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Yasuhiro TAMBA †

Abstract

This paper presents the tree construction approach to pricing a Bermudan swaption. The Bermudan swaption is an option, which at each date in a schedule of exercise dates gives the holder the right to enter an interest swap, provided that this right has not been exercised at any previous time in the schedule. Assuming a common diffusion short rate dynamics, the Hull–White model, we propose a dynamic programming approach for their risk neutral evaluation. This framework is suited to a calibration from an observed initial yield curve and market price data of discount bonds and European swaptions.

Keywords: Bermudan swaption, swap rate, risk neutral evaluation, dynamic programming, Hull–White model, calibration.
JEL classification: G13, G15, G21

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1 Introduction

As various types of financial instruments have been developed, more flexible instruments are needed for hedging their risk. Exotic interest rate derivatives are flexible financial instruments which satisfy such demand. One of the most traded exotic interest rate derivatives is a Bermudan swaption. The Bermudan swaption is an option, which at each date in a schedule of exercise dates gives the holder the right to enter an interest swap, provided that this right has not been exercised at any previous time in the schedule. In this paper we deal with the Bermudan swaption with a constant length of payoff periods after the exercise. Because of its usefulness as hedges for callable bonds, the Bermudan swaption is probably the most liquid interest rate instrument with a built-in early exercise feature.

There are many papers for pricing the Bermudan swaption because of it's popularity in the market. But pricing the instruments with the early exercise features is more complicate than other plain type instruments. The pricing method used in most papers is a Monte Carlo simulation because of its simplicity and applicability for a multifactor model and long maturity instruments. In spite of its usefulness, the Monte Carlo simulation has some weaknesses to be applied for pricing exotic derivatives. The most important drawback is that the Monte Carlo simulation has difficulty in dealing with derivatives that contain early exercise features, like an American option and the Bermudan option. So we need some improvement of the Monte Carlo method for pricing the above early exercise derivatives. Longstaff and Schwartz (1998) developed the least square method to overcome the weakness of the Monte Carlo simulation for pricing American option. Andersen (1999) used the Monte Carlo simulation for pricing the Bermudan swaption, and derived a lower bound on the Bermudan swaption prices considering less advantage exercise strategies. Broadie and Glasserman (1997a, 1997b) developed the stochastic mesh method. Carr and Yang (1997) developed a method based on the stratification technique.

Another choice of the pricing method is the dynamic programming approach. Pedersen and Sidenius (1998) uses an optimality equation of the dynamic programming approach to price a chooser flexible cap, another exotic interest rate derivative. This method is rather appropriate for pricing exotic derivatives because we can solve the problem by a backward induction setting up a recombining tree. In Ito, Ohnishi and Tamba (2004) we extend the method to price the chooser flexible cap focusing on it's calibration method so that the derived price reflects real market data. To price Bermudan options, many banks use one- or two-factor short rate model, in which the dynamic programming approach is used. But we have seen no paper discussing theoretically appropriate calibration method.

In this paper we focus on the pricing problem of the Bermudan swaption based on the observed market prices of rather simple interest rate derivatives such as an European swaption. We use the same method as Ito, Ohnishi and Tamba (2004), in which we extend the pricing method in some points compared with Pedersen and Sidenius (1998). Because the Bermudan swaption has different features compared to the chooser flexible cap such as an exercisable number, payoff number and exercise decision–making of the option, we need to consider another procedure of the pricing calculation. One of the main features of this paper is that deriving the theoretical prices of bond and European swaptions we use these prices for a calibration. Another feature is that we use the short rate for the tree construction instead of the LIBOR. This point has advantage in that the short rate model like Hull–White has some convinced features matching the interest rate movement in the real world such as a mean reversion property. We also show numerical examples, comparative statics and derive a non early exercise property of the Bermudan swaption.

The paper is organized as follows. In Section 2, we introduce the various notations about interest rate. In Section 3, we derive the theoretical no-arbitrage price of the discount bond and the European swaption in the Hull–White model. In Section 4, we discuss the pricing method of the Bermudan swaption with the optimality equations. We also show a calibration method of model parameters. In Section 5, we describe a construction of a trinomial tree for the Bermudan swaption pricing. We also show a numerical example and discuss comparative statics. In Section 6 we derive theoretical conditions under which the option holder does not exercise the Bermudan swaption. Section 7 concludes the paper.

2 Notations of Interest Rates

2.1 Notations of Interest Rates

In this subsection we explain notations of various interest rates. Let $D(t,T)$ $0 \leq t \leq T \leq T^*$ be the time t price of the discount bond (or zero–coupon bond) with maturity T, in brief T–bond, which pays 1–unit of money at the maturity T (where $D(T,T) = 1$ for any $T \in \mathbb{T}^*$). For $0 \leq t \leq S < T \leq T^*$,

$$R(t;S,T) := -\frac{\ln D(t,T) - \ln D(t,S)}{T-S} \tag{1}$$

is the (continuous compounding based) forward rate prevailing at time t which covers time interval $(S,T]$. For $0 \leq t < T \leq T^*$,

$$Y(t,T) := R(t;t,T) = -\frac{1}{T-t} \ln D(t,T) \tag{2}$$

is the (continuous compounding based) spot rate prevailing at current time t or yield which covers time interval $(t,T]$. The map $T \mapsto Y(t,T)$ is called the yield curve at time t. For $0 \leq t \leq T \leq T^*$,

$$f(t,T) := \lim_{U \downarrow T} R(t;T,U) = -\frac{\partial}{\partial T} \ln D(t,T) \tag{3}$$

is the (instantaneous) forward rate prevailing at current time t with the maturity time T. The map $T \mapsto f(t,T)$ is called the forward rate curve at time t. For $t \in \mathbb{T}^*$,

$$r(t) := f(t,t) = \lim_{T \downarrow t} Y(t,T) = -\left.\frac{\partial}{\partial T} \ln D(t,T)\right|_{T=t} \tag{4}$$

is the short rate at time t. For $0 \leq t \leq T \leq T^*$,

$$B(t,T) := \exp\left\{\int_t^T r(s)\mathrm{d}s\right\} \tag{5}$$

is the risk–free bank account at time T with unit investment capital at time t (where $B(t,t) = 1$).

