# Discussion Papers In Economics And Business

Heterogeneous Yield Curves and Basis Swaps

Keiichi Tanaka

Discussion Paper 03-12

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

# Heterogeneous Yield Curves and Basis Swaps

Keiichi Tanaka

Discussion Paper 03-12

June 2003

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

# Heterogeneous Yield Curves and Basis Swaps *

Keiichi Tanaka †

Graduate School of Economics, Osaka University

June 3, 2003

**Abstract**

We present a framework of heterogeneous yield curves of agents based on the pricing kernel approach in order to model LIBOR and basis swap rates. Each yield curve may imply different prices of assets but is consistent with swap rates, basis swap rates and foreign exchange rates. We show three conditions that gurantee the no-arbitrage and the consistency with these rate processes. The introduction of contributors and the Market Representative Agent ("MRA") leads to an explanation of a non-zero basis swap rate as a swap rate priced by one of the MRAs.

*Keywords*: Interest rate swap, Basis swap, Pricing kernel, Risk premium, LIBOR

*JEL classification*: G1; D4; C6

*We are grateful to Kazuhiko Nishina, Yasuhiko Tanigawa, Masamitsu Ohnishi, Shinsuke Ikeda, Jun Sekine, Toshiki Yotsuzuka, Makoto Saito, and seminar participants at Osaka University and the 2002 Japan Finance Association meeting for their thoughtful comments. All errors are solely our responsibility.

†Correspondence: 4-3-30-401 Higashigotanda Shinagawa-ku Tokyo 141-0022 Japan, E-mail : tanakak@uranus.dti.ne.jp

# 1 Introduction

In typical interest rate swap transactions, the floating rates such as LIBOR (London InterBank Offered Rate) are averaged rates quoted by multi agents who may be heterogeneous in their available funding rates. The heterogeneity has not been considered in the literature since LIBOR has been treated as a single rate when modelling swap rates. As a result any theoretical explanation of non-zero basis swap rates between two currencies has not been given. Since many practitioners notice that the basis swap rate indicates the relative demand of involved currencies and some credit quality difference in the actual markets, we have the motivation to model the basis swap rates by heterogeneous yield curves and we are the first to do so, to our knowledge.

Pricing kernels are useful in evaluating the future cash flow and they are characterised by the instantaneous short rate and the risk premium (market price of risk). Therefore, heterogeneous yield curves can be derived by assuming different short rates and risk premia between agents though they may not be guranteed to be consistent with the products traded with other agents, such as interest rate swaps.

In this paper we explore how heterogeneous yield curves are built consistently in the arbitrage-free setting so that LIBOR is modeled and nonzero basis swap rates are derived. We study an economy where several "rate products" (interest rate swaps, basis swaps and foreign exchanges) are traded between agents who may be heterogeneous in the short rate and the risk premium. Then the agents build their own yield curve by the pricing kernel and we show three conditions under which there are no arbitrage within each agent's "market" and each yield curve is consistent with the tradable products. Also by regarding LIBOR as the short rate of a fictious agent "Market Representative Agent"(MRA), we show that the basis swap rate is a fixed rate in a swap transaction, priced by the pricing kernel of one of the two MRAs.

One of our basic ideas is that a swap transaction can be identified with the exchange of two different kinds of bonds whose present values are equivalent but the value is not necessarily a par. The difference from a par must represent some heterogeniety. Another critical point is our assumption that each agent can trade only the rate products with other agents externally and trade internally the money market account, which is a "price product". This assumption implies that there are no secondary markets for swap transactions but rules out the arbitrage opportunities with other agents.

The relationship between a swap rate and prices of zero coupon bonds are well-known. Swap rates and LIBOR are modeled in the class of market models which are explored by Jamshidian (1997) and Brace et al. (1997). Under these models, each swap transaction involve a finite number of the cash flow. We study ideal swaps whose cash flows take place continuously thus we have to treat a continum of assets, a family of price processes parametrised by the maturity. Such trading strategies are studied by Björk et al. (1997). In our discussion a formula of the SDE of a process which is defined by two families of processes is established. The formula is a generalisation of a well-known lemma cited in Heath et al. (1992) and Musiela and Rutkowski (1997).

Results of our work are close to ones in studies of spreads between swap rates and bond yields, for example Grinblatt (2001) and He (2001) in default-free settings, and Collin-Dufresne and Solnik (2001) in a defaultable setting. Risk neutral measures in these papers are constructed by considering one rate as risk-free and another rate as risky. Our methodology is similar in that an agent considers their own short rate as risk-free but LIBOR risky.

The remainder of the paper is organised as follows. Section 2 defines the market of each agent and presents the basic setup of the model. Section 3 argues the arbitrage-free and consistent conditions for the market. In Section 4 we give a specification which will be useful in intuitive understanding. Section 5 summarizes these results and makes concluding remarks.

# 2 Setup of the Model

## 2.1 Notations

We consider our model in a finite time horizon $[0, T^*]$ ($T^* < \infty$) on a probability space ($\Omega$, $\mathcal{F}$, $\mathbb{P}$) where $\mathbb{P}$ is some objective probability measure. The filtration $\{\mathcal{F}^W_t\}$ is the augmented filtration generated by an $n$-dimensional Brownian motion $W(t)$, $\mathcal{F}^W_t = \sigma(W(u) : 0 \le u \le t)$. All processes are bounded and continuous semimartingale with respect to the filtration $\{\mathcal{F}^W_t\}$. All processes containing a maturity parameter $T$ are assumed to be smooth enough so that they are differentiable and integrable with respect to $T$ and Fubini's theorem holds.

All agents $i \in \mathcal{I} = \{1, 2, \dots, I\}$ are supposed to be financial institutions engaged in trading swaps, deposits and foreign exchange rates subject to market risk of interest rates and foreign exchange rates. The set of currencies to be considered is denoted by $\mathcal{J} = \{1, 2, \dots, q\}$. Each agent $i$ is endowed with and characterised by the set of pairs of $\{\mathcal{F}^W_t\}$-adapted processes $\{(r^i_j, \lambda^i_j) : j \in \mathcal{J}\}$ where $r^i_j(t) \in \mathbb{R}$ is the instantaneous short rate applicable to the agent $i$ at time $t$ in currency $j$, and $\lambda^i_j(t) \in \mathbb{R}^n$ represents the risk premium of the agent $i$ in currency $j$ and satisfies $\mathbb{E}\Big[\mathcal{E}_{T^*}\Big(-\int_0^\cdot \lambda^i(s)\, dW(s)\Big)\Big] = 1$.

