Discussion Papers In Economics And Business

A Periodic Review Inventory Model in a Random Environment

Hirotaka Matsumoto
Yoshio Tabata

Discussion Paper 03-01

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

A Periodic Review Inventory Model in a Random Environment

Hirotaka Matsumoto
Yoshio Tabata

Discussion Paper 03-01

January 2003

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

A Periodic Review Inventory Model in a Random Environment

Hirotaka Matsumoto
Osaka University

Yoshio Tabata
Osaka University

January 2003

Abstract

We consider a single product, periodic review inventory model with variable capacity, random yield, and uncertain demand in a random environment. All model parameters and distributions depend on environmental fluctuations. It is assumed that the environmental process follows a discrete-time Markov chain. The optimal inventory policy to minimize the total discounted expected cost is derived via dynamic programming. For the finite-horizon model, we show that the objective function is quasi-convex and that the structure of the optimal policy is characterized by a single environmental-dependent critical number for the initial inventory level at each period. Expressions for solving the critical number and the optimal planned ordering are obtained. We further show that the solution for the finite-horizon model converges to that of the infinite-horizon model.

1 Introduction

Inventory systems are often subject to randomly changing exogenous environment-conditions that affect the demand for the product, the supply, and the cost structure.

But, the inventory control has long focused on managing certain specific types of probability distributions in the demand for the products. Since many models include a purely random component in the demand process, it will be difficult to describe the inventory control of products sensitive to economic conditions, products subject to obsolescence, and new products.

Inventory models operating in random environments are only scarcely considered in earlier papers. For example, Kalymon[11] considers a discrete-time inventory purchasing model, in which the unit cost of the item is determined by a Markov process, and the distribution of demand in each period depends on the current cost. Feldman[7] models the demand environment as a continuous-time Markov chain. Given the state of environment, the demand forms a compound-Poisson process. But he studies only the stationary distribution of the inventory position. Song and Zipkin[17] derive some basic characteristics of optimal policies and develop algorithms for computing them in a continuous-review inventory model where the demand process is a Markov modulated Poisson process. Özekici and Parlar[14] consider infinite-horizon periodic-review inventory models with unreliable suppliers where the demand, the supply and cost parameters change with respect to a randomly changing environment.

The effect of a randomly changing environment in other stochastic models in operations research is discussed in following papers. Çinlar and Özekici[3] studied a model in reliability and maintenance where the failure rates of the components of a device depend on a semi-Markov environment process.

Eisen and Tainiter[6], Neuts[12], and Prabhu and Zhu[15] introduced a model where the arrival and service rates depend on a randomly changing environment.

Another issue in this paper is that of supply process. As uncertainties in supply process, we consider two categories that effect the products by the different ways, variable capacity and random yield. In a given time period, the planned order is fully received if the order quantity is less than the realized capacity. Otherwise, only part of it can be received. On the other hand, yield is random due to the random proportion of defectives received in a lot. So, in this paper, we use the stochastically proportional yield model, as defined in Henig and Gerchak[8]. That is, the received quantity is the product of the order quantity and a random fraction, called yield rate, which is independent of the order quantity.

The earliest model of random supply variable was explored by Karlin[1]. Yano and Lee[20] gave several ways of modeling random yield. Henig and Gerchak[8] analyzed a periodic review inventory policy with stochastically proportional yield. In the single period situation, they showed that there exists a reorder point that does'nt depend on uncertain yield. In multi-period problem, they showed the existence of a reorder point, nonorder-up-to optimal policy, and convexity of the cost function. Ciarallo et al.[2] model capacity as random variable with a known distribution function. In a periodic review model, they show that the objective function is quasi-convex, and the optimal policy is a base-stock policy where the order-up-to level is a constant. Further, Wang and Gerchak[19] simultaneously incorporate variable capacity and stochastically proportional yield in a periodic review setting. They show that the objective function is quasi-convex, and the optimal policy is not an order-up-to type where is characterized by a single critical number at each period.

In this paper, we introduce a periodic review inventory model that incorporates random environment, variable capacity, and stochastically proportional yield. The supply and the demand processes depend on environmental fluctuations of which process follows a discrete-time Markov chain. Furthermore, all of the cost parameters are affected by the environmental process. The main advantage of the Markov-chain approach is that it provides a natural and flexible framework for formulating various changes described above.

The purpose of this paper is to show that the objective function is quasi-convex, and the structure of the optimal policy is characterized by a single environmental-dependent critical number for the initial inventory level at each period. We further show that the solution for the finite-horizon model converges to that of the infinite-horizon model. In this paper, in particular, we focus on the finite-horizon analysis, since it gives us concrete and realistic insights.

This paper is organized as follows: Section 2 presents the formulation of the general problem as a dynamic programming model. Section 3, 4, and 5 provide analyses for the single-period, finite-horizon, and infinite-horizon problems, respectively. The paper concludes with some final remarks in Section 6.

2 Assumption and Notation

Consider a single-product periodic-review inventory system for N-periods. Let the period be numbered such that the final period is denoted as period 1, while the first period is denoted as period N.

The state of the environment observed at the beginning of period n ($n = 1, 2, \dots, N$) is represented by I_n and we assume that $I = \{I_n; n \geq 0\}$ is a Markov chain on a countable state space E with a given transition matrix $P = \{P(i,j)\} = \{P[I_{n-1} = j | I_n = i]\}$. Let X_n denote the inventory level observed at the beginning of period n. The basic assumption of this model is that the cost-parameters, the demand and the supply distributions at any period depend on the state of the environment at the beginning of that period. Therefore, the decision maker observes both the inventory level and the environment state to decide on the optimal order quantity which is delivered immediately.

If D_n is the total demand during period n, then the demand process $D = \{D_n; n \geq 0\}$ depends on the Markov chain I so that its conditional distribution function is $A_i(z_n) = P[D_n \leq z_n | I_n = i]$, with the probability density function $a_i(z_n)$. Also, we assume $A_i(0) = 0, a_i(\cdot) > 0$. Let W_n be a random variable representing the uncertain capacity in period n. Its conditional distribution function is $F_i(w_n) = P[W_n \leq w_n | I_n = i]$, with the probability density function $f_i(w_n)$. Also, we assume $F_i(0) = 0, f_i(\cdot) > 0$. Let R_n be a random variable representing the random yield in period n. Its conditional distribution function is $Q_i(r_n) = P[R_n \leq r_n | I_n = i]$, with the probability density function $q_i(r_n)$ and a mean ν_i. Also, we assume $Q_i(0) = 0, q_i(\cdot) > 0$

We consider the following four types of costs: if the environmental state is i, a fixed ordering cost K_i independent of the order quantity, a unit ordering cost c_i, a unit holding cost h_i incurred at the end of period, and a unit shortage cost p_i incurred at the end of period. To motivate ordering, we assume that $\nu_i p_i > c_i$ as in standard models. Also, we assume that unsatisfied demands are fully backlogged.