For $N \in \mathbb{Z}_+$, let

$$0 \leq T_0 < T_1 < \cdots < T_i < T_{i+1} < \cdots < T_{N-1} < T_N \leq T^* \tag{6}$$

be the sequence of setting times and payment times of floating interest rates, that is, for $i = 0, \cdots, N-1$, the floating interest rate which covers time interval $(T_i, T_{i+1}]$, is set at time T_i and paid at time T_{i+1}. For convenience, we let

$$T_{i+1} - T_i = \delta \ (= \text{constant} \in \mathbb{R}_{++}), \quad i = 0, \cdots, N-1. \tag{7}$$

For $i = 0, \cdots, N-1$, we define the simple (or simple compounding based) interest rate which covers time interval $(T_i, T_{i+1}]$ by

$$L_{T_i}(T_i) := \frac{1}{\delta}\left\{\frac{1}{D(T_i, T_{i+1})} - 1\right\}. \tag{8}$$

This amount is set at time T_i, paid at time T_{i+1}, and is conventionally called as a spot LIBOR (London Inter–Bank Offer Rate). For $i = 0, \cdots, N-1$,

$$L_{T_i}(t) := \frac{1}{\delta}\left\{\frac{D(t, T_i)}{D(t, T_{i+1})} - 1\right\} \tag{9}$$

is the simple (or simple compounding based) interest rate prevailing at time t $(\in [0, T_i])$ which covers time interval $(T_i, T_{i+1}]$, and is called as a forward LIBOR.

An interest rate swap is a contract where two parties agree to exchange a set of floating interest rate, LIBOR, payments for a set of fixed interest rate payments. In the market, swaps are not quoted as prices for different fixed rates K, but only the fixed rate K is quoted for each swap such that the present value of the swap is equal to zero. This rate, called the par swap rate $S(t)$ at t, with the payments from T_1 to T_n is calculated as

$$S(t) = \frac{D(t, T_0) - D(t, T_n)}{\sum_{k=1}^{n} \delta D(t, T_k)}. \tag{10}$$

2.2 European Swaption

An European swaption gives the holder the right to enter at time T_N into a swap with fixed rate K. The value of the European swaption at T_N with n payments from T_{N+1} to T_{N+n} is

$$\delta(S(T_N) - K)^{+} \sum_{k=N+1}^{N+n} D(T_N, T_k). \tag{11}$$

3 No–Arbitrage Prices of the Discount Bond and the European Swaption and Calibration

3.1 No–Arbitrage Prices of Discount Bonds in Hull–White Model

We consider a continuous trading economy with a finite time horizon given by $\mathbb{T}^* := [0, T^*]$ $(T^* \in \mathbb{R}_{++})$. The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. In this notation, Ω denotes a sample space with elements $\omega \in \Omega$; $\mathcal{F}$ denotes a σ-algebra on Ω ; and $\mathbb{P}$ denotes a probability measure on $(\Omega, \mathcal{F})$. The uncertainty is resolved over $\mathbb{T}^*$ according to a 1–dimensional Brownian (motion) filtration $\mathbb{F} := (\mathcal{F}(t) : t \in \mathbb{T}^*)$ satisfying the usual conditions. $W := (W(t) : t \in \mathbb{T}^*)$ denotes a 1–dimensional standard $(\mathbb{P};\mathbb{F})$–Brownian motion. Consistent with the no-arbitrage and complete market paradigm, we assume the existence of the risk neutral equivalent martingale measure $\mathbb{P}^*$ with a bank account as a numéraire in this economy.

The Hull–White model, Hull and White (1990), is one of the most popular short rate models with the Affine Term Structure (ATS) feature in practice because it has desirable characters of the interest rate such a mean reversion property. It can also be fitted with an observable initial term structure. The Hull–White model assumes that, under the risk neutral probability measure $\mathbb{P}^*$, the short rate process $r = (r(t) : t \in \mathbb{T}^*)$ satisfies the following special form of SDE with the ATS property:

$$\mathrm{d}r(t) = \{\alpha(t) - \beta r(t)\}\mathrm{d}t + \sigma \mathrm{d}W^*(t), \quad t \in \mathbb{T}^*, \tag{12}$$

that is,

$$m_0(t) = \alpha(t), \qquad m_1(t) = -\beta; \tag{13}$$

$$s_0(t) = \frac{\sigma^2}{2}, \qquad s_1(t) = 0. \tag{14}$$

Under the Hull–White model, the time t price of T–bond can be explicitly derived as follows:

$$D(t,T;r(t)) = \exp\{-a(t,T) - b(t,T)r(t)\}, \quad 0 \le t \le T \le T^*, \tag{15}$$

where

$$a(t,T) = -\frac{\sigma^2}{2}\int_t^T \{b(s,T)\}^2 \mathrm{d}s + \int_t^T \alpha(s)b(s,T)\mathrm{d}s; \tag{16}$$

$$b(t,T) = \frac{1 - \mathrm{e}^{-\beta(T-t)}}{\beta}. \tag{17}$$

Then, the initial forward rate can be derived explicitly as:

$$\begin{aligned} f(0,T;r(0)) &= \frac{\partial}{\partial T}a(0,T) + r(0)\frac{\partial}{\partial T}b(0,T) \\ &= -\frac{\sigma^2}{2\beta^2}(\mathrm{e}^{-\beta T} - 1)^2 + \int_0^T \alpha(s)\frac{\partial}{\partial T}b(s,T)\mathrm{d}s \\ &\quad + r(0)\frac{\partial}{\partial T}b(0,T), \quad 0 \le T \le T^*. \end{aligned} \tag{18}$$

3.2 No–Arbitrage Price of the European swaption in the Hull–White Model

To calculate the price of the Bermudan swaption in the Hull–White model, we need to decide the values of the model parameters. The parameters are decided as observable simple derivatives (the European swaption) in the market fit theoretical prices derived by the model. So, for the calibration we need to derive theoretical prices of the European swaption based on the Hull–White model.