For each currency $j \in \mathcal{J} = \{1, 2, \dots, q\}$ there is a set of contributors [1]

[1] In practice the members of the set of contributors are selected among agents by an association and may be changed from time to time to sustain a certain credit quality in average: $\mathcal{L}_{jt} = \{i(1)_{jt}, i(2)_{jt}, \dots, i(L_j)_{jt}\}$, and the arithmetic average is taken after removing the highest $L_j/4$ quotations and the lowest $L_j/4$ quotations. However we assume that the set of contributors are fixed through our time-horizon: $\mathcal{L}_{jt} = \mathcal{L}_j$, and that the reference rates are calculated as the arithmetic average of quotations across all contributors to simplify our analysis.

given by $\mathcal{L}_j = \{i(1)_j, i(2)_j, \ldots, i(L_j)_j\} \subset \mathcal{I} \quad (\sharp\{\mathcal{L}_j\} = L_j)$. We set

$$r_k^{M_j} \equiv \frac{1}{L_j} \sum_{i \in \mathcal{L}_j} r_k^i, \quad \lambda_k^{M_j} \equiv \frac{1}{L_j} \sum_{i \in \mathcal{L}_j} \lambda_k^i. \tag{2.1}$$

The reference rate $r_j^{M_j}(t)$ of currency $j$ at time $t$ is an arithmetic average of the instantaneous short rates in currency $j$ at time $t$ across the contributors. With the definition (2.1) we can treat $M_j$ as a fictitious agent and call it the Market Representative Agent ("MRA") of currency $j$. The extended set of agents is denoted by $\widetilde{\mathcal{I}} = \mathcal{I} \cup \{M_1, M_2, \cdots, M_q\}$.

The funding spread process $\varphi_j^i$ of an agent $i$ is the difference of the instantaneous short rate with MRA in currency $j$ definied by $\varphi_j^i(t) \equiv r_j^{M_j}(t) - r_j^i(t)$ with a condition $\sum_{i \in \mathcal{L}_j} \varphi_j^i(t) = 0$.

$Q_{jk}(t)$ denotes the spot foreign exchange rate at time $t$ of a currency pair $(j, k)$ which is quoted as units of currency $k$ per unit of currency $j$.

$S_j(t, T)$ is the fixed rate (the interest swap rate) to be exchanged against the floating rate $r_j^{M_j}(s)$ $(t \leq s \leq T)$, on a contract entered into at time $t$ with maturity $T$ in currency $j$. We assume that all agents are allowed to trade a continum of interest rate swaps at time $t$ of maturities between $u$ and $T$ $(t \leq u \leq T)$ for all currencies.

$U_{jk}(t, T)$ denotes the basis swap rate of a contract which is entered into at time $t$ and where the cash flow $r_j^{M_j}(s) + U_{jk}(t, T)$ in currency $j$ on the notional amount of 1 and $r_k^{M_k}(s)$ in currency $k$ on the notional amount of $Q_{jk}(t)$ are exchanged at time $s$ $(t \leq s \leq T)$ in addition to the principal exchange. In both the interest swap transactions and the basis swap transactions the coupon payments take place continuously.

These rate processes $Q_{jk}, S_j(\cdot, T), U_{jk}(\cdot, T)$ are assumed to be exogeneously given as follows. First, we assume that swap rates are strictly positive and satisfy

$$dS_j(t, T) = \alpha_j^S(t, T)\, dt + \sigma_j^S(t, T) dW(t), \tag{2.2}$$

where $\alpha_j^S(\cdot, T)$ and $\sigma_j^S(\cdot, T)$ are $\{\mathcal{F}_t^W\}$-adapted processes with sutable dimensions. Secondly, the basis swap rates and the foreign exchange rates are assumed to follow

$$\begin{aligned} dU_{jk}(t, T) &= \alpha_{jk}^U(t, T)\, dt + \sigma_{jk}^U(t, T) dW(t), && (2.3) \\ \frac{dQ_{jk}(t)}{Q_{jk}(t)} &= \alpha_{jk}^Q(t, T)\, dt + \sigma_{jk}^Q(t, T) dW(t), && (2.4) \end{aligned}$$

where $\alpha_{jk}^U(\cdot, T)$, $\sigma_{jk}^U(\cdot, T)$, $\alpha_{jk}^Q(\cdot, T)$ and $\sigma_{jk}^Q(\cdot, T)$ are $\{\mathcal{F}_t^W\}$-adapted processes with sutable dimensions.

## 2.2 Market

In this subsection for each agent $i \in \widetilde{\mathcal{I}}$ we define the filtration $\{\mathcal{G}_t^i\}$ and the market $\mathcal{M}^i$ of the agent $i$.

The filtration $\{\mathcal{G}_t^i\}$ of the agent $i$ is the augmented filtration defined as

$$\begin{aligned}\mathcal{G}_t^i &\equiv \sigma(S_j(u,T), U_{jk}(u,T), Q_{jk}(u), r_j^i(u), \lambda_j^i(u), \lambda_j^{M_j}(u) \\ &\quad : 0 \le u \le t \le T \le T^*, j,k \in \mathcal{J}).\end{aligned}$$

Note that the reference rate $r_j^{M_j}$ and the funding spread $\varphi_j^i$ are $\{\mathcal{G}_t^i\}$-adapted since $r_j^{M_j}(t) = S_j(t,t)$ and $\varphi_j^i(t) = r_j^{M_j}(t) - r_j^i(t)$.

Write a $\{\mathcal{G}_t^i\}$-martingale $W^i(t) \equiv \mathbb{E}\Big[W(t) \mid \mathcal{G}_t^i\Big]$ which is the projection of the Brownian motion $W(t)$ into the $\sigma$-field $\mathcal{G}_t^i$. Then the above three rate processes are rewritten in the form of

$$dS_j(t,T) = \alpha_j^{i,S}(t,T)\,dt + \sigma_j^{i,S}(t,T)dW^i(t),$$

etc. where the coefficients are $\{\mathcal{G}_t^i\}$-adapted processes with sutable dimensions.

To make the discussion simple, we assume that $W^i = W$, that is, the filtration $\{\mathcal{G}_t^i\}$ coincides with the Brownian filtration $\{\mathcal{F}_t^W\}$, [2] $\{\mathcal{G}_t^i\} = \{\mathcal{F}_t^W\}$. Then above coefficients coincide with the original ones ; $\alpha_j^{i,S}(t,T) = \alpha_j^S(t,T)$, $\sigma_j^{i,S}(t,T) = \sigma_j^S(t,T)$ etc.

So far we have not had any price processes. Price processes of contingent claims for the agent $i$ are defined as follows. Let the money market account $C_j^i(t) = \exp\bigl(\int_0^t r_j^i(s)\,ds\bigr)$ of currency $j$. The pricing kernel $Z_j^i$ is defined as

$$dZ_j^i(t) = Z_j^i(t)\Big[-r_j^i(t)\,dt - \lambda_j^i(t)dW(t)\Big] \tag{2.5}$$

with $Z_j^i(0) = 1$.

