「Econometrics II」

Homework # 1 (Due: December 23, 2025, AM10:20)

- The answer should be written in English or Japanese.
- Your name and student ID number should be included in your answer sheet.
- Bring your answer sheet in this classroom and hand it to me.
- DO NOT send me the answer sheet by email.

$\boxed{1}$ Consider that n random variables X_1, X_2, $\cdots$, X_n are mutually independently and identically distributed as the density function $f(x;\theta)$, where θ is an unknown parameter to be estimated. For simplicity, θ is a scalar. Let $s(X)$ be an unbiased estimator of θ for $X = (X_1, X_2, \cdots, X_n)$.

(1) Suppose that $L(\theta; x) = \prod_{i=1}^n f(x_i;\theta)$, which is called the likelihood function. Show the following two equalities:

$$\mathrm{E}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right) = 0.$$

$$-\mathrm{E}\left(\frac{\partial^2 \log L(\theta; X)}{\partial \theta^2}\right) = \mathrm{E}\left(\Big(\frac{\partial \log L(\theta; X)}{\partial \theta}\Big)^2\right) = \mathrm{V}\left(\frac{\partial \log L(\theta; X)}{\partial \theta}\right).$$

(2) Show that the following inequality holds.

$$\mathrm{V}(s(X)) \geq (I(\theta))^{-1},$$

where $I(\theta) = -\mathrm{E}\Big(\dfrac{\partial^2 \log L(\theta; X)}{\partial \theta^2}\Big)$, which is called Fisher's information matrix.

(3) Suppose that $s(X) = \tilde{\theta}$ is a maximum likelihood estimator. Then, show that

$$\sqrt{n}(\tilde{\theta} - \theta) \longrightarrow N(0, \sigma^2),$$

where $\sigma^2 = \lim_{n\to\infty}\Big(\dfrac{I(\theta)}{n}\Big)^{-1}$.

$\boxed{2}$ Consider the following regression model:

$$y_i^* = X_i\beta + u_i,$$

where X_i is assumed to be exogenous and nonstochastic, and $u_1, u_2, \cdots, u_n$ are mutually independent errors.

Let $f(x)$ be the density function of u_i and $F(x)$ be the cumulative distribution function of u_i, i.e., $F(x) = \int_{-\infty}^{x} f(z)\mathrm{d}z$.

(a) Let us define:

$$y_i = \begin{cases} 1, & \text{if } y_i^* > 0, \\ 0, & \text{if } y_i^* \le 0, \end{cases}$$

i.e., y_i^* is not observed and we know the sign of y_i^* (i.e., positive or negative). $y_i$is assigned to be one when $y_i^* > 0$, while it is zero when $y_i \le 0$.

(1) What is $\mathrm{E}(y_i)$?

(2) Obtain the likelihood function.

(3) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.

(4) Discuss how to estimate β and σ^2.

(5) What is the asymptotic distribution of the maximum likelihood estimator of $\beta^* = \dfrac{\beta}{\sigma}$?

(b) Let us define:

$$y_i^* = y_i, \qquad \text{if } y_i > 0,$$

i.e., y_t^* is not observed when $y_t \le 0$ and $y_t^* = y_t$ is observed when $y_t > 0$.

(6) What is $\mathrm{E}(y_i|y_i > 0)$?

(7) Obtain the likelihood function.

(8) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.

(9) Discuss how to estimate β and σ^2.

(10) What are the asymptotic distributions of the maximum likelihood estimators of β and σ^2?

(c) Let us define:

$$y_i^* = \begin{cases} y_i, & \text{if } y_i > 0, \\ 0, & \text{if } y_i \le 0, \end{cases}$$

i.e., $y_t^* = 0$ is observed when $y_t \leq 0$ and $y_t^* = y_t$ is observed when $y_t > 0$.

(11) Obtain the likelihood function.

(12) Assuming that the density function of u_i is $f(\cdot)$, derive the first-order condition.

(13) Discuss how to estimate β and σ^2.

(14) What are the asymptotic distributions of the maximum likelihood estimators of β and σ^2?

$\boxed{3}$ Suppose that the probability function of y_i is Poisson with parameter λ_i for $i = 1, 2, \cdots, n$.

(1) What is $\mathrm{E}(y_i)$?

(2) Assuming $\lambda_i = \exp(X_i\beta)$, obtain the likelihood function, where β is a unknown parameter vector to be estimated.

(3) Derive the first-order condition for maximization of the log-likelihood function.

(4) Discuss how to estimate β.

(5) What are the asymptotic distribution of the maximum likelihood estimator of β?