Using the same calculation in Ito, Ohnishi and Tamba (2004), we can obtain that $\overline{L}_{T_i}(t)(:= L_{T_i}(t) + \frac{1}{\delta})$ follows the SDE

$$\frac{\mathrm{d}\overline{L}_{T_i}(t)}{\overline{L}_{T_i}(t)} = h(t,T_i,T_{i+1})\sigma \mathrm{d}W^{T_{i+1}}(t), \quad 0 \le t \le T_i, \tag{19}$$

where $W^{T_{i+1}}(t)$ is the Brownian Motion under the forward neutral probability $\mathbb{P}^{T_{i+1}}$. See Appendix for the derivation. By utilizing the swap neutral evaluation method, we know that the fair (no–arbitrage) price of the European swaption at time t ($\in [0,T_i]$), $ES_{T_i}(t)$, is given by the expectation under the forward swap probability measure $\mathbb{P}^s$, under which the swap rate follows log–normal martingale (See Filipović (2002)):

$$ES_{T_i}(t) = \delta \sum_{k=i+1}^{i+n} D(t,T_k) E^S\left[(S(T_i)-K)^+ \middle| \mathcal{F}(t) \right], \quad 0 \le t \le T_i. \tag{20}$$

Using the same calculation as the Black–Scholes Formula and the approximation of the Black–like swaption volatility by Rebonato (1998), we obtain

$$ES_{T_i}(t) = \delta \sum_{k=i+1}^{i+n} D(t,T_k)[S(t)\Phi(d_{T_i}(t)) - K\Phi(d_{T_i}(t) - \nu_{T_i}(t))], \tag{21}$$

where $d_{T_i}(t)$, $\Phi(d)$ and $\nu_{T_i}(t)$ are defined as follows:

$$\begin{aligned}
d_{T_i}(t) &:= \frac{\log(S(t)/K)}{\nu_{T_i}(t)} + \frac{\nu_{T_i}(t)}{2}; \\
\Phi(d) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-x^2/2} dx; \\
\nu_{T_i}^2(t) &:= \sum_{k,l=i+1}^{i+n} \frac{w_k(0) w_l(0) L_{T_{k-1}}(0) L_{T_{l-1}}(0) \rho_{k,l}}{S(0)^2} \int_0^{T_i} \sigma^2 h(s,T_k,T_{k+1}) h(s,T_l,T_{l+1}) ds; \\
w_l(t) &:= \frac{\prod_{j=i+1}^{l} \frac{1}{1+\delta L_{T_{j-1}}(t)}}{\sum_{k=i+1}^{i+n} \prod_{j=i+1}^{k} \frac{1}{1+\delta L_{T_{j-1}}(t)}} = \frac{D(t,T_l)}{\sum_{k=i+1}^{i+n} D(t,T_k)}; \\
dW^{T_i} dW^{T_j} &:= d\langle W^{T_i}, W^{T_j} \rangle_t = \rho_{i,j} dt.
\end{aligned} \tag{22}$$

3.3 Calibration

Unknown parameters $\alpha(\cdot)$, β, σ could be estimated as follows. In the Hull–White model by solving the theoretical value formula of the forward rate (18) with respect to the function $\alpha(\cdot)$ and substituting the observed forward rate curve at current time 0, $f_{\mathrm{mkt}}(0,T)$, we have

$$\begin{aligned}
\alpha_{\mathrm{mkt}}(T) &:= \frac{\partial}{\partial T}\left[f_{\mathrm{mkt}}(0,T) + \frac{\sigma^2}{2\beta^2}(\mathrm{e}^{-\beta T} - 1)^2 \right] \\
&\quad -\beta \left[f_{\mathrm{mkt}}(0,T) + \frac{\sigma^2}{2\beta^2}(\mathrm{e}^{-\beta T} - 1)^2 \right], \quad 0 \le T \le T^*.
\end{aligned} \tag{23}$$

This function includes unknown parameters β and σ. Accordingly, the remaining unknown parameters β and σ could be determined, for example, by the minimizers of the following criterion function:

$$\begin{aligned}
C(\beta,\sigma) &:= w_1 \sum_i \left| D(t,T_i)_{\mathrm{mkt}} - D(t,T_i)_{\mathrm{mdl}} \right|^2 \\
&\quad + w_2 \sum_i \left| ES_{T_i}(t)_{\mathrm{mkt}} - ES_{T_i}(t)_{\mathrm{mdl}} \right|^2,
\end{aligned} \tag{24}$$

where $D(t,T_i)_{\rm mkt}$, $ES_{T_i}(t)_{\rm mkt}$: the observed prices of the i–th discount bond and the European swaption, respectively; $D(t,T_i)_{\rm mdl}$, $ES_{T_i}(t)_{\rm mdl}$: the theoretical prices of the i–th discount bond and the European swaption, respectively; w_1, w_2 ($\in \mathbb{R}_+$) ($w_1 + w_2 = 1$): weighting coefficients. We can take weighting coefficients in other ways like taking summation after we put weighting coefficients on each difference between the market data and the theoretical price of each i-th instrument. We can also use other financial instruments for this calibration.