We call $B_j^i(t,T)$ the discount factor, corresponding to the zero coupon bond, and $\Gamma_j^i(t,T)$ the price of the floating rate note (FRN) with coupon of $r_j^{M_j}(s)$ $(t \le s \le T)$ in currency $j$, defined by

$$B_j^i(t,T) \equiv Z_j^i(t)^{-1}\mathbb{E}\big[Z_j^i(T) \mid \mathcal{G}_t^i\big], \tag{2.6}$$

$$\Gamma_j^i(t,T) \equiv Z_j^i(t)^{-1}\mathbb{E}\Big[Z_j^i(T) + \int_t^T Z_j^i(s) r_j^{M_j}(s)\,ds \mid \mathcal{G}_t^i\Big]. \tag{2.7}$$

Note that if a $T$-maturity contingent claim $X$ in currency $j$ pays the continuous dividened $h^X$ and the price process $X$ satisfies $dX(t) = \alpha^X(t)\,dt + \sigma^X(t)\,dW(t)$, then it holds that

$$\alpha^X(t) + h^X(t) - \lambda_j^i(t)\sigma^X(t) = r_j^i(t)X(t) \tag{2.8}$$

[2] A simple example is $I = 2, q = 1, \lambda^1(t) = \lambda^2(t) = 0, S(t,T) = \exp\Big(W^{(1)}(t)\Big)$, $\varphi^1(t) = W^{(2)}(t), \varphi^2(t) = -W^{(2)}(t)$, where $W(t) = (W^{(1)}(t), W^{(2)}(t))$ is a two dimensional Brownian motion.

which can be derived by the definition of pricing kernels. We will often see this fundamental consequence of arbitrage-free later.

The market $\mathcal{M}^i$ of the agent $i$ is a set of $\{\mathcal{G}_t^i\}$-adapted price processes of these "bonds" and the money market account $C^i$

$$\mathcal{M}^i = \Big\{B_j^i(\cdot, T), \Gamma_j^i(\cdot, T), C_j^i : 0 \leq T \leq T^*, j \in \mathcal{J}\Big\}. \tag{2.9}$$

For each market $\mathcal{M}^i$ we can define the self-financing strategies, the arbitrage startegies. Note that a continuum of price processes can be regarded as an admissible strategy.[3] Our purpose is to provide conditions of the risk premia and the short rates under which each agent's market $\mathcal{M}^i$ is arbitrage-free and consistent with given market rates $S_j(t, T), U_{jk}(t, T), Q_{jk}(t)$.

It is worth of noting that each agent $i \in \mathcal{I}$ can trade only "rates products", not "price products" of which the price represents a kind of the present value of future cash flows, with other agents. The price products include a deposit transaction and a termination of an existing transaction. Moreover all agents know the prices of the market $\mathcal{M}^{M_j}$ of MRAs but such market does not exist physically. Therefore we do not discuss the "arbitrage opportunity" between different markets.

# 3 Arbitrage-free and Consistent Market

We derive the conditions under which each agent's market $\mathcal{M}^i$ is arbitrage-free and consistent with given market rates $S_j(t, T), U_{jk}(t, T), Q_{jk}(t)$.

## 3.1 Valuation of Floating Leg on Interest Rate Swaps

We study consistent discount factors with the interest rate swap rates of a specific currency in this and next subsection. Although we omit the subscript to show the currency in the processes in these subsections, our discussion is valid for all currencies.

We start with a study of the valuation of the floating leg of an interest rate swap in this subsection. Let us denote by $\gamma^i(t, T)$ the price of the funding spread $\varphi^i(t) = r^M(t) - r^i(t)$ with the pricing kernel

$$\gamma^i(t, T) \equiv Z^i(t)^{-1} \mathbb{E}\big[Z^i(T)\varphi^i(T) \mid \mathcal{G}_t^i\big]. \tag{3.1}$$

Since $\gamma^i(t, T)Z^i(t)$ is a $\{\mathcal{G}_t^i\}$-martingale, by the martingale representation theorem there exists a $\{\mathcal{G}_t^i\}$-predictable process $\phi^i(\cdot, T)$ by which the dynamics of $\gamma^i(t, T)$ can be written in the form of

$$d\gamma^i(t, T) = r^i(t)\gamma^i(t, T)\, dt + \phi^i(t, T)\, \big[dW(t) + \lambda^i(t)\, dt\big].$$

[3] See Björk, Kabanov, and Runggaldier (1997).

Now we consider a FRN which has a floating coupon of $r^M$ and is redeemed at the face value at maturity $T$. We denote by $\Gamma^i(t,T)$ the price process of the FRN. Then by definition

$$\begin{aligned}\Gamma^i(t,T) &= Z^i(t)^{-1}\mathbb{E}\big[Z^i(T) + \int_t^T Z^i(s) r^M(s)\,ds \mid \mathcal{G}_t^i\big] \\ &= 1 + \int_t^T \gamma^i(t,u)\,du,\end{aligned} \tag{3.2}$$

where we used the Fubini's theorem and the definition of the pricing kernel.

For a process $g(t,T)$ we define a process $g^*(\cdot,T)$ as $g^*(t,T) = \int_t^T g(t,u)\,du$. Next lemma holds by using the Fubini's theorem.

**Lemma 3.1.** *If $V(t,T)$ follows $dV(t,T) = \alpha^V(t,T)\,dt + \sigma^V(t,T)\,dW(t)$, then $V^*(t,T) = \int_t^T V(t,u)\,du$ satisfies*

$$dV^*(t,T) = \big[\alpha^{V*}(t,T) - V(t,t)\big]\,dt + \sigma^{V*}(t,T)\,dW(t).$$

*Proof.* See Heath et al. (1992) and Musiela and Rutkowski (1997). □

By Lemma 3.1 the dynamics of $\Gamma^i(\cdot,T)$ is given by

$$d\Gamma^i(t,T) = \alpha^{\Gamma^i}(t,T)\,dt + \sigma^{\Gamma^i}(t,T)\,dW(t) \tag{3.3}$$

where

$$\begin{aligned}\alpha^{\Gamma^i}(t,T) &= r^i(t)\Gamma^i(t,T) - r^M(t) + \lambda^i(t)\sigma^{\Gamma^i}(t,T), \\ \sigma^{\Gamma^i}(t,T) &= \int_t^T \phi^i(t,u)\,du.\end{aligned}$$

## 3.2 Discount Factors Consistent with Swap Rates

The discount factor and FRN will be arbitrage-free by construction with the pricing kernel but are not guranteed to be consistent with swap rates of the currency. In this subsection, we study the process of the discount factors which satisfy the consistency with swap rates. An interest swap is equivalent to an exchange of a fixed coupon bond and a floating coupon bond *with the same price* which is not necessary a par. The consistency means the equation

$$B^i(t,T) + S(t,T)\int_t^T B^i(t,u)\,du = \Gamma^i(t,T) \tag{3.4}$$

holds for all $t \le T$ with given $\Gamma^i(t,T)$. We will derive the SDEs of the discount factors and then show the condition on $r^i, \lambda^i$ under which the discount factors are arbitrage-free with FRNs and the money market account.