Let $U_n(i, x_n)$ be the order quantity if the environment is i and the inventory level is x_n at the beginning of period n. The admissibility condition requires that $U_n(i, x_n) \geq 0$ since we do not allow for discarding of any inventory without satisfying demand. It is noted that, for any u_n, the inventory level X_n is a Markov chain, where

$$X_{n-1} = x_n + R_n \min\{u_n(i, x_n), W_n\} - D_n$$

for $n \geq 0$. Figure 1 illustrates the behavior of the inventory level.

Now, let $V_i^n(x_n)$ be the minimum expected total discount cost of operating for n-period with the state of the environment i and the initial inventory level x_n, under the best ordering decision is used at period n through period 1. Then, a dynamic programming equation (DPE) for the problem can be given by

$$V_i^0(x_0) \equiv 0,$$

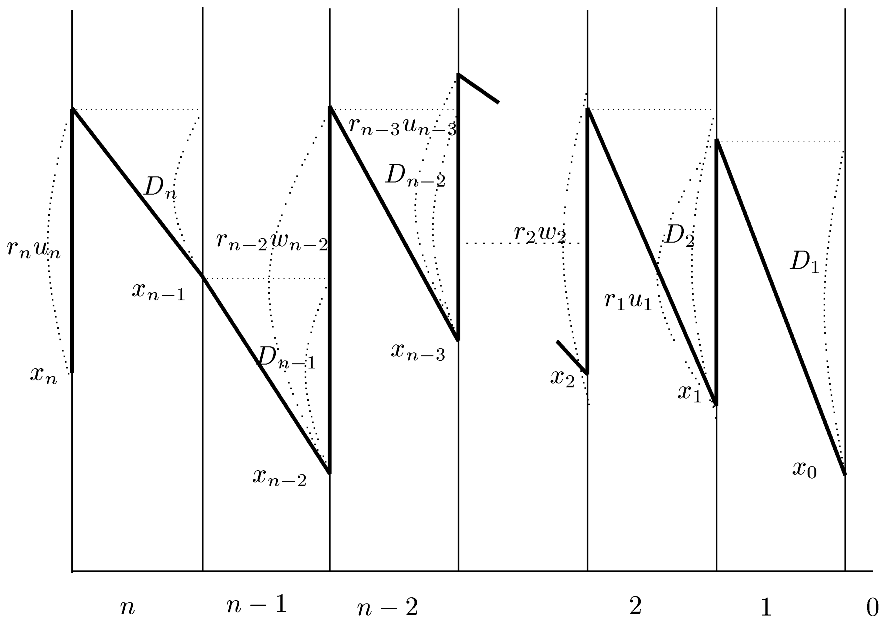

Figure 1: The behavior of the inventory level

and

$$V_i^n(x_n) = \min_{u_n \geq 0} \{K_i \delta(u_n) + G_i^n(x_n, u_n)\}, \; n > 0, i \in E, \tag{1}$$

where

$$\delta(u_n) = \begin{cases} 1 & \text{if } u_n > 0, \\ 0 & \text{if } u_n = 0, \end{cases}$$

and

$$\begin{aligned} G_i^n(x_n, u_n) \;\; = \;\; & c_i \int_0^{u_n} w_n dF_i(w_n) + \int_0^{\infty} \int_0^{u_n} \Big\{ L_i^n(x_n + r_n w_n) + \alpha \sum_{j \in E} P(i,j) \\ & \times \int_0^{\infty} V_j^{n-1}(x_n + r_n w_n - z_n) dA_i(z_n) \Big\} dF_i(w_n) dQ_i(r_n) \\ & + \big[1 - F_i(u_n)\big] \Big[c_i u_n + \int_0^{\infty} \Big\{ L_i^n(x_n + r_n u_n) \\ & + \alpha \sum_{j \in E} P(i,j) \int_0^{\infty} V_j^{n-1}(x_n + r_n u_n - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \Big] \end{aligned} \tag{2}$$

with the expected holding and shortage cost function at period n

$$L_i^n(y) \;\; = \;\; h_i \int_0^y (y - z_n) dA_i(z_n) + p_i \int_y^{\infty} (z_n - y) dA_i(z_n)$$

and the discount factor α per period. The first and second derivatives of $L_i^n(y)$ are

$$\begin{aligned} L_i'^n(y) &= (h_i + p_i) A_i(y) - p_i \\ L_i''^n(y) &= (h_i + p_i) a_i(y) > 0 \end{aligned}$$

The decision variable in this model is u_n, so (2) plays a central role to find the optimal value u_n^*.

We assume that all parameters and costs are nonnegative, and that all relevant functions are differentiable.

3 Single-Period Analysis

In this section we analyze the single-period problem for the model introduced in the last section. This analysis will provide important insights in understanding the n-period analysis. We begin by rewriting (1) and (2) as

$$\begin{aligned} V_i^1(x_1) &= \min_{u_1 \geq 0} \{K_i \delta(u_1) + G_i^1(x_1, u_1)\} && (3) \\ G_i^1(x_1, u_1) &= c_i \int_0^{u_1} w_1 dF_i(w_1) + \int_0^\infty \int_0^{u_1} L_i^1(x_1 + r_1 w_1) dF_i(w_1) dQ_i(r_1) \\ &\quad + \big[1 - F_i(u_1)\big] \bigg\{ c_i u_1 + \int_0^\infty L_i^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} && (4) \end{aligned}$$

We first investigate the properties of (4) since it plays a central role in the minimization in (3). We obtain the first two derivatives of (4) as follows:

$$\begin{aligned} \frac{\partial G_i^1(x_1, u_1)}{\partial u_1} &= \big[1 - F_i(u_1)\big] \bigg\{ c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} && (5) \\ \frac{\partial^2 G_i^1(x_1, u_1)}{\partial u_1^2} &= \big[1 - F_i(u_1)\big] \int_0^\infty r_1^2 L_i''^1(x_1 + r_1 u_1) dQ_i(r_1) \\ &\quad - f_i(u_1) \bigg\{ c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} && (6) \end{aligned}$$

It should be noted that $L_i'^1(\cdot)$ is increasing,

$$\begin{aligned} \lim_{u_1 \to \infty} \bigg\{ c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} &= c_i + \nu_i h_i > 0, && (7) \\ \lim_{u_1 \to -\infty} \bigg\{ c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} &= c_i - \nu_i p_i < 0. && (8) \end{aligned}$$

For given x_1, therefore, there exists a finite and unique solution such that

$$c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) = 0 \tag{9}$$

Let $u_i^1(x_1)$ solve (9), then $u_1 = u_i^1(x_1)$ satisfies the first-order condition for minimizing (4). To satisfy the second-order conditions, notice that (6) is nonnegative when

$$\begin{aligned} &\big[1 - F_i(u_1)\big] \int_0^\infty r_1^2 L_i''^1(x_1 + r_1 u_1) dQ_i(r_1) \\ &\qquad \geq f_i(u_1) \bigg\{ c_i + \int_0^\infty r_1 L_i'^1(x_1 + r_1 u_1) dQ_i(r_1) \bigg\} && (10) \end{aligned}$$

Because the left hand side of (10) is always positive, the inequality is always satisfied when the right-hand side is nonpositive. Thus, if $u_1 < u_i^1(x_1)$, then