4 Pricing the Bermudan swaption

4.1 Pricing the Bermudan swaption under the Risk Neutral Probability $\mathbb{P}^*$

Let $W(T_i, r(T_i)), i = 0, \cdots, N-1$ be the fair (no–arbitrage) price of the Bermudan swaption at time T_i. The optimal equation can be derived by using the Bellman Principle. We can use the short rate as the state variable instead of the swap rate because the swap rate is the increasing function in the short rate and these are one to one relation as we see below in equation (31). From (10) with the Hull–White model swap rate, explicitly shown as the function of t and $r(t)$, is represented as

$$\begin{aligned} S(t,r(t)) &= \frac{D(t,T_0;r(t)) - D(t,T_n;r(t))}{\sum_{k=1}^{n} \delta D(t,T_k;r(t))} \\ &= \frac{\exp\{-a(t,T_0) - b(t,T_0)r(t)\} - \exp\{-a(t,T_n) - b(t,T_n)r(t)\}}{\sum_{k=1}^{n} \delta \exp\{-a(t,T_k) - b(t,T_k)r(t)\}} \\ &:= \frac{g_0(t,r(t)) - g_n(t,r(t))}{\delta f(t,r(t))}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} f(t,r(t)) &= \sum_{k=1}^{n} \exp\{-a(t,T_k) - b(t,T_k)r(t)\}; \\ &= \sum_{k=1}^{n} g_k(t,r(t)) \end{aligned} \tag{26}$$

$$g_k(t,r(t)) = \exp\{-a(t,T_k) - b(t,T_k)r(t)\}, \quad k = 0,1,\cdots,n. \tag{27}$$

Then we have

$$\frac{\partial}{\partial r(t)} f(t,r(t)) = -\sum_{k=1}^{n} b(t,T_k) g_k(r(t)); \tag{28}$$

$$\frac{\partial}{\partial r(t)} g_k(t,r(t)) = -b(t,T_k) g_k(r(t)). \tag{29}$$

$$\begin{aligned} \frac{\partial}{\partial r(t)} S(t,r(t)) &= \tfrac{1}{\delta f^2(r(t))}[\{-b(t,T_0)g_0(t,r(t)) + b(t,T_n)g_n(t,r(t))\} \textstyle\sum_{k=1}^{n} g_k(t,r(t)) \\ &\quad + \{g_0(t,r(t)) - g_n(t,r(t))\} \textstyle\sum_{k=1}^{n} b(t,T_k) g_k(t,r(t))] \\ &= \tfrac{1}{\delta f^2(r(t))}[g_0(t,r(t))\{-b(t,T_0) \textstyle\sum_{k=1}^{n} g_k(t,r(t)) + \sum_{k=1}^{n} b(t,T_k) g_k(t,r(t))\} \\ &\quad + g_n(t,r(t))\{b(t,T_n) \textstyle\sum_{k=1}^{n} g_k(t,r(t)) - \sum_{k=1}^{n} b(t,T_k) g_k(t,r(t))\}] \end{aligned} \tag{30}$$

From (17) we know $b(t, T_k) < b(t, T_{k+1}) \quad k = 0, 1, \cdots, n-1$. As the result we could prove

$$\frac{\partial}{\partial r(t)} S(t) > 0, \tag{31}$$

which means that the swap rate is the increasing function in the short rate and these are one to one relation between them.

Next we derive the optimality equation under the risk neutral probability measure $\mathbb{P}^*$ with a bank account as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(S(T_{N-1})) \;=\; \delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k}) \tag{32}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} W(S(T_i)) \;=\; & \max\Big\{ E^*\Big[\frac{\delta[S(T_i) - K]_+ \sum_{k=i+1}^{i+n} D(T_{i+1}, T_k)}{B(T_i, T_{i+1})}\Big| S(T_i)\Big], \\ & E^*\Big[\frac{W(S(T_{i+1}))}{B(T_i, T_{i+1})}\Big| S(T_i)\Big]\Big\} \end{aligned} \tag{33}$$

4.2 Pricing the Bermudan swaption under the Forward Neutral Probability $\mathbb{P}^{T_N}$

We can also derive the optimality equation under the forward neutral probability $\mathbb{P}^{T_N}$ with a T_N–bond as a numéraire.

Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(S(T_{N-1})) \;=\; \delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k}) \tag{34}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} W(S(T_i)) \;=\; & \max\Big\{ \delta[S(T_i) - K]_+ \sum_{k=i+1}^{i+n} D(T_i, T_k), \\ & D(T_i, T_N) E^{T_N}\Big[\frac{W(S(T_{i+1}))}{D(T_{i+1}, T_N)}\Big| S(T_i)\Big]\Big\} \end{aligned} \tag{35}$$

4.3 Pricing the Bermudan swaption under Varying Forward Neutral Probabilities $\mathbb{P}^{T_i}(1 \leq i \leq N)$

In this subsection we write the optimality equation under forward neutral probabilities $\mathbb{P}^{T_i}$ varying at each period with a T_i–bond as a numéraire. This optimality equation is different from the both optimality equations of Subsection 4.1 and 4.2 that have the fixed probability measures at all periods.
Optimality Equation:

(i) For $i = N-1$ (Terminal Condition):

$$W(S(T_{N-1})) = \delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k}) \tag{36}$$

(ii) For $i = N-2, \cdots, 0$:

$$\begin{aligned} W(S(T_i)) = \max\Big\{ & \delta[S(T_i) - K]_+ \sum_{k=i+1}^{i+n} D(T_i, T_k), \\ & D(T_i, T_{i+1}) E^{T_{i+1}} \Big[W(S(T_{i+1})) \Big| S(T_i) \Big] \Big\} \end{aligned} \tag{37}$$