Equation (3.4) can be regarded as an integral equation with respect to $B^i(t,T)$ as a function of $T$ with given families of swap rates and $\Gamma^i(t,T)$ for each $t$. It is easy to have the solution with a condition $B^i(T,T)=1$ as

$$B^i(t,T) = \Gamma^i(t,T) - S(t,T)\int_t^T \Gamma^i(t,u)\exp\Big(-\int_u^T S(t,v)\,dv\Big)\,du. \tag{3.5}$$

Then the forward rates are given by

$$f^i(t,T) = S(t,T) + \frac{A^i(t,T)}{B^i(t,T)}\frac{\partial S(t,T)}{\partial T} - \frac{\gamma^i(t,T)}{B^i(t,T)} \tag{3.6}$$

where

$$A^i(t,T) \equiv \int_t^T B^i(t,u)\,du = \int_t^T \Gamma^i(t,u)\exp\Big(-\int_u^T S(t,v)\,dv\Big)\,du \tag{3.7}$$

is the price of annuity whose coupons of 1 are paid continuously until time $T$. Thus we have $r^i(t) = r^M(t) - \varphi^i(t)$ as expected.

It is worth of noting that equation (3.5) represents a replicating strategy of $B^i(t,T)$ with swap rates and floating coupon bonds. A long position of $B^i(t,T)$ can be replicated with

- Receive swap $S(t,T)$ with notional amount 1
- Buy a FRN at $\Gamma^i(t,T)$ for notional amount 1
- Pay swap $S(t,u)$ with notional amount $S(t,T)\exp\Big(-\int_u^T S(t,v)\,dv\Big)\,du$ for all $u\in[t,T]$
- Sell FRNs at $\Gamma^i(t,T)$ for notional amount $S(t,T)\exp\Big(-\int_u^T S(t,v)\,dv\Big)\,du$ for all $u\in[t,T]$.

Now we move on to have the dynamics of $B^i(t,T)$ and $A^i(t,T)$. Ito's formula cannot be directly applied to equations (3.5) and (3.7) each of which contains a continum of processes. To find the dynamics it is useful to define the operators $\mathcal{R}$ and $\mathcal{A}$ as follows. Given two families of processes $\{X(\cdot,T)\}_T$ and $\{Y(\cdot,T)\}_T$ parametrised by $T$, we define the two processes $\mathcal{R}\{X,Y\}(\cdot,T)$ and $\mathcal{A}\{X,Y\}(\cdot,T)$ as

$$\mathcal{R}\{X,Y\}(t,T) \equiv X(t,T) - Y(t,T)\int_t^T X(t,u)\exp\Big(-\int_u^T Y(t,v)\,dv\Big)\,du, \tag{3.8}$$

$$\mathcal{A}\{X,Y\}(t,T) \equiv \int_t^T X(t,u)\exp\Big(-\int_u^T Y(t,v)\,dv\Big)\,du. \tag{3.9}$$

Then $\mathcal{R}\{X,Y\}(t,T)$ and $\mathcal{A}\{X,Y\}(t,T)$ satisfy the following equations

$$\mathcal{R}\{X,Y\}(t,T) + Y(t,T)\mathcal{A}\{X,Y\}(t,T) = X(t,T),$$
$$\mathcal{R}\{X,Y\}(t,T) = \frac{\partial}{\partial T}\mathcal{A}\{X,Y\}(t,T).$$

Therefore it is obvious that $B^i(t,T) = \mathcal{R}\{\Gamma^i,S\}(t,T)$ and $A^i(t,T) = \mathcal{A}\{\Gamma^i,S\}(t,T)$. By using integration by parts we have the next Lemma that generalises a result of Lemma 3.1; $V^*(t,T) = \mathcal{A}\{V,0\}(t,T)$.

**Lemma 3.2.** *If $X(\cdot,T)$ and $Y(\cdot,T)$ satisfies SDEs*

$$\begin{aligned} dX(t,T) &= \alpha^X(t,T)\,dt + \sigma^X(t,T)\,dW(t), \\ dY(t,T) &= \alpha^Y(t,T)\,dt + \sigma^Y(t,T)\,dW(t), \end{aligned}$$

*then it holds that*

$$\begin{aligned} d\mathcal{R}\{X,Y\}(t,T) &= \alpha^R(t,T)\,dt + \sigma^R(t,T)\,dW(t), && (3.10) \\ d\mathcal{A}\{X,Y\}(t,T) &= \alpha^A(t,T)\,dt + \sigma^A(t,T)\,dW(t), && (3.11) \end{aligned}$$

*where*

$$\begin{aligned} \alpha^R(t,T) &= \mathcal{R}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \sigma^A\sigma^Y, Y\}(t,T) + X(t,t)\mathcal{R}\{Y,Y\}(t,T), \\ \alpha^A(t,T) &= \mathcal{A}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \sigma^A\sigma^Y, Y\}(t,T) + X(t,t)(\mathcal{A}\{Y,Y\}(t,T) - 1), \\ \sigma^R(t,T) &= \mathcal{R}\{\sigma^X - \mathcal{A}\{X,Y\}\sigma^Y, Y\}(t,T), \\ \sigma^A(t,T) &= \mathcal{A}\{\sigma^X - \mathcal{A}\{X,Y\}\sigma^Y, Y\}(t,T). \end{aligned}$$

*Proof.* See Appendix A. □

By Lemma 3.1 and $\Gamma^i(t,t) = 1$ we have the SDEs

$$\begin{aligned} dB^i(t,T) &= \mathcal{R}\{\alpha^{\Gamma^i} - A^i\alpha^S - \sigma^{A^i}\sigma^S + S, S\}(t,T)\,dt + \sigma^{B^i}(t,T)\,dW(t), \\ dA^i(t,T) &= \Big[\mathcal{A}\{\alpha^{\Gamma^i} - A^i\alpha^S - \sigma^{A^i}\sigma^S + S, S\}(t,T) - 1\Big]\,dt + \sigma^{A^i}(t,T)\,dW(t), \end{aligned}$$

where $\sigma^{B^i}(t,T) = \mathcal{R}\{\sigma^{\Gamma^i} - A^i\sigma^S, S\}(t,T)$, $\quad \sigma^{A^i}(t,T) = \mathcal{A}\{\sigma^{\Gamma^i} - A^i\sigma^S, S\}(t,T)$. We know that it holds that

$$\alpha^{\Gamma^i}(t,T) - \lambda^i(t)\sigma^{\Gamma^i}(t,T) = r^i(t)\Gamma^i(t,T) - r^M(t)$$

by equation (3.3). Thus if the condition

$$(\mathcal{S}) \quad A^i(t,T)\big[\alpha^S(t,T) - \lambda^i(t)\sigma^S(t,T)\big] + \sigma^{A^i}(t,T)\sigma^S(t,T) = S(t,T) - r^M(t)$$

is satisfied, we have

$$\big[\alpha^{\Gamma^i} - A^i\alpha^S - \sigma^{A^i}\sigma^S + S\big](t,T) = r^i(t)\Gamma^i(t,T) + \lambda^i(t)\big[\sigma^{\Gamma^i} - A^i\sigma^S\big](t,T).$$

