before.

Choose *p* by AIC or SBIC.

Use $N(0, 1)$ to test $H_0 : \ \delta_j = 0$ against $H_1 : \ \delta_j \neq 0$ for $j = 1, 2, \cdots, p - 1$.

Reference

Kurozumi (2008) "Economic Time Series Analysis and Unit Root Tests: Development and Perspective," *Japan Statistical Society*, Vol.38, Series J, No.1, pp.39 – 57.
Download the above paper from:

http://ci.nii.ac.jp/vol_issue/nels/AA11989749/ISS0000426576_ja.html

7.2 Cointegration (共和分)

1. For a scalar y_t, when $(1 - L)^d y_t$ is stationary, we write $y_t \sim I(d)$.

   When $\Delta y_t = y_t - y_{t-1}$ is stationary, we write $\Delta y_t \sim I(0)$ or $y_t \sim I(1)$.

2. **Definition of Cointegration:**

   Suppose that each series in a $g \times 1$ vector y_t is $I(1)$, i.e., each series has unit root, and that a linear combination of each series (i.e, $a'y_t$ for a nonzero vector a) is $I(0)$, i.e., stationary.

   Then, we say that y_t has a cointegration.

3. **Example:**

   Suppose that $y_t = (y_{1,t},\ y_{2,t})'$ is the following vector autoregressive process:

$$y_{1,t} = \gamma y_{2,t} + \epsilon_{1,t},$$

$$y_{2,t} = y_{2,t-1} + \epsilon_{2,t}.$$

Then,

$$\Delta y_{1,t} = \gamma \epsilon_{2,t} + \epsilon_{1,t} - \epsilon_{1,t-1}, \quad \text{(MA(1) process)},$$

$$\Delta y_{2,t} = \epsilon_{2,t},$$

where both $y_{1,t}$ and $y_{2,t}$ are $I(1)$ processes.

The linear combination $y_{1,t} - \gamma y_{2,t}$ is $I(0)$.

In this case, we say that $y_t = (y_{1,t}, \ y_{2,t})'$ is cointegrated with $a = (1, \ -\gamma)$.

$a = (1, \ -\gamma)$ is called the **cointegrating vector (共和分ベクトル)**, which is not unique. Therefore, the first element of a is set to be one.

4. Suppose that $y_t \sim I(1)$ and $x_t \sim I(1)$.

For the regression model $y_t = x_t\beta + u_t$, OLS does not work well if we do not have the β which satisfies $u_t \sim I(0)$.

$\implies$ **Spurious regression (見せかけの回帰)**

5. Suppose that $y_t \sim I(1)$, y_t is a $g \times 1$ vector and $y_t = \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$.

   $y_{2,t}$ is a $k \times 1$ vector, where $k = g - 1$.

   Consider the following regression model:

   $$y_{1,t} = \alpha + \gamma' y_{2,t} + u_t, \qquad t = 1, 2, \cdots, T.$$

   OLSE is given by:

   $$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} \sum y_{1,t} \\ \sum y_{1,t} y_{2,t} \end{pmatrix}.$$

   Next, consider testing the null hypothesis $H_0 : R\gamma = d$, where R is a $G \times k$

matrix ($G \leq k$) and r is a $G \times 1$ vector. G denotes the number of the linear restrictions.

The F statistic, denoted by F, is given by:

$$F = \frac{1}{G}(R\hat{\gamma} - d)' \left(s^2 \begin{pmatrix} 0 & R \end{pmatrix} \begin{pmatrix} T & \sum y'_{2,t} \\ \sum y_{2,t} & \sum y_{2,t} y'_{2,t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R' \end{pmatrix} \right)^{-1} (R\hat{\gamma} - d),$$

where

$$s^2 = \frac{1}{T - g} \sum_{t=1}^{T} (y_{1,t} - \hat{\alpha} - \hat{\gamma}' y_{2,t})^2.$$

When we have the γ such that $y_{1,t} - \gamma y_{2,t}$ is stationary, OLSE of γ, i.e., $\hat{\gamma}$, is not statistically equal to zero.

When the sample size T is large enough, H_0 is rejected by the F test.

6. Phillips, P.C.B. (1986) "Understanding Spurious Regressions in Econometrics," *Journal of Econometrics*, Vol.33, pp.95 – 131.