5 Construction of the Trinomial Tree of the Bermudan Swaption Price, Numerical Example and Comparative Statics

5.1 Backward Induction of the Bermudan Swaption Price

Using a trinomial tree of the short rate $r(t)$ that follows the Hull–White model, we construct the trinomial tree of the short rate and the Bermudan swaption price in the discrete time setting under the risk neutral probability $\mathbb{P}^*$. We can also construct the trees under the forward neutral probabilities $\mathbb{P}^{T_N}$ and $\mathbb{P}^{T_i}$ in the same way. We construct the short rate trinomial tree based on Hull–White (1994). For the simplicity of the calculation, we set $\Delta t = \delta = T_{i+1} - T_i$ as constant for all i. (i, j) represents a node at $t = i\Delta t$ and $r = j\Delta r$. We define $D(i, i+1, j)$ as the $(i+1)\Delta t$-bond price at (i, j), $L(i, i, j)$ as the spot LIBOR at T_i with the payment time $(i+1)\Delta t$, $L(i, i+k, j)$ as the forward LIBOR from T_{i+k} to T_{i+k+1} at T_i, and $r_{i,j}$ as the short rate value at the node (i, j). Under the

Hull–White model from (16) and (17) we have

$$\begin{aligned}
a(T_i, T_{i+1}) &= -\frac{\sigma^2}{2}\int_{T_i}^{T_{i+1}} \{b(s, T_{i+1})\}^2 \mathrm{d}s + \int_{T_i}^{T_{i+1}} \alpha(s) b(s, T_{i+1}) \mathrm{d}s \\
&\cong -\frac{\sigma^2}{2}\int_{T_i}^{T_{i+1}} \{b(s, T_{i+1})\}^2 \mathrm{d}s + \alpha_i \int_{T_i}^{T_{i+1}} b(s, T_{i+1}) \mathrm{d}s \\
&= -\frac{\sigma^2}{2\beta^2}\Big((T_{i+1} - T_i) + \frac{2}{\beta} e^{-\beta(T_{i+1}-T_i)} - \frac{1}{2\beta} e^{-2\beta(T_{i+1}-T_i)} - \frac{3}{2\beta}\Big) \\
&\quad +\alpha_i \frac{1}{\beta^2}\Big(\beta(T_{i+1} - T_i) + e^{-\beta(T_{i+1}-T_i)} - 1\Big) \\
&=: a(i, i+1), \qquad (38) \\
b(T_i, T_{i+1}) &= \frac{1 - \mathrm{e}^{-\beta(T_{i+1}-T_i)}}{\beta} =: b(i, i+1). \qquad (39)
\end{aligned}$$

Then the bond price $D(i, i{+}1, j)$, the spot LIBOR $L(i, i, j)$, the forward LIBOR $L(i, i{+}k, j)$ and the swap rate $S(T_i)$ are represented respectively as

$$D(i, i+1, j) = e^{-a(i,i+1)-b(i,i+1)r_{i,j}}; \qquad (40)$$

$$L(i, i, j) = -\frac{1}{\delta} + \frac{1}{\delta D(i, i+1, j)}; \qquad (41)$$

$$L(i, i+k, j) = -\frac{1}{\delta} + \frac{D(i, i+k, j)}{\delta D(i, i+k+1, j)}; \qquad (42)$$

$$S(T_i) \approx \sum_{k=0}^{n-1} w_{i+k+1}(0) L(i, i+k, j), \qquad (43)$$

where

$$w_i(0) = \frac{D(0, i, j)}{\sum_{k=0}^{n-1} D(0, i+k+1, j)}. \qquad (44)$$

See Brigo and Mercurio (2000) for the above approximation of the swap rate. The bank account $B(T_i, T_{i+1})$ with $r(T_i)$ transiting from a node (i, j) to a node $(i+1, j')$ can be approximated utilizing the trapezoidal rule as

$$B(T_i, T_{i+1}) = \exp\Big\{ \int_{T_i}^{T_{i+1}} r(s) \mathrm{d}s \Big\} \cong \exp\Big\{ \frac{r_{i,j} + r_{i+1,j'}}{2} \delta \Big\}. \qquad (45)$$

We define $p_u(i, j), p_m(i, j), p_d(i, j)$ as the transition probabilities from the node (i, j) to up, same, and down states at $t = (i+1)\Delta t$ respectively, $W(i, j)$ as the Bermudan swaption price at the node (i, j), and $W(i+1, j+1), W(i+1, j), W(i+1, j-1)$ as the Bermudan swaption prices at each state, $j+1$, j, and $j-1$, at time $(i+1)\Delta t$ with above corresponding transition probabilities from (i, j), and $B_u(i, j), B_m(i, j), B_d(i, j)$ as the bank account values at each state, $j+1$, j, and $j-1$, at time $(i+1)\Delta t$ with above corresponding transition probabilities from (i, j). The Bermudan swaption price at (i, j) , $W(i, j)$, can be derived as

$$
\begin{aligned}
W(i,j) \;\; = \;\; & \max\Big\{\delta(S(T_i)-K)^{+}\Big(p_u(i,j)\frac{\sum_{k=i+1}^{i+n} D(i+1,k,j+1)}{B_u(i,j)} \\
& +p_m(i,j)\frac{\sum_{k=i+1}^{i+n} D(i+1,k,j)}{B_m(i,j)} + p_d(i,j)\frac{\sum_{k=i+1}^{i+n} D(i+1,k,j-1)}{B_d(i,j)}\Big), \\
& p_u(i,j)\frac{W(i+1,j+1)}{B_u(i,j)} + p_m(i,j)\frac{W(i+1,j)}{B_m(i,j)} + p_d(i,j)\frac{W(i+1,j-1)}{B_d(i,j)}\Big\}.
\end{aligned}
\tag{46}
$$

By the backward induction from the terminal condition, we can calculate the current Bermudan swaption price at $(0, 0)$.