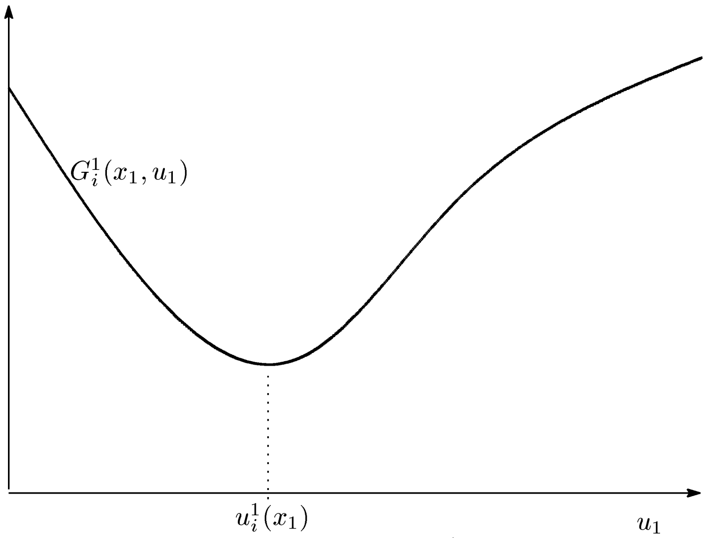

Figure 2: The form of $G_i^1(x_1,u_1)$

$\partial^2 G_i^1(x_1,u_1)/\partial u_1^2 > 0$, hence $G_i^1(x_1,u_1)$ is convex and decreasing. For values of u_1 close to $u_i^1(x_1)$, still $\partial^2 G_i^1(x_1,u_1)/\partial u_1^2 > 0$, hence $G_i^1(x_1,u_1)$ is convex. But for large values of $u_1 \geq u_i^1(x_1)$, $\partial^2 G_i^1(x_1,u_1)/\partial u_1^2 < 0$, hence $G_i^1(x_1,u_1)$ is concave and increasing. The behavior of $G_i^1(x_1,u_1)$ for given x_1 is shown graphically in Figure 2.

From these observations, it is clear that $G_i^1(x_1,u_1)$ is quasi-convex in u_1 and attains its global minimum at $u_1 = u_i^1(x_1)$ for given x_1.

Now, differentiating (9) with respect to x_1, we get

$$u_i'^1(x_1) = -\frac{\int_0^\infty r_1 L_i''^1(x_1 + r_1 u_i^1(x_1))dQ_i(r_1)}{\int_0^\infty r_1^2 L_i''^1(x_1 + r_1 u_i^1(x_1))dQ_i(r_1)} < 0. \tag{11}$$

So, when $R_1 \equiv 1$, i.e., without random yield, we see from (11) that $u_i'^1(x_1) = -1$. Then, the optimal policy is of an order-up-to type. But in case with random yield, in general, $u_i'^1(x_1) \neq -1$. That is, the optimal policy is not of an order-up-to type.

Next, we investigate the properties of $G_i^1(x_1, u_i^1(x_1))$. We obtain the first two derivatives of $G_i^1(x_1, u_i^1(x_1))$ in x_1 as follows:

$$\begin{aligned}
G_i'^1(x_1, u_i^1(x_1)) &= \partial G_i^1(x_1, u_i^1(x_1))/\partial x_1 + \partial G_i^1(x_1, u_i^1(x_1))/\partial u_1 \cdot u_i'^1(x_1) \\
&= \partial G_i^1(x_1, u_i^1(x_1))/\partial x_1 \\
&= \int_0^\infty \int_0^{u_i^1(x_1)} L_i'^1(x_1 + r_1 w_1) dF_i(w_1) dQ_i(r_1) \\
&\quad + \big[1 - F_i(u_i^1(x_1))\big] \int_0^\infty L_i'^1(x_1 + r_1 u_i^1(x_1)) dQ_i(r_1) \qquad (12)
\end{aligned}$$

$$\begin{aligned}
G_i''^1(x_1, u_i^1(x_1)) = \int_0^\infty \int_0^{u_i^1(x_1)} L_i''^1(x_1 + r_1 w_1) dF_i(w_1) dQ_i(r_1) \\
+ \big[1 - F_i(u_i^1(x_1))\big] \int_0^\infty L_i''^1(x_1 + r_1 u_i^1(x_1))(1 + r_1 u_i'^1(x_1) dQ_i(r_1)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^{u_i^1(x_1)} L_i''^1(x_1 + r_1 w_1) dF_i(w_1) dQ_i(r_1) + \left[1 - F_i(u_i^1(x_1))\right] \\
&\times \Bigg[\int_0^\infty L_i''^1(x_1 + r_1 u_i^1(x_1)) dQ_i(r_1) \cdot \int_0^\infty r_1^2 L_i''^1(x_1 + r_1 u_i^1(x_1)) dQ_i(r_1) \\
&- \left\{ \int_0^\infty r_1 L_i''^1(x_1 + r_1 u_i^1(x_1)) dQ_i(r_1) \right\}^2 \Bigg] \\
&\Big/ \int_0^\infty r_1^2 L_i''^1(x_1 + r_1 u_i^1(x_1)) dQ_i(r_1) \qquad (13)
\end{aligned}
$$

The second multiplicand of the second term in (13) is positive by the Cauchy-Schwartz inequality. Then, we see that all the terms in (13) are positive, so $G_i''^1(x_1, u_i^1(x_1)) > 0$. That is, $G_i^1(x_1, u_i^1(x_1))$ is convex in x_1.

Now, if $u_1 = 0$, (9) becomes

$$c_i + \nu_i L_i'^1(x_1) = 0. \qquad (14)$$

Let $\hat{x_1}$ solve (14), that is,

$$\hat{x_1} = A_i^{-1}\left[\frac{p_i - c_i/\nu_i}{h_i + p_i}\right].$$

$\hat{x_1}$ is nonnegative and finite because $0 < \left[\frac{p_i - c_i/\nu_i}{h_i + p_i}\right] < 1$ with $(p_i - c_i/\nu_i) > 0$ and $(p_i - c_i/\nu_i) < (h_i + p_i)$. That is, if $x_1 = \hat{x_1}$, then $G_i^1(x_1, u_i^1(x_1)) = G_i^1(x_1, 0)$. Otherwise, $G_i^1(x_1, u_i^1(x_1)) < G_i^1(x_1, 0)$. Here, it should be noted that the planned order quantity can not be negative. From (9) we see that $u_i^1(x_1)$ is decreasing in x_1, since $L_i'^1(\cdot)$ is increasing. So, if $x_1 \geq \hat{x_1}$, then $u_i^1(x_1) \leq 0$. That is, from the quasi-convexity of $G_i^1(x_1, u_1)$ in u_1, the optimal order quantity is 0. Furthermore, for $x_1 < \hat{x_1}$, there exists a unique solution such that

$$K_i + G_i^1(x_1, u_i^1(x_1)) = G_i^1(x_1, 0) \qquad (15)$$

Let s_i^1 solve (15), then it follows that

$$
\begin{aligned}
K_i + G_i^1(x_1, u_i^1(x_1)) &\leq G_i^1(x_1, 0) \text{ for } x_1 \leq s_i^1, \qquad (16) \\
K_i + G_i^1(x_1, u_i^1(x_1)) &> G_i^1(x_1, 0) \text{ for } s_i^1 < x_1 \leq \hat{x_1}. \qquad (17)
\end{aligned}
$$

The behavior of $G_i^1(x_1, u_i^1(x_1))$, $K_i + G_i^1(x_1, u_i^1(x_1))$, and $G_i^1(x_1, 0)$ is shown graphically in Figure 3.