Then, by taking $\mathcal{R}\{\cdot, S\}$ and $\mathcal{A}\{\cdot, S\}$ on the above equation, the dynamics can be reduced to

$$\begin{aligned} dB^i(t,T) &= r^i(t)B^i(t,T)\,dt + \sigma^{B^i}(t,T)\,\big[dW(t) + \lambda^i(t)\,dt\big], && (3.12) \\ dA^i(t,T) &= \Big[r^i(t)A^i(t,T) - 1\Big]\,dt + \sigma^{A^i}(t,T)\,\big[dW(t) + \lambda^i(t)\,dt\big], && (3.13) \end{aligned}$$

which implies that discounted wealth processes $B^i(\cdot,T)/C^i$ and $(\Gamma^i(\cdot,T) + \int_0^\cdot r^M(s)\,ds)/C^i_j$ follow martingales under the spot martingale measure $\mathbb{P}^i_j$ defined by $\dfrac{d\mathbb{P}^i_j}{d\mathbb{P}} = \exp\Big(-\displaystyle\int_0^\cdot \lambda^i_j(s)\,ds\Big)$, where the right hand side follows $\mathbb{P}$-martingale by assumption. Therefore, we have the following Proposition.

**Proposition 3.1.** *If the market premium $\lambda^i$ satisfies the condition $(\mathcal{S})$ for all $t \le T$, then there are no arbitrage between $\{B^i(t,T) : t \le T \le T^*\}$, $\{\Gamma^i(t,T) : t \le T \le T^*\}$ and $C^i(t)$, and they are consistent with swap rates $S(t,T)$.*

Swap rates can be recovered by the processes constucted above as follows

$$S(t,T) = \mathcal{A}\Big\{\frac{B^i f^i - \gamma^i}{A^i}, \frac{B^i}{A^i}\Big\}(t,T), \quad t < T. \tag{3.14}$$

When the condition $(\mathcal{S})$ is satisfied, the dynamics follows

$$\begin{aligned} dS(t,T) &= A^i(t,T)^{-1}\Big[S(t,T) - r^M(t) - \sigma^{A^i}(t,T)\sigma^S(t,T)\Big]\,dt \\ &\quad + \sigma^S(t,T)\,\big[dW(t) + \lambda^i(t)\,dt\big]. \end{aligned}$$

When the market $\mathcal{M}^M$ satisfies the condition $(\mathcal{S})$, the equivalent condition to the $(\mathcal{S})$ for $\mathcal{M}^i$ is

$$\begin{aligned} \Big(\lambda^M(t) - \lambda^i(t)\Big)\sigma^S(t,T) &= \Big(\frac{\sigma^{A^M}(t,T)}{A^M(t,T)} - \frac{\sigma^{A^i(t,T)}}{A^i(t,T)}\Big)\sigma^S(t,T) \\ &\quad - \Big(\frac{1}{A^M(t,T)} - \frac{1}{A^i(t,T)}\Big)\Big(S(t,T) - r^M(t)\Big). \end{aligned}$$

## 3.3 Foreign Exchange Rates

The consistency with foreign exchange rates is equivalent to a change of numéraire[4] between the currencies. The consistency requires no arbitrage between assets in a currency and synthetic assets which are originally denominated in another currency and converted to the currency in question by the foreign exchange rate. In terms of pricing kernels it is equivalent to $Z^i_k(t) = Z^i_j(t)Q_{jk}(t)/Q_{jk}(0)$.

The following condition $(\mathcal{Q})$ is well-known and the proof of the Proposition is quite similar to one in Musiela and Ruskowski (1997).

**Proposition 3.2.** *Suppose that the condition $(\mathcal{S})$ holds for currency $j$ and $k$. If the condition*

$$\begin{aligned} (\mathcal{Q}) \qquad \lambda^i_k(t) &= \lambda^i_j(t) + \sigma^Q_{jk}(t), \\ r^i_k(t) &= r^i_j(t) + \alpha^Q_{jk}(t) - \sigma^Q_{jk}(t)\lambda^i_k(t) \end{aligned}$$

*holds, then there are no arbitrage between $\{B^i_k(t,T) : t \le T \le T^*\}$, $\{Q_{jk}(t)B^i_j(t,T) : t \le T \le T^*\}$, $\{Q_{jk}(t)\Gamma^i_j(t,T) : t \le T \le T^*\}$ and $Q_{jk}(t)C^i_j(t)$, and they are consistent with the foreign exchange rates $Q_{jk}(t)$.*

[4]The change of numéraire is discussed, for examaple, in El Karoui et al. (1995) and Rogers (1997).

The condition $(\mathcal{Q})$ implies $Z^i_k(t) = Z^i_j(t)Q_{jk}(t)/Q_{jk}(0)$. Therefore, the pricing kernel (equivalently, the short rate and the risk premium) of currency $k$ is identified by the pricing kernel of currency $j$ and the foreign exchange rate. Another implication is the relationship between the return and the risk

$$\Big\{\mathbb{E}_t\Big[\frac{dQ_{jk}(t)}{Q_{jk}(t)}\Big] + r^i_j(t)\,dt\Big\} - r^i_k(t)\,dt = -\mathrm{Cov}_t\Big[\frac{dQ_{jk}(t)}{Q_{jk}(t)}, \frac{dZ^i_k(t)}{Z^i_k(t)}\Big]. \quad (3.15)$$

By regarding the exchange rate as an asset of currency $k$, the term in the bracket represents the instantaneous return plus the dividedned. The above equation also can be viewed as the uncovered interest parity modified by the covariance term.

## 3.4 Basis Swap Rates

The consistency with the basis swap rate $U_{jk}(t,T)$ is

$$\Gamma^i_k(t,T) = \Gamma^i_j(t,T) + U_{jk}(t,T)A^i_j(t,T). \quad (3.16)$$

By comparing the drift term and volatility term we get the following Proposition.