5.2 Numerical Example and Comparative Statics

Following Hull and White (1994), we construct the trinomial tree of the short rate and the swap rate and calculate the Bermudan swaption price. We discuss how each parameter and an initial condition affect the price of the Bermudan swaption. The Table 1 shows the Bermudan swaption price calculated with only one parameter changed where other parameters and initial conditions are fixed on the bench mark values, β=0.3, R : increasing 0.0025 in each period starting from 0.1, $\sigma = 0.01$, $N = 16$, $T = 4$, $\delta = 0.25$ and $K = 0.8$. We set the time interval $\delta = 0.25$ for both of the short rate and the spot LIBOR in this example. But we can set different time intervals for each rate. For example, we can set the shorter time interval for the short rate tree than the spot LIBOR. We use MATLAB for the calculation.

The result shows that as n(number of payoff periods) is bigger, the Bermudan swaption price is bigger. This is because the total amount of the payoff gets bigger. The Bermudan swaption price is smaller as β(parameter of HW model) is bigger. The value of β affects the level and speed of the mean reversion of the short rate. The larger β is, the smaller mean reversion level is. This keeps the values of the short rate and swap rate small, and causes the small Bermudan swaption price. The Bermudan swaption price is larger as σ(parameter of HW model) is bigger. The reason is the same as the result of the simple Black-Scholes formula. As σ is bigger, the value of the instruments, with which we can hedge the floating interest rate risk, is highly evaluated by buyers of the option. The Bermudan swaption price is smaller as K(exercise rate) is bigger. This result is caused by the payoff function, $\delta(S(T_i) - K)_+$. As the value of K increases, the payment gets smaller. The Bermudan swaption price is larger as T(option maturity) is bigger. Because we can replicate the Bermudan swaption with the short maturity by the longer one, the longer one is at least more expensive than the shorter one. Finally, we examine six patterns of the initial yield curve such as increasing curves and decreasing curves at different yield levels. Among the six initial yield curve examples, the price of the case of R3(the increasing, 0.01 at each period, curve starting from 0.1) is most expensive, and the prices of R2 case is cheapest in this example. But the prices with other parameter set cases do not necessarily result in the same pattern. Although the increasing initial yield curves cause the bigger values of the swap rate than the decreasing one, the bigger values cause not only the bigger values of payments but also the bigger values of the discount bonds, which are

Table 1. Comparative Statics[a]

	Price	Change of the Price
$n = 4$	0.5375	n bigger $\Rightarrow$ price bigger
$n = 2$	0.0072	
$\beta = 0.3$	0.5375	β bigger $\Rightarrow$ price smaller
$\beta = 0.6$	0	
$\sigma = 0.01$	0.5375	σ bigger $\Rightarrow$ price bigger
$\sigma = 0.5$	1.8732	
$K = 0.8$	0.5375	K bigger $\Rightarrow$ price smaller
$K = 0.1$	0.7996	
$T = 4, N = 16, \delta = 0.25$	0.5375	$T(orN)$ bigger $\Rightarrow$ price bigger
$T = 1, N = 4, \delta = 0.25$	0.1964	
$R0$: gently increasing from 0.1	0.5375	$R3 \Rightarrow$ price biggest $R2 \Rightarrow$ price smallest
$R1$: flat at 0.1	0.5322	
$R2$: decreasing from 0.14	0.4777	
$R3$: increasing from 0.06	0.5916	
$R4$: decreasing from 0.1	0.5135	
$R5$: increasing from 0.1	0.5404	

[a] The Table 1 shows the Bermudan swaption prices calculated with only one parameter changed where other parameters and initial conditions are fixed on the bench mark values, n (payment number)$= 4$, β (parameter of HW model)$= 0.3$, σ (parameter of HW model)$= 0.01$, K (exercise rate)$= 0.8$, T (option maturity)$= 4$, N (number of time periods)$= 16$, δ (time interval)$= 0.25$, and R (initial yield curve): increasing 0.0025 in each period starting from 0.1. The column of "Price" shows the Bermudan swaption price with each parameter. The column of "Change of the Price" shows the change of the Bermudan swaption price when each corresponding parameter is changed. $R0$ is the gently increasing, 0.005 at each period, initial yield curve from 0.1. $R1$ is the flat curve at 0.1. $R2$ is the decreasing, 0.01 at each period, curve from 0.14. $R3$ is the increasing, 0.01 at each period, curve from 0.06. $R4$ is the decreasing, 0.01 at each period, curve from 0.1. $R5$ is the increasing, 0.01 at each period, curve from 0.1.

the discount values used on the backward calculation. As we calculate the values of the Bermudan swaption backwardly, the Bermudan swaption values are discounted more in the case of the increasing case than the decreasing case. As the result, the price differences between the increasing and decreasing cases get smaller and smaller on the backward calculation. As the result in some cases, we see that the option price with the decreasing yield curve may be more expensive than the increasing case.

6 Conditions for Non Early Exercise of the Bermudan Swaption

In this section we derive theoretical conditions under which the option holder does not exercise the Bermudan swaption at T_i for $i = 0, \cdots, N-2$.