Based upon the state of the environment i and the initial inventory level x_1, the optimal policy can now be characterized in terms of the single critical number s_i^1. For $x_1 \leq s_i^1$, the expected savings $\{G_i^1(x_1, 0) - G_i^1(x_1, u_i^1(x_1))\}$ gained by ordering $u_i^1(x_1)$ units can offset the fixed ordering cost K_i provided one plans to order. This follows from (16). On the other hand, for $s_i^1 < x_1 < \hat{x_1}$, it is not worthwhile to order because the fixed ordering cost K_i will offset the expected savings $\{G_i^1(x_1, 0) - G_i^1(x_1, u_i^1(x_1))\}$ derived from ordering $u_i^1(x_1)$ units. This follows from (17). Since the planned order quantity < 0 for $x_1 \geq \hat{x_1}$, it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period 1 and the property of $G_i^1(x_1, u_1)$.

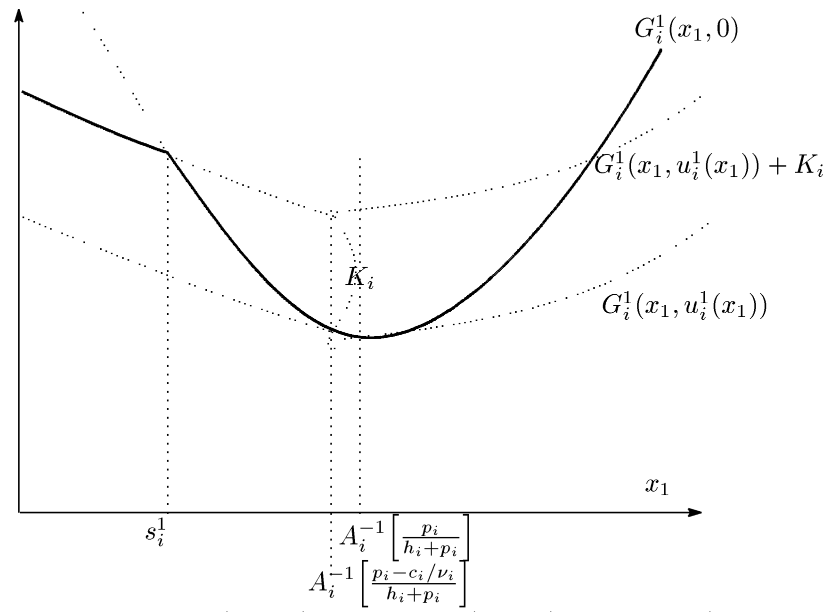

Figure 3: The form of $G_i^1(x_1,u_i^1(x_1))$, $K_i+G_i^1(x_1,u_i^1(x_1))$, and $G_i^1(x_1,0)$

(1) the optimal policy for period 1 is given by

$$U_1^*(i,x_1)=\begin{cases} u_i^1(x_1) & \text{if } x_1\le s_i^1,\\ 0 & \text{if } x_1>s_i^1,\end{cases}$$

where critical number s_i^1 is a solution to Eq.(15) and is the reorder point.

(2) For given x_1,

$$G_i^1(x_1,u_1) \text{ is quasi-convex and } \begin{cases} \text{decreasing in } u_1 & \text{if } u_1\le u_i^1(x_1),\\ \text{increasing in } u_1 & \text{if } u_1>u_i^1(x_1).\end{cases}$$

Therefore, the expected cost $V_i^1(x_1)$ under the optimal policy is obtained by substituting $U_1^*(i,x_1)$ into (3):

$$V_i^1(x_1) \;=\; \begin{cases} K_i+G_i^1(x_1,u_i^1(x_1)) & \text{if } x_1\le s_i^1,\\ G_i^1(x_1,0)=L_i^1(x_1) & \text{if } x_1>s_i^1.\end{cases} \tag{18}$$

And its first two derivatives are

$$V_i'^1(x_1) \;=\; \begin{cases} G_i'^1(x_1,u_i^1(x_1)) & \text{if } x_1\le s_i^1\\ L_i'^1(x_1) & \text{if } x_1>s_i^1\end{cases} \tag{19}$$

$$V_i''^1(x_1) \;=\; \begin{cases} G_i''^1(x_1,u_i^1(x_1)) & \text{if } x_1\le s_i^1\\ L_i''^1(x_1) & \text{if } x_1>s_i^1\end{cases} >0. \tag{20}$$

So, $V_i^1(x_1)$ is convex in x_1.

4 n-Period Analysis

In this section, we analyze the n-period problem for the model introduced in Section 2.

To use induction, we assume that the following properties hold for the $(n-1)$-period problem, where the state of the environment is $j \in E$.

$$V_j^{n-1}(x_{n-1}) \text{ is convex. That is, } V_j''^{n-1}(x_{n-1}) > 0.$$

$$\begin{aligned}
\lim_{x_{n-1}\to\infty} V_j'^{n-1}(x_{n-1}) &= h_j + \alpha \sum_{k\in E} P(j,k)h_k + \alpha^2 \sum_{k,l\in E} P(j,k)P(k,l)h_l + \\
&\quad \cdots + \alpha^{n-2} \sum P(j,k)P(k,l)\cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0 \\
\lim_{x_{n-1}\to-\infty} V_j'^{n-1}(x_{n-1}) &= -p_j - \alpha \sum_{k\in E} P(j,k)p_k - \alpha^2 \sum_{k,l\in E} P(j,k)P(k,l)p_l - \\
&\quad \cdots - \alpha^{n-2} \sum P(j,k)P(k,l)\cdots P(\chi,\psi)P(\psi,\omega)p_\omega < 0
\end{aligned}$$

For an n-period problem, the DPE is given by (1). We investigate the property of (2) since it plays a central role in the minimization in (1). We obtain the first two derivatives of (2) as follows:

$$\begin{aligned}
&\frac{\partial G_i^n(x_n,u_n)}{\partial u_n} \\
&\quad = \big[1 - F_i(u_n)\big]\Big[c_i + \int_0^\infty r_n\Big\{L_i'^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n)dA_i(z_n)\Big\}dQ_i(r_n)\Big]
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\frac{\partial^2 G_i^n(x_n,u_n)}{\partial u_n^2} \\
&\quad = \big[1 - F_i(u_n)\big] \int_0^\infty r_n^2\Big\{L_i''^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_n - z_n)dA_i(z_n)\Big\}dQ_i(r_n) \\
&\qquad - f_i(u_n)\Big[c_i + \int_0^\infty r_n\Big\{L_i'^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n)dA_i(z_n)\Big\}dQ_i(r_n)\Big]
\end{aligned} \tag{22}$$