**Proposition 3.3.** *If the condition* $(\mathcal{Q})$ *and*

$$\begin{aligned}(\mathcal{U}) \qquad (i)\quad & r^i_k(t)\Gamma^i_k(t,T) - r^{M_k}_k(t) - \sigma^{\Gamma^i}_k(t,T)\sigma^Q_{jk}(t)\\ &= r^i_j(t)\big[\Gamma^i_j(t,T) + U_{jk}(t,T)A^i_j(t,T)\big] - r^{M_j}_j(t) - U_{jk}(t,T)\\ &+ A^i_j(t,T)\big[\alpha^U_{jk}(t,T) - \lambda^i_j(t)\sigma^U_{jk}(t,T)\big] + \sigma^U_{jk}(t,T)\sigma^{A^i}_j(t,T)\\ (ii)\quad & \sigma^{\Gamma^i}_k(t,T) = \sigma^{\Gamma^i}_j(t,T) + A^i_j(t,T)\sigma^U_{jk}(t,T) + U_{jk}(t,T)\sigma^{A^i}_j(t,T)\end{aligned}$$

*hold, then* $\{\Gamma^i_j(t,T) : t \le T \le T^*\}$ *and* $\{\Gamma^i_k(t,T) : t \le T \le T^*\}$ *are consistent with the basis swap rates* $U_{jk}(t,T)$.

The condition $(\mathcal{U})$ is equivalent to equation (3.16). Note that equation (3.16) implies

$$U_{jk}(t,T) = \mathcal{A}\Big\{\frac{\gamma^i_k - \gamma^i_j}{A^i_j}, \frac{B^i_j}{A^i_j}\Big\}(t,T), \quad t < T. \quad (3.17)$$

We will show that the basis swap rates are "priced" by MRAs. Suppose that the conditions $(\mathcal{Q})$ and $(\mathcal{U})$ holds. Differentiating equation (3.16) with respect to $T$ yields

$$\gamma^i_k(t,T) = \gamma^i_j(t,T) + U_{jk}(t,T)B^i_j(t,T) + \Big[\frac{\partial}{\partial T}U_{jk}(t,T)\Big]A^i_j(t,T)$$

and $\varphi^i_k(t) = \varphi^i_j(t) + U_{jk}(t,t)$. On the other hand from the condition $(\mathcal{Q})$ we have $\varphi^i_k(t) = r^{M_k}_k(t) - r^i_k(t) = \varphi^i_j(t) - \varphi^{M_k}_j(t) - \mu^i_k(t)\sigma^Q_{jk}(t)$ where $\mu^i_j(t) = \lambda^{M_j}_j(t) -$

$\lambda^i_j(t)$. By comparing the above two equations, $U_{jk}(t,t) = -\varphi^{M_k}_j(t) - \mu^i_k(t)\sigma^Q_{jk}(t)$ must hold. By letting $i = M_k$ we have

$$U_{jk}(t,t) = -\varphi^{M_k}_j(t) \tag{3.18}$$

and $\mu^i_k(t)\sigma^Q_{jk}(t) = 0$ for all $i \in \widetilde{\mathcal{I}}$. When $L_j = L_k = L$, since the short basis swap rate $U_{jk}(t,t)$ can be written as

$$U_{jk}(t,t) = r^{M_k}_j(t) - r^{M_j}_j(t) = \frac{1}{L}\sum_{i\in\mathcal{L}_k\setminus\mathcal{L}_j} r^i_j(t) - \frac{1}{L}\sum_{i\in\mathcal{L}_j\setminus\mathcal{L}_k} r^i_j(t),$$

we can say that the short basis swap rate is caused by the possibly different average rates of short rates in currency $j$ between two sets of contributors, $\mathcal{L}_k \setminus \mathcal{L}_j$ and $\mathcal{L}_j \setminus \mathcal{L}_k$.

Also equation (3.16) for $i = M_k$ yields

$$U_{jk}(t,T) = \frac{1 - \Gamma^{M_k}_j(t,T)}{A^{M_k}_j(t,T)} = \frac{\int_t^T \mathbb{E}\big[Z^{M_k}_j(s)U_{jk}(s,s) \mid \mathcal{G}^{M_k}_t\big]\, ds}{\int_t^T \mathbb{E}\big[Z^{M_k}_j(s) \mid \mathcal{G}^{M_k}_t\big]\, ds}. \tag{3.19}$$

Therefore, the basis swap rate $U_{jk}(t,T)$ represents the fixed rate to be exchanged with a floating rate $U_{jk}(s,s)$ in currency $j$ from the $M_k$'s perspective, in addition to the meaning of a weighted average of the funding spread of $M_k$ in currency $j$. It follows that if the sets of contributors of two currencies coincide, the basis swap rates are always zero in our framework.

## 3.5 Arbitrage-free and Consistent Conditions

As we see, the condition $(\mathcal{S})$ relates the market $\mathcal{M}^i$ to the swap rates $S(t,T)$, $(\mathcal{Q})$ does to $Q(t)$ and $(\mathcal{U})$ does to $U(t,T)$. These conditions can also be viewed as the restriction of the drifts of these rate processes. Summarising Proposition 3.1-3.3, we conclude

**Theorem 3.1.** *For each $i \in \widetilde{\mathcal{I}}$, if all of the conditions $(\mathcal{S})$, $(\mathcal{Q})$ and $(\mathcal{U})$ hold for all currencies, then the market $\mathcal{M}^i$ is arbitrage-free and consistent with the swap rates, the basis swap rates and the foreign exchanged rates.*

While the pricing kernel is constructed by two parameters, we have three conditions $(\mathcal{S})$, $(\mathcal{Q})$ and $(\mathcal{U})$. The reason of additional factor is the existence of the funding spreads between the short rate of the agent and MRA.

Once the market $\mathcal{M}^i$ is known to be arbitrage-free, it is easy to find yield curves in all currencies. Concentrate on an agent $i$ and assume that the funding spread $\varphi^i_1(t)$ of currency 1 is given. Then the FRN price $\Gamma^i_1(t,T)$ can be calculated, and by observing swap rates $S_1(t,T)$ of the currency 1 the discount factors $B^i_1(t,T)$ are given by equation (3.5). Then the condition $(\mathcal{Q})$ determines the short rate and the risk premium of other currencies by those of currency

1 and the foreign exchange rates. And equation (3.16) determines the FRN price of other currencies by the basis swap rates. Hence the discount factors of other currencies can be calculated. They are arbitrage-free and consistent by the construction.

# 4 A Specification

In this section we consider the case that all agents' risk premium and funding spread satisy

$$\varphi^i_j(t) = -\mu^i_j(t)\lambda^{M_j}_j(t) \tag{4.1}$$

in addition to the conditions $(\mathcal{S})$, $(\mathcal{Q})$, $(\mathcal{U})$ and develop the further implications. The first condition states that the funding spread of a currency is determined by the risk premium of the agent and MRA of the currency.