Proposition 6.1. *The holder of the Bermudan swaption does not exercise the Bermudan swaption at $t = T_{N-2}$ under the conditions*

$$S(T_{N-2}) < \frac{1}{1+\delta L_{N+l-2}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 1, 2; \tag{47}$$

$$S(T_{N-2}) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{N+s-3}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 3, 4, \cdots, n. \tag{48}$$

Proposition 6.2. *The holder of the Bermudan swaption does not exercise the option at $t = T_i$ under the conditions*

$$S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 1, 2; \tag{49}$$

$$S(T_i) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s-1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 3, 4, \cdots, n. \tag{50}$$

Proof. See Appendix. □

7 Conclusion

In this paper we propose the pricing method of the Bermudan swaption. We have mainly four contributions in this paper. Firstly, we utilize the dynamic programming approach with the short rate model, in particular the Hull–White model. Secondly, deriving the theoretical prices of bond and the European swaption, we use the theoretical prices for the calibration. Thirdly, we show the numerical examples and discuss comparative statics. Fourthly, we derive the conditions for non early exercise of the Bermudan swaption.

The future plan of the research is to price the Bermudan swaption after the calibration thorough real market data. Furthermore we should verify if prices derived by the model suit the Bermudan swaption prices of the real market. One factor model like this paper has only one driving Brownian motion and it implies perfect correlation of all forward rates with different maturity dates. Furthermore, by working with the one factor short rate as the model primitive, it is difficult to allow for a precise fit to processes of quoted instruments (bond and European swaption) because of low degrees of freedom. We can extend the one factor model of this paper to a multi factor model.

A The derivation of the stochastic process of $\overline{L}_{T_i}(t)$

We assume that the short rate $r(t)$ follows the Hull–White model (12). From the results (15), (16) and (17) the forward LIBOR $L_{T_i}(t)$ can be represented as

$$\begin{aligned} L_{T_i}(t) &:= \frac{D(t,T_i) - D(t,T_{i+1})}{\delta D(t,T_{i+1})} \\ &= -\frac{1}{\delta} + \frac{1}{\delta}\exp\Big\{ - a(t,T_i) + a(t,T_{i+1}) - b(t,T_i) + b(t,T_{i+1})\Big\} r(t). \end{aligned} \tag{51}$$

Defining

$$g(t,T_i,T_{i+1}) := \frac{1}{\delta}\exp\Big\{ - a(t,T_i) + a(t,T_{i+1})\Big\}; \tag{52}$$

$$h(t,T_i,T_{i+1}) := -b(t,T_i) + b(t,T_{i+1}); \tag{53}$$

$$\overline{L}_{T_i}(t) := L_{T_i}(t) + \frac{1}{\delta}, \tag{54}$$

we can represent $\overline{L}_{T_i}(t)$ as

$$\overline{L}_{T_i}(t) = g(t,T_i,T_{i+1})\mathrm{e}^{h(t,T_i,T_{i+1})r(t)}. \tag{55}$$

From the Itô's Lemma $\overline{L}_{T_i}(t)$ follows the SDE

$$\begin{aligned} \frac{\mathrm{d}\overline{L}_{T_i}(t)}{\overline{L}_{T_i}(t)} &= \Big\{ \frac{g(t,T_i,T_{i+1})_t}{g(t,T_i,T_{i+1})} + h(t,T_i,T_{i+1})_t r(t) + h(t,T_i,T_{i+1})(\alpha(t) - \beta r(t)) \\ &\quad + \frac{1}{2}h(t,T_i,T_{i+1})^2\sigma^2\Big\}\mathrm{d}t + h(t,T_i,T_{i+1})\sigma dW^*(t), \end{aligned} \tag{56}$$

where $g(t,T_i,T_{i+1})_t$ and $h(t,T_i,T_{i+1})_t$ are partial differentials of $g(t,T_i,T_{i+1})$ and $h(t,T_i,T_{i+1})$ with respect to t. $g(t,T_i,T_{i+1})_t$ and $h(t,T_i,T_{i+1})_t$ can be calculated as

$$g(t,T_i,T_{i+1})_t = \Big[\frac{\sigma^2}{2}\Big\{b(t,T_i) + b(t,T_{i+1})\Big\} - \alpha(t)\Big] h(t,T_i,T_{i+1}) g(t,T_i,T_{i+1}); \tag{57}$$

$$h(t,T_i,T_{i+1})_t = h(t,T_i,T_{i+1})\beta. \tag{58}$$

Hence, we obtain

$$\begin{aligned} \frac{\mathrm{d}\overline{L}_{T_i}(t)}{\overline{L}_{T_i}(t)} &= \Big[\frac{h(t,T_i,T_{i+1})\sigma^2}{2}\Big\{b(t,T_i) + b(t,T_{i+1})\Big\} + \frac{1}{2}h(t,T_i,T_{i+1})^2\sigma^2\Big]\mathrm{d}t \\ &\quad + h(t,T_i,T_{i+1})\sigma dW^*(t) \\ &= h(t,T_i,T_{i+1})\sigma\Big[\Big\{\frac{\sigma}{2}\Big(b(t,T_i) + b(t,T_{i+1})\Big) + \frac{1}{2}h(t,T_i,T_{i+1})\sigma\Big\}\mathrm{d}t + dW^*(t)\Big] \\ &= h(t,T_i,T_{i+1})\sigma\Big\{\sigma b(t,T_{i+1})\mathrm{d}t + dW^*(t)\Big\}. \end{aligned} \tag{59}$$

Changing the probability measure by

$$\mathrm{d}W^{T_{i+1}}(t) = \sigma b(t, T_{i+1})\mathrm{d}t + \mathrm{d}W^*(t), \tag{60}$$

we obtain that $\overline{L}_{T_i}(t)$ follows the SDE

$$\frac{\mathrm{d}\overline{L}_{T_i}(t)}{\overline{L}_{T_i}(t)} = h(t, T_i, T_{i+1})\sigma \mathrm{d}W^{T_{i+1}}(t), \quad 0 \le t \le T_i, \tag{61}$$

where $W^{T_{i+1}}(t)$ is the Brownian Motion under the forward neutral probability $\mathbb{P}^{T_{i+1}}$.