It should be noted that $L_i'^n(\cdot)$ and $V_j'^{n-1}(\cdot)$ are increasing,

$$\begin{aligned}
&\lim_{u_n\to\infty}\Big[c_i + \int_0^\infty r_n\Big\{L_i'^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n)dA_i(z_n)\Big\}dQ_i(r_n)\Big] \\
&\quad = c_i + \nu_i\Big\{h_i + \alpha \sum_{j\in E} P(i,j)h_j + \alpha^2 \sum_{j,k\in E} P(i,j)P(j,k)h_k + \quad \cdots
\end{aligned}$$

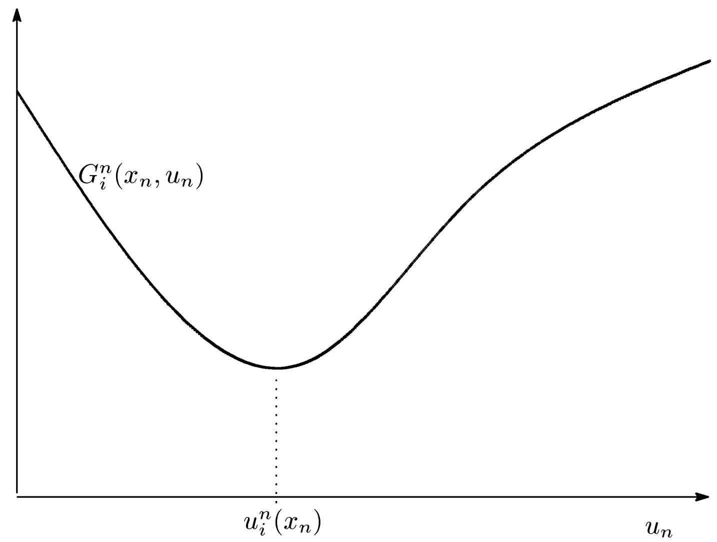

Figure 4: The form of $G_i^n(x_n, u_n)$

$$\cdots + \alpha^{n-1} \sum P(i,j)P(j,k) \cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0 \tag{23}$$

$$\begin{aligned}
&\lim_{u_n \to -\infty} \Bigg[c_i + \int_0^\infty r_n \Big\{ L_i'^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \Bigg] \\
&= c_i - \nu_i \Big\{ p_i + \alpha \sum_{j \in E} P(i,j)p_j + \alpha^2 \sum_{j,k \in E} P(i,j)P(j,k)p_k + \quad \cdots \\
&\quad \cdots + \alpha^{n-1} \sum P(i,j)P(j,k) \cdots P(\chi,\psi)P(\psi,\omega)p_\omega < 0.
\end{aligned} \tag{24}$$

For given x_n, therefore, there exists a finite and unique solution such that

$$\begin{aligned}
&c_i + \int_0^\infty r_n \Big\{ L_i'^n(x_n + r_n u_n) + \alpha \sum_{j \in E} P(i,j) \\
&\qquad \times \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n) dA_i(z_n) \Big\} dQ_i(r_n) = 0.
\end{aligned} \tag{25}$$

Let $u_i^n(x_n)$ solve (25), then $u_n = u_i^n(x_n)$ satisfies the first-order condition for minimizing (2). To satisfy the second-order conditions, notice that (22) is non-negative when

$$\begin{aligned}
&\big[1 - F_i(u_n)\big] \int_0^\infty r_n^2 \Big\{ L_i''^n(x_n + r_n u_n) \\
&\qquad + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_n - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \\
&\geq f_i(u_n) \Bigg[c_i + \int_0^\infty r_n \Big\{ L_i'^n(x_n + r_n u_n)
\end{aligned}$$

$$+\alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_n - z_n) dA_i(z_n)\Big\} dQ_i(r_n)\Big]. \tag{26}$$

Because the left hand side of (26) is always positive, the inequality is always satisfied when the right-hand side is nonpositive. Thus, if $u_n < u_i^n(x_n)$, then $\partial^2 G_i^n(x_n, u_n)/\partial u_n^2 > 0$, hence $G_i^n(x_n, u_n)$ is convex and decreasing. For values of u_n close to $u_i^n(x_n)$, still $\partial^2 G_i^n(x_n, u_n)/\partial u_n^2 > 0$, hence $G_i^n(x_n, u_n)$ is convex. But for large values of $u_n \geq u_i^n(x_n)$, $\partial^2 G_i^n(x_n, u_n)/\partial u_n^2 < 0$, hence $G_i^n(x_n, u_n)$ is concave and increasing. The behavior of $G_i^n(x_n, u_n)$ for given x_n is shown graphically in Figure 4.

From these observations, it is clear that $G_i^n(x_n, u_n)$ is quasi-convex in u_n and attains its global minimum at $u_n = u_i^n(x_n)$ for given x_n.

Now, differentiating (25) with respect to x_n, we get

$$\begin{aligned}
u_i'^n(x_n) \quad = \quad & -\int_0^\infty r_n \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) \\
& +\alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n)\Big\} dQ_i(r_n) \\
& \Big/ \int_0^\infty r_n^2 \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) \\
& +\alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n)\Big\} dQ_i(r_n) \\
& < 0.
\end{aligned} \tag{27}$$

So, when $R_n \equiv 1$, i.e., without random yield, we see from (27) that $u_i'^n(x_n) = -1$. Then, the optimal policy is of an order-up-to type. But in case with random yield, in general, $u_i'^n(x_n) \neq -1$. That is, the optimal policy is not of an order-up-to type.

Next, we investigate the properties of $G_i^n(x_n, u_i^n(x_n))$. We obtain the first two derivatives of $G_i^n(x_n, u_i^n(x_n))$ in x_n as follows:

$$\begin{aligned}
& G_i'^n(x_n, u_i^n(x_n)) \\
& \qquad = \partial G_i^n(x_n, u_i^n(x_n))/\partial x_n + \partial G_i^n(x_n, u_i^n(x_n))/\partial u_n \cdot u_i'^n(x_n) \\
& \qquad = \partial G_i^n(x_n, u_i^n(x_n))/\partial x_n \\
& \qquad = \int_0^\infty \int_0^{u_i^n(x_n)} \Big\{ L_i'^n(x_n + r_n w_n) + \alpha \sum_{j\in E} P(i,j) \\
& \qquad\quad \times \int_0^\infty V_j'^{n-1}(x_n + r_n w_n - z_n) dA_i(z_n)\Big\} dF_i(w_n) dQ_i(r_n) \\
& \qquad\quad + \big[1 - F_i(u_i^n(x_n))\big] \int_0^\infty \Big\{ L_i'^n(x_n + r_n u_i^n(x_n)) \\
& \qquad\quad +\alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n)\Big\} dQ_i(r_n)
\end{aligned} \tag{28}$$

$$
\begin{aligned}
&G_i''^n(x_n, u_i^n(x_n)) \\
&\quad = \int_0^\infty \int_0^{u_i^n(x_n)} \Big\{ L_i''^n(x_n + r_n w_n) + \alpha \sum_{j \in E} P(i,j) \\
&\quad \times \int_0^\infty V_j''^{n-1}(x_n + r_n w_n - z_n) dA_i(z_n) \Big\} dF_i(w_n) dQ_i(r_n) \\
&\quad + \big[1 - F_i(u_i^n(x_n))\big] \Bigg[\int_0^\infty \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) \\
&\quad + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \\
&\quad \times \int_0^\infty r_n^2 \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) \\
&\quad + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \\
&\quad - \Big[\int_0^\infty r_n \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) + \alpha \sum_{j \in E} P(i,j) \\
&\quad \times \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n) \Big\} dQ_i(r_n) \Big]^2 \Bigg] \\
&\quad \Big/ \int_0^\infty r_n^2 \Big\{ L_i''^n(x_n + r_n u_i^n(x_n)) \\
&\quad + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j''^{n-1}(x_n + r_n u_i^n(x_n) - z_n) dA_i(z_n) \Big\} dQ_i(r_n)
\end{aligned} \tag{29}
$$