The definition of the pricing kernel of agent $i$

$$Z^i_j(t) = \exp\Big(-\int_0^t \big(r^i_j(s) + \frac{1}{2}|\lambda^i_j(s)|^2\big)\,ds - \int_0^t \lambda^i_j(s)\,dW(s)\Big)$$

yields to a relation with one of MRA

$$Z^i_j(t) = Z^{M_j}_j(t)\exp\Big(\int_0^t \big(\varphi^i_j(s) + \mu^i_j(s)\lambda^{M_j}_j(s) - \frac{1}{2}|\mu^i_j(s)|^2\big)\,ds + \int_0^t \mu^i_j(s)\,dW(s)\Big).$$

Plugging the assumption it can be rewritten as

$$Z^i_j(t) = Z^{M_j}_j(t)L^i_j(t) \tag{4.2}$$

where

$$L^i_j(t) = \exp\Big(-\frac{1}{2}\int_0^t |\mu^i_j(s)|^2\,ds + \int_0^t \mu^i_j(s)\,dW(s)\Big)$$

is the Doléan's exponential.

When $L^i_j$ is a martingale, by using equation (4.2) and a definition $Z^i_j(t)B^i_j(t,T) = \mathbb{E}\big[Z^i_j(T) \mid \mathcal{G}^i_t\big]$, we have

$$B^i_j(t,T) = B^{M_j}_j(t,T) + \mathrm{Cov}\Big[\frac{Z^{M_j}_j(T)}{Z^{M_j}_j(t)}, \frac{L^i_j(T)}{L^i_j(t)} \mid \mathcal{G}^i_t\Big] \tag{4.3}$$

hence $\mathcal{R}\{\Gamma^i_j, S_j\}(t,T) = \mathrm{Cov}\Big[\frac{Z^{M_j}_j(T)}{Z^{M_j}_j(t)}, \frac{L^i_j(T)}{L^i_j(t)} \mid \mathcal{G}^i_t\Big]$.

# 5 Concluding Remarks

In order to focus on the heterogeneous aspects involved in LIBOR we construct a framework in which multi agents have possibly different short rates and risk premium and each market of agents is well-defined via the pricing kernel of the agent. The discount factors are calculated by "bootstrapping" swap rates. The arbitrage-free and consistent conditions of the market are provided for each rate processes through the study of the dynamics of the products in the market. The situation that all agents observe and trade only rate processes in the "public market" gives each agent the freedom to choose the short rate and the risk premium. It results in the different price of the floating rate notes with coupon of MRA's short rate.

Once the suitable short rate and risk premium processes are selected *in a currency*, the arbitrage-free and consistent conditions determine the ones *in any other currencies*. This fact supports the efficiency of the global (rather than local) risk management of interest rate related products in the financial instituitions.

The basis swap market has been developing in accordance with active international investment and capital raising via debt instruments and derivatives. Nevertheless no academic study has ever been tried, to the best of our knowledge, since it cannot co-exist with risk-free rate. Our approach through a set of contributors and MRA leads to a simple analysis of basis swap rates. Similar discussion will be available with other spreads within a currency, such as TIBOR-LIBOR spread, by setting $\sigma^Q = 0$.

Although we define the short rate and risk premium of MRA as the arithmetic average of ones of the contributors, different definition from ours is possible and possibly different implications will be achieved. Especially specifications or modelling of the funding spread need more fundamental research in the relationship between the short rate and the risk premium though our discussions are based on exogeneously given short rates of the agent. The funding spread may be the result of a projection of the default likelihood in a defaultable market onto a default-free market. These topics are left to future research.

# Appendix

## A Proof of Lemma 3.2

Since the SDE of $\mathcal{R}\{X,Y\}(\cdot,T)$ can be derived by one of $\mathcal{A}\{X,Y\}(\cdot,T)$ and Ito's formula, it is enough to show the SDE of $\mathcal{A}\{X,Y\}(\cdot,T)$.

By letting $E(t,T) = \exp\Big(\int_t^T Y(t,u)\,du\Big)$, $\mathcal{A}\{X,Y\}(\cdot,T)$ can be rewritten as

$$\mathcal{A}\{X,Y\}(t,T) = E(t,T)^{-1}\int_t^T X(t,u)E(t,u)\,du.$$

We will derive the SDEs of the integrand step by step. By Lemma 3.1 we have

$$dE(t,T) = E(t,T)\big(\alpha^{Y*}(t,T) + \frac{1}{2}|\sigma^{Y*}(t,T)|^2 - Y(t,t)\big)\,dt + E(t,T)\sigma^{Y*}(t,T)\,dW(t).$$

Next we set $F(t,T) = X(t,T)E(t,T)$ which follows by Ito's formula

$$\begin{aligned} dF(t,T) &= \big[E\big(\alpha^X + \sigma^X\sigma^{Y*} + X(\alpha^{Y*} + \frac{1}{2}|\sigma^{Y*}|^2)\big)(t,T) - E(t,T)X(t,T)Y(t,t)\big]\,dt \\ &\quad + E\big(\sigma^X + X\sigma^{Y*}\big)(t,T)\,dW(t) \\ &\equiv \alpha^F(t,T)\,dt + \sigma^F(t,T)\,dW(t). \end{aligned}$$

Then again, Lemma 3.1 gives the SDE of $G(t,T) = \int_t^T F(t,u)\,du$ as

$$dG(t,T) = \big[\alpha^{F*}(t,T) - X(t,t)\big]\,dt + \sigma^{F*}(t,T)\,dW(t)$$

since $F(t,t) = X(t,t)$.
Finally, since $\mathcal{A}\{X,Y\}(t,T) = E(t,T)^{-1}G(t,T)$, it follows that

$$\begin{aligned} d\mathcal{A}\{X,Y\}(t,T) &= \Big[\Big(E^{-1}\alpha^{F*} + E^{-1}G\big(-\alpha^{Y*} + \frac{1}{2}|\sigma^{Y*}|^2\big) - E^{-1}\sigma^{F*}\sigma^{Y*}\Big)(t,T) \\ &\quad + E^{-1}(t,T)G(t,T)Y(t,t) - E^{-1}(t,T)X(t,t)\Big]\,dt \\ &\quad + \big[E^{-1}\sigma^{F*} - E^{-1}G\sigma^{Y*}\big](t,T)\,dW(t) \\ &\equiv \alpha^A(t,T)\,dt + \sigma^A(t,T)\,dW(t). \end{aligned}$$