The second multiplicand of the second term in (29) is positive by the Cauchy-Schwartz inequality. Then, we see that all the terms in (29) are positive, so $G_i''^n(x_n, u_i^n(x_n)) > 0$. That is, $G_i^n(x_n, u_i^n(x_n))$ is convex in x_n.

Now, if $u_n = 0$, (25) becomes

$$
c_i + \nu_i \Big\{ L_i'^n(x_n) + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j'^{n-1}(x_n - z_n) dA_i(z_n) \Big\} = 0. \tag{30}
$$

Let $\hat{x_n}$ solve (30). That is, if $x_n = \hat{x_n}$, then $G_i^n(x_n, u_i^n(x_n)) = G_i^n(x_n, 0)$. Otherwise, $G_i^n(x_n, u_i^n(x_n)) < G_i^n(x_n, 0)$. Here, it should be noted that the planned order quantity can not be negative. From (25) we see that $u_i^n(x_n)$ is decreasing in x_n, since $L_i'^n(\cdot)$ and $V_j'^{n-1}(\cdot)$ are increasing. So, if $x_n \geq \hat{x_n}$, then $u_i^n(x_n) \leq 0$. That is, from the quasi-convexity of $G_i^n(x_n, u_n)$ in u_n, the optimal order quantity is 0. Furthermore, for $x_n < \hat{x_n}$, there exists a unique solution such that

$$
K_i + G_i^n(x_n, u_i^n(x_n)) = G_i^n(x_n, 0) \tag{31}
$$

Let s_i^n solve (31), then it follows that

$$
K_i + G_i^n(x_n, u_i^n(x_n)) \leq G_i^n(x_n, 0) \text{ for } x_n \leq s_i^n, \tag{32}
$$
$$
K_i + G_i^n(x_n, u_i^n(x_n)) > G_i^n(x_n, 0) \text{ for } s_i^n < x_n \leq \hat{x_n}. \tag{33}
$$

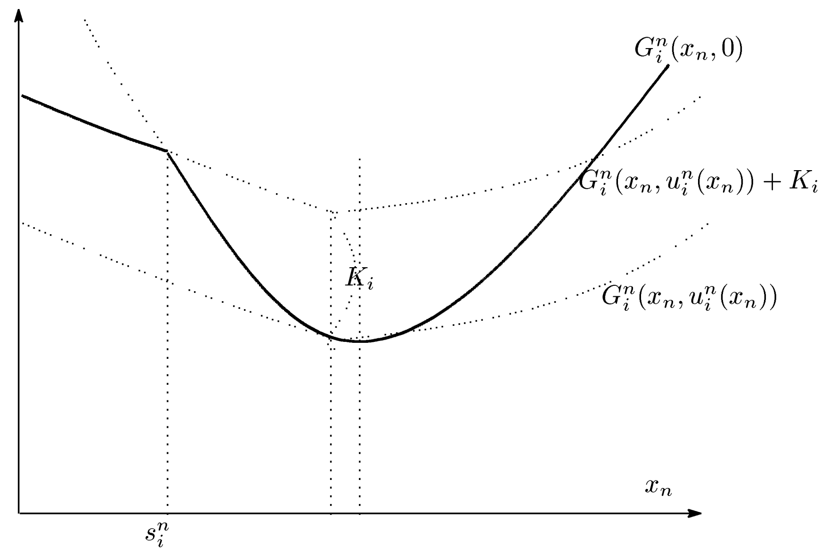

Figure 5: The form of $G_i^n(x_n,u_i^n(x_n))$, $K_i+G_i^n(x_n,u_i^n(x_n))$, and $G_i^n(x_n,0)$

The behavior of $G_i^n(x_n,u_i^n(x_n))$, $K_i+G_i^n(x_n,u_i^n(x_n))$, and $G_i^n(x_n,0)$ is shown graphically in Figure 5.

Based upon the state of the environment i and the initial inventory level x_n, the optimal policy can now be characterized in terms of the single critical number s_i^n. For $x_n \le s_i^n$, the expected savings $\{G_i^n(x_n,0)-G_i^n(x_n,u_i^n(x_n))\}$ gained by ordering $u_i^n(x_n)$ units can offset the fixed ordering cost K_i provided one plans to order. This follows from (32). On the other hand, for $s_i^n < x_n < \hat{x_n}$, it is not worthwhile to order because the fixed ordering cost K_i will offset the expected savings $\{G_i^n(x_n,0)-G_i^n(x_n,u_i^n(x_n))\}$ derived from ordering $u_i^n(x_n)$ units. This follows from (33). Since the planned order quantity <0 for $x_n \ge \hat{x_n}$, it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period n and the property of $G_i^n(x_n,u_n)$.
(1) the optimal policy for period n is given by

$$U_n^*(i,x_n)=\begin{cases} u_i^n(x_n) & \text{if } x_n \le s_i^n, \\ 0 & \text{if } x_n > s_i^n, \end{cases}$$

where critical number s_i^n is a solution to Eq.(31) and is the reorder point.

(2) For given x_n,

$$G_i^n(x_n,u_n) \text{ is quasi-convex and } \begin{cases} \text{decreasing in } u_n & \text{if } u_n \le u_i^n(x_n), \\ \text{increasing in } u_n & \text{if } u_n > u_i^n(x_n). \end{cases}$$

Therefore, the expected cost $V_i^n(x_n)$ under the optimal policy is obtained by substituting $U_n^*(i,x_n)$ into (1):

$$V_i^n(x_n) \quad = \quad \begin{cases} K_i+G_i^n(x_n,u_i^n(x_n)) & \text{if } x_n \le s_i^n, \\ G_i^n(x_n,0)=L_i^n(x_n)+\alpha\sum_{j\in E}P(i,j) & \\ \qquad \times\int_0^\infty V_j^{n-1}(x_n-z_n)dA_i(z_n) & \text{if } x_n > s_i^n. \end{cases} \tag{34}$$