Then we have

$$\begin{aligned} \sigma^A(t,T) &= E^{-1}(t,T)\int_t^T \big(\sigma^X + X\sigma^{Y*}\big)(t,u)E(t,u)\,du - E^{-1}G\sigma^{Y*}(t,T) \\ &= \mathcal{A}\{\sigma^X + X\sigma^{Y*}, Y\}(t,T) - \mathcal{A}\{X,Y\}(t,T)\sigma^{Y*}(t,T) \\ &= \mathcal{A}\{\sigma^X - \mathcal{A}\{X,Y\}\sigma^Y, Y\}(t,T) \end{aligned}$$

where in the last equation we used integration by parts

$$
\begin{aligned}
& \mathcal{A}\{\mathcal{A}\{X,Y\}\sigma^Y, Y\}(t,T) \qquad\qquad (\text{A.1})\\
= \ & E(t,T)^{-1}\int_t^T \Big[\frac{\partial \sigma^{Y*}(t,u)}{\partial u}\Big] E(t,u)^{-1}\Big(\int_t^u X(t,v)E(t,v)\,dv\Big)E(t,u)\,du \\
= \ & \mathcal{A}\{X,Y\}(t,T)\sigma^{Y*}(t,T) - \mathcal{A}\{X\sigma^{Y*},Y\}(t,T).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
& \alpha^A(t,T) \\
= \ & \mathcal{A}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \mathcal{A}\{\sigma^X,Y\}\sigma^Y + X\frac{1}{2}|\sigma^{Y*}|^2, Y\}(t,T) - E^{-1}(t,T)X(t,t) \\
& + \mathcal{A}\{X,Y\}(t,T)\frac{1}{2}|\sigma^{Y*}(t,T)|^2 - \mathcal{A}\{X\sigma^{Y*},Y\}(t,T)\sigma^{Y*}(t,T)
\end{aligned}
$$

where we used integration by parts

$$
\begin{aligned}
\mathcal{A}\{\mathcal{A}\{X,Y\}\alpha^Y,Y\}(t,T) &= \mathcal{A}\{X,Y\}(t,T)\alpha^{Y*}(t,T) - \mathcal{A}\{X\alpha^{Y*},Y\}(t,T), \\
\mathcal{A}\{\mathcal{A}\{\sigma^X,Y\}\sigma^Y,Y\}(t,T) &= \mathcal{A}\{\sigma^X,Y\}(t,T)\sigma^{Y*}(t,T) - \mathcal{A}\{\sigma^X\sigma^{Y*},Y\}(t,T),
\end{aligned}
$$

and the equations

$$
\begin{aligned}
E^{-1}(t,T)\alpha^{F*}(t,T) &= \mathcal{A}\{\alpha^X + \sigma^X\sigma^{Y*} + X(\alpha^{Y*} + \frac{1}{2}|\sigma^{Y*}|^2), Y\}(t,T) \\
& \quad - Y(t,t)\mathcal{A}\{X,Y\}(t,T), \\
E^{-1}(t,T)\sigma^{F*}(t,T) &= \mathcal{A}\{\sigma^X + X\sigma^{Y*}, Y\}(t,T).
\end{aligned}
$$

To rearrange the terms of $\alpha^A(t,T)$, with integration by parts

$$
\mathcal{A}\{X\sigma^{Y*},Y\}(t,T)\sigma^{Y*}(t,T) = \mathcal{A}\{\mathcal{A}\{X\sigma^{Y*},Y\}\sigma^Y + X|\sigma^{Y*}|^2, Y\}(t,T)
$$

and multiplying $\sigma^{Y*}(t,T)$ on both sides of equation $(A.1)$ we have

$$
\begin{aligned}
& \mathcal{A}\{X,Y\}(t,T)|\sigma^{Y*}(t,T)|^2 \\
= \ & \mathcal{A}\{2\mathcal{A}\{\mathcal{A}\{X,Y\}\sigma^Y,Y\}\sigma^Y + 2\mathcal{A}\{X\sigma^{Y*},Y\}\sigma^Y + X|\sigma^{Y*}|^2, Y\}(t,T).
\end{aligned}
$$

Then it holds that

$$
\begin{aligned}
& \mathcal{A}\{X\frac{1}{2}|\sigma^{Y*}|^2, Y\}(t,T) + \mathcal{A}\{X,Y\}(t,T)\frac{1}{2}|\sigma^{Y*}(t,T)|^2 - \mathcal{A}\{X\sigma^{Y*},Y\}(t,T)\sigma^{Y*}(t,T) \\
= \ & \mathcal{A}\{\mathcal{A}\{\sigma^X,Y\}\sigma^Y - \sigma^A\sigma^Y, Y\}(t,T).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\alpha^A(t,T) &= \mathcal{A}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \mathcal{A}\{\sigma^X,Y\}\sigma^Y, Y\}(t,T) \\
& \quad - E^{-1}(t,T)X(t,t) + \mathcal{A}\{\mathcal{A}\{\sigma^X,Y\}\sigma^Y - \sigma^A\sigma^Y, Y\}(t,T) \\
&= \mathcal{A}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \sigma^A\sigma^Y, Y\}(t,T) - E^{-1}(t,T)X(t,t).
\end{aligned}
$$

Observing that $\mathcal{A}\{Y,Y\}(t,T) = 1 - E(t,T)^{-1}$, we achieve the result

$$
\alpha^A(t,T) = \mathcal{A}\{\alpha^X - \mathcal{A}\{X,Y\}\alpha^Y - \sigma^A\sigma^Y, Y\}(t,T) + X(t,t)(\mathcal{A}\{Y,Y\}(t,T) - 1).
$$

# References


[1] Björk, T., G. Di Masi, Y. Kabanov, and W. Runggaldier (1997): Towards a general theory of bond markets, *Finance Stoch.* 1, 141-174.

[2] Brace, A., D. Gatarek and M. Musiela (1997): The Market Model of Interest Rate Dynamics, *Math. Finance* 7, 127-155.

[3] Collin-Dufresne, P. and B. Solnik (2001): On the Term Structure of Default Premia in the Swap and LIBOR Markets, *J. Finance* 56, 1095-1115.

[4] Duffie, D. and M. Huang (1996): Swap Rates and Credit Quality, *J. Finance* 51, 921-949.

[5] El Karoui, N., H. Geman, and J.-C. Rochet (1995): Changes of Numéraire, Changes of Probability Measure, and Option Pricing, *J. Appl. Probability* 32, 443-458.

[6] Grinblatt, M. (2001): An Analytical Solution for Interest Rate Swap Spreads, *International Review of Finance* 2, 113-149

[7] He, H. (2001): Modeling Term Structures of Swap Spreads, Working Paper, Yale School of Management.

[8] Heath, D., R. Jarrow, and A. Morton (1992): Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation, *Econometrica* 60, 77-105.

[9] Hunt, P.J. and J.E. Kennedy (2000): *Financial Derivatives in Theory and Practice*. West Sussex: John Wiley & Sons Ltd.

[10] Jamshidian, F. (1997): Libor and swap market models and measures, *Finance Stoch.* 1, 293-330.

[11] Musiela, M. and M. Rutkowski (1997): *Martingale Methods in Finacial Modelling*. Berlin Heidelberg New York: Springer-Verlag.

[12] Rogers, L.C.G. (1997): The Potential Approach to the Term Structure of Interest Rates and Foreign Exchange Rates, *Math. Finance* 7, 157-176.