And its first two derivatives are

$$V_i'^n(x_n) = \begin{cases} G_i'^n(x_n, u_i^n(x_n)) & \text{if } x_n \le s_i^n \\ L_i'^n(x_n) + \alpha \sum_{j\in E} P(i,j) & \\ \quad \times \int_0^\infty V_j'^{n-1}(x_n - z_n) dA_i(z_n) & \text{if } x_n > s_i^n \end{cases} \tag{35}$$

$$V_i''^n(x_n) = \begin{cases} G_i''^n(x_n, u_i^n(x_n)) & \text{if } x_n \le s_i^n \\ L_i''^n(x_n) + \alpha \sum_{j\in E} P(i,j) & \\ \quad \times \int_0^\infty V_j''^{n-1}(x_n - z_n) dA_i(z_n) & \text{if } x_n > s_i^n \end{cases} > 0. \tag{36}$$

So, $V_i^n(x_n)$ is convex in x_n, where,

$$\begin{aligned} \lim_{x_n\to\infty} V_j'^n(x_n) &= h_i + \alpha \sum_{j\in E} P(i,j)h_j + \alpha^2 \sum_{j,k\in E} P(i,j)P(j,k)h_k + \\ &\quad \cdots + \alpha^{n-1} \sum P(i,j)P(j,k)\cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0 \\ \lim_{x_n\to-\infty} V_j'^n(x_n) &= -p_i - \alpha \sum_{j\in E} P(i,j)p_j - \alpha^2 \sum_{j,k\in E} P(i,j)P(j,k)p_k - \\ &\quad \cdots - \alpha^{n-1} \sum P(i,j)P(j,k)\cdots P(\chi,\psi)P(\psi,\omega)p_\omega < 0 \end{aligned}$$

5 Infinite-Horizon Analysis

In this section, we consider the case where $n \to \infty$ for the model introduced in section 1. For an infinite-horizon problem, the DPE, which is equivalent to (1), can be written as

$$V_i(x) = \min_{u\ge 0}\{K_i\delta(u) + G_i(x,u)\}, \tag{37}$$

where

$$\begin{aligned} G_i(x,u) &= c_i \int_0^u w dF_i(w) + \int_0^\infty \int_0^u \Big[L_i(x+rw) \\ &\quad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j(x+rw-z)dA_i(z)\Big] dF_i(w)dQ_i(r) \\ &\quad + \big[1-F_i(u)\big]\Big\{c_i u + \int_0^\infty \Big[L_i(x+ru) \\ &\quad + \alpha \sum_{j\in E} P(i,j) \int_0^\infty V_j(x+ru-z)dA_i(z)\Big] dQ_i(r)\Big\} \end{aligned} \tag{38}$$

Our purpose in this section is to show that the DPE $V_i^n(x)$ of the finite-horizon problem converges to a limit function $V_i(x)$, which satisfies (37), (38), and that the reorder point s_i^n of the finite-horizon problem also converges to s_i where s_i specifies the optimal ordering policy for (37).

The following gives the proof.

Consider the convergence of the functional sequence $\{V_i^n(x)\}_{n=1}^\infty$ defined as

$$\begin{aligned} V_i^1(x) &= \min_{u\ge 0}\{K_i\delta(u) + c_i \int_0^u w dF_i(w) + \int_0^\infty \int_0^u L_i(x+rw)dF_i(w)dQ_i(r) \\ &\quad + \big[1-F_i(u)\big]\Big\{c_i u + \int_0^\infty L_i(x+ru)dQ_i(r)\Big\} \end{aligned}$$

$$
\begin{aligned}
V_i^n(x) &= \min_{u\geq 0}\{K_i\delta(u) + c_i\int_0^{u_n} w_n dF_i(w_n) + \int_0^\infty\int_0^{u_n}\Big[L_i^n(x_n + r_n w_n) \\
&\quad +\alpha\sum_{j\in E}P(i,j)\int_0^\infty V_j^{n-1}(x_n + r_n w_n - z_n)dA_i(z_n)\Big]dF_i(w_n)dQ_i(r_n) \\
&\quad +\big[1-F_i(u_n)\big]\Big\{c_i u_n + \int_0^\infty\Big[L_i^n(x_n + r_n u_n) \\
&\quad +\alpha\sum_{j\in E}P(i,j)\int_0^\infty V_j^{n-1}(x_n + r_n u_n - z_n)dA_i(z_n)\Big]dQ_i(r_n)\Big\} \qquad (39)
\end{aligned}
$$

Note that $V_i^1(x)$ is continuous in x from (18) and so is $V_i^n(x)$ for each n recursively from (39). And, for any x,

$$V_i^2(x) \geq V_i^1(x) \geq 0,$$

therefore, $\{V_i^n(x)\}_{n=1}^\infty$ is nondecreasing recursively from (39), i.e.,

$$0 \leq V_i^1(x) \leq V_i^2(x) \leq \cdots\cdots \leq V_i^n(x) \leq \cdots .$$

Since $\{V_i^n(x) - V_i^{n-1}(x)\} \geq 0$ is also continuous in x for sufficiently large $X > 0$, there exists a maximum value in the closed interval $-X \leq x \leq X$ from the Weierstrass M test. Let the value be

$$b_i^n = \max_{-X\leq x\leq X}\{V_i^n(x) - V_i^{n-1}(x)\},\ n = 2,3,\cdots ,$$

then, since

$$
\begin{aligned}
|V_i^{n+1}(x) - V_i^n(x)| &\leq \alpha\sum_{j\in E}P(i,j)\min_{u\geq 0}\Big\{\int_0^\infty\int_0^u\int_0^\infty |V_j^n(x + rw - z) \\
&\quad -V_j^{n-1}(x + rw - z)|dA_i(z)dF_i(w)dQ_i(r) \\
&\quad +\big[1-F_i(w)\big]\int_0^\infty\int_0^\infty |V_j^n(x + ru - z) \\
&\quad -V_j^{n-1}(x + ru - z)|dA_i(z)dQ_i(r)\Big\},
\end{aligned}
$$

$$b_i^{n+1} \quad \leq \quad \alpha\sum_{j\in E}P(i,j)b_j^n,\ n = 1,2,\cdots .$$

Hence,

$$b_i^n \leq \alpha\sum_{j_1\in E}P(i,j_1)b_{j_1}^{n-1} \leq \alpha^2\sum_{j_1,j_2\in E}P(i,j_1)P(j_1,j_2)b_{j_2}^{n-2} \leq \cdots\cdots$$

$$\Downarrow$$

$$b_i^n \leq \alpha^{n-1}\sum_{j_1,\cdots,j_{n-1}\in E}P(i,j_1)P(j_1,j_2)\cdots P(j_{n-2},j_{n-1})b_{j_{n-1}}^1, n = 2,3,\cdots .$$

Since $\sum_{n=2}^{\infty} b_i^n$ is convergent, the positive term series $\sum_{n=2}^{\infty}\{V_i^{n+1}(x) - V_i^n(x)\}$ is uniformly convergent for $x \leq |X|$.
Therefore, because

$$V_i^n(x) = V_i^1(x) + \{V_i^2(x) - V_i^1(x)\} + \cdots\cdots + \{V_i^n(x) - V_i^{n-1}(x)\},$$

the functional sequence $\{V_i^n(x)\}_{n=1}^{\infty}$ is uniformly convergent for $x \leq |X|$. Let the limitting function be $V_i(x)$, then $V_i(x)$ is continuous because of its uniform convergency, and

$$\begin{aligned} V_i(x) &= \min_{u \geq 0}\{K_i\delta(u) + c_i \int_0^u w dF_i(w) + \int_0^{\infty}\int_0^u \Big[L_i(x+rw) \\ &\quad +\alpha \sum_{j\in E} P(i,j) \int_0^{\infty} V_j(x+rw-z)dA_i(z)\Big] dF_i(w) dQ_i(r) \\ &\quad +\big[1-F_i(u)\big]\Big\{c_i u + \int_0^{\infty}\Big[L_i(x+ru) \\ &\quad +\alpha \sum_{j\in E} P(i,j) \int_0^{\infty} V_j(x+ru-z)dA_i(z)\Big] dQ_i(r)\Big\} \\ &= \min_{u\geq 0}\{K_i\delta(u) + G_i(x,u)\}, \end{aligned}$$

where

$$\begin{aligned} G_i(x,u) &= c_i \int_0^u w dF_i(w) + \int_0^{\infty}\int_0^u \Big[L_i(x+rw) \\ &\quad +\alpha \sum_{j\in E} P(i,j) \int_0^{\infty} V_j(x+rw-z)dA_i(z)\Big] dF_i(w) dQ_i(r) \\ &\quad +\big[1-F_i(u)\big]\Big\{c_i u + \int_0^{\infty}\Big[L_i(x+ru) \\ &\quad +\alpha \sum_{j\in E} P(i,j) \int_0^{\infty} V_j(x+ru-z)dA_i(z)\Big] dQ_i(r)\Big\} \end{aligned}$$

Hence,

$$\begin{aligned} V_i(x) &= \lim_{n\to\infty} V_i^n(x) \\ G_i(x,u) &= \lim_{n\to\infty} G_i^n(x,u). \end{aligned}$$

Since the functional sequence which is uniformly convergent and continuous is partial differentiable with term by term,

$$\frac{\partial G_i(x,u)}{\partial u} = \lim_{n\to\infty} \frac{\partial G_i^n(x,u)}{\partial u}$$

Hence,

$$u_i(x) = \lim_{n\to\infty} u_i^n(x),$$

and let solve $K_i + G_i(x, u_i(x)) = G_i(x, 0)$, then

$$s_i = \lim_{n\to\infty} s_i^n.$$

6 Concluding Remarks

In this paper, we simultaneously consider an internal uncertainty and an external uncertainty. First, we analyze a finite-horizon periodic-review inventory model that incorporates random environment, variable capacity, and stochastically proportional yield. We show that the objective function is quasi-convex, and the structure of the optimal policy is characterized by a single environmental-dependent critical number for the initial inventory level at each period. We further show that the solution for the finite-horizon model converges to that of the infinite-horizon model.

We discuss the limitation of our model as well as possible extensions.

Nonstationary transition probability: In this paper, the probability of environment changing is decreasing in time(periods), since we assume that the environment is Markov chain. But, there is a case that it is increasing in time, like the obsolescence of products. To solve the contradiction, we must introduce a nonstastionary transition probability.

Another policy: In our model, we consider the optimal policy that let the order-up-to point be the decision variable. So, we next time consider the optimal policy that let the order quantity, the reorder-point, and the order intervals be the decision variable. Thereby, we decide the real optimal policy by comparing them.

Endogenous factors: In this paper, we analyze the inventory model that depends on the exogenous factors. So, we present the inventory model that depends on the endogenous factors where the demand be influenced by the order quantity for example.

Another uncertain element: We will introduce to our model the uncertain leadtime, since these are depend on the environment.

Production and distribution: Recently, like KANBAN of TOYOTA, zero-inventory policy becomes main topic. So, hereafter, we focus on the inventory management to minimize the cost in the system combined with production or distribution.

References

[1] K. J. Arrow, S. Karlin, H. Scarf, "Studies in the Mathematical Theory of Inventory and Production," Stanford University Press, Stanford, CA, 1958.

[2] F. W. Ciarallo, R. Akella, T. M. Morton, "A Periodic Review Production Planning Model with Uncertain Capacity and Uncertain Demand-Optimality of Extended Myopic Policies," *Management Science*, Vol. 40, No. 3, March 1994, 320-332.

[3] E. Çinlar, S. Özekici, "Reliability of complex devices in random environment," *Probability in the Engineering and Informational Science*, 1 (1987), 97-115.

[4] I. David, E. Greenshtein, A. Mehrez, "A dynamic-programming approach to continuous-review obsolescent inventory problems," *Naval Reserch Logistics*, Vol. 44 (1997), 757-774.

[5] I. David, A. Mehrez, "An inventory model with exogenous failures," *Operations Research*, 43 (1995), 902-903.

[6] M. Eisen, M. Tainiter, "Stochastic variations in queueing processes," *Operations Research*, 11 (1963), 922-927.

[7] R. Feldman, "A continuous review (s,S) inventory system in a random environment," *Journal of Applied Probability*, 15 (1978), 654-659.

[8] M. Henig, Y. Gerchak, "The Structure of Periodic Review Policies in the Presence of Random Yield," *Operations Research*, 38 (1990), 634-643.

[9] J. Hwang, M. R. Singh, "Optimal production policies for multi-stage systems with setup costs and uncertain capacities," *Management Science*, Vol. 44, No. 9, September 1998, 1279-1294.

[10] P. Joglekar, P. Lee, "An exact formulation of inventory costs and optimal lot-size in face of sudden obsolescence," *Operations Research Letters*, 14 (1993), 283-290.

[11] B. Kalymon, "Stochastic prices in a single item inventory purchasing model," *Operations Research*, 19 (1971), 1434-1458.

[12] M. F. Neuts, "The M/M/1 queue with randomly varying arrival and service rates," *Opsearch*, 15 (1978), 139-157.

[13] S. Özekici, "Optimal maintenance policies in random environment," *European Journal of Operations Rsearch*, 82 (1995) 283-294.

[14] S. Özekici, M. Parlar, "Inventory models with unreliable suppliers in a random environment," *Annals of Operations Research*, 91 (1999), 123-136.

[15] N. U. Prabhu, Y. Zhu, "Markov-modulated queueing systems," *Queueing Systems*, 5 (1989), 215-246.

[16] S. P. Sethi, F. Cheng, "Optimality of (s,S) Policies in inventory models with Markovian demand," *Operations Research*, 45 (1997) 931-939.

[17] J. S. Song, P. Zipkin, "Inventory control in fluctuating demand environment," *Operations Research*, 41 (1993), 351-370.

[18] S. Sugiyama, "Dynamic programming," NIKKAGIREN (1976).

[19] Y. Wang, Y. Gerchak, "periodic review production models with variable capacity, random yield, and uncertain demand," *Management Science*, Vol. 42, No. 1, January 1996, 130-137.

[20] C. A. Yano, H. L. Lee, "Lot Sizing with Random Yield: A Review," *Operations Research*, 43 (1995), 311-334.