B Proof of the Proposition 1

Proof. We derive theoretical conditions under which the option holder does not exercise the Bermudan swaption at T_{N-2}. The value of the Bermudan swaption at the terminal period, T_{N-1}, is

$$W(S(T_{N-1})) = \delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k}). \tag{62}$$

The optimality equation at T_{N-2} is

$$\begin{aligned} W(S(T_{N-2})) &= \max\Big\{\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}), \\ &\qquad D(T_{N-2}, T_N)E^{T_N}\Big[\frac{W(S(T_{N-1}))}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\} \\ &= \max\Big\{\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}), \\ D(T_{N-2}, T_N)E^{T_N}&\Big[\frac{\delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k})}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\}. \end{aligned} \tag{63}$$

The condition under which the option holder does not exercises the Bermudan swaption at T_{N-2} is

$$\begin{aligned} &\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}) \\ &< D(T_{N-2}, T_N)E^{T_N}\Big[\frac{\delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k})}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\}. \end{aligned} \tag{64}$$

Using an approximation,

$$S(T_{N-1}) \approx \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-1}), \tag{65}$$

where

$$w_i(t) = \frac{D(t, T_i)}{\sum_{k=0}^{n-1} D(t, T_{i+k})}, \tag{66}$$

we have the first term of the summation on the right hand side of (64), A, as

$$\begin{aligned}
A &= D(T_{N-2}, T_N) E^{T_N} \Big[\delta [S(T_{N-1}) - K]_+ \Big| S(T_{N-2}) \Big] \\
&\approx D(T_{N-2}, T_N) E^{T_N} \Big[\delta [\sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-1}) - K]_+ \Big| S(T_{N-2}) \Big] \\
&\geq \delta D(T_{N-2}, T_N) [\sum_{k=0}^{n-1} w_{N+k}(0) E^{T_N} \Big[L_{N+k-1}(T_{N-1}) \Big| S(T_{N-2}) \Big] - K]_+ .
\end{aligned} \tag{67}$$

We evaluate the expectation terms of the equation (67). The first term of the summation is evaluated as

$$E^{T_N} \Big[L_{N-1}(T_{N-1}) \Big| S(T_{N-2}) \Big] = L_{N-1}(T_{N-2}). \tag{68}$$

Next we evaluate the second term of the summation, $E^{T_N} \Big[L_N(T_{N-1}) \Big| S(T_{N-2}) \Big]$. We consider the payoff $L_N(T_{N-1})$ given at T_N. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N}$ and $\mathbb{P}^{T_{N+1}}$. We define P as the evaluated value at T_{N-2} corresponding to the payoff.

$$\frac{P}{D(T_{N-2}, T_N)} = E^{T_N} \Big[\frac{L_N(T_{N-1})}{D(T_N, T_N)} \Big| S(T_{N-2}) \Big] \tag{69}$$

$$\frac{P}{D(T_{N-2}, T_{N+1})} = E^{T_{N+1}} \Big[\frac{L_N(T_{N-1})}{D(T_N, T_{N+1})} \Big| S(T_{N-2}) \Big] \tag{70}$$

So we have

$$E^{T_N} \Big[L_N(T_{N-1}) \Big| S(T_{N-2}) \Big] = \frac{D(T_{N-2}, T_{N+1})}{D(T_{N-2}, T_N)} E^{T_{N+1}} \Big[\frac{L_N(T_{N-1})}{D(T_N, T_{N+1})} \Big| S(T_{N-2}) \Big]. \tag{71}$$

Utilizing the facts that

$$D(T_N, T_{N+1}) = \frac{1}{1 + \delta L_N(T_N)} \tag{72}$$

and the function

$$f(x) := x(1 + \delta x) \tag{73}$$

is convex in x, we evaluate the expectation term in the equation (71) as

$$\begin{aligned}
E^{T_{N+1}} \Big[\frac{L_N(T_{N-1})}{D(T_N, T_{N+1})} \Big| S(T_{N-2}) \Big] &= E^{T_{N+1}} \Big[L_N(T_{N-1}) \{1 + \delta L_N(T_N)\} \Big| S(T_{N-2}) \Big] \\
&\geq L_N(T_{N-2})(1 + \delta L_N(T_{N-2})).
\end{aligned} \tag{74}$$

Hence we have the relationship

$$E^{T_N}\left[L_N(T_{N-1})\middle|S(T_{N-2})\right] \geq L_N(T_{N-2}). \tag{75}$$

Next we evaluate the third term of the summation, $E^{T_N}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right]$, in the same way. We consider the payoff $L_{N+1}(T_{N-1})$ at T_N. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N}$ and $\mathbb{P}^{T_{N+1}}$. We define Q as the evaluated value at T_{N-2} corresponding to the payoff.

$$\frac{Q}{D(T_{N-2},T_N)} = E^{T_N}\left[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_N)}\middle|S(T_{N-2})\right] \tag{76}$$

$$\frac{Q}{D(T_{N-2},T_{N+1})} = E^{T_{N+1}}\left[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right] \tag{77}$$

So we have

$$\begin{aligned} & E^{T_N}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right] \\ = & \tfrac{D(T_{N-2},T_{N+1})}{D(T_{N-2},T_N)} E^{T_{N+1}}\left[\tfrac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right]. \end{aligned} \tag{78}$$

Assuming that the following Brownian motions are uncorrelated

$$\mathrm{d}W^{T_{N+2}}(t)\mathrm{d}W^{T_{N+1}}(t) = 0, \tag{79}$$

where

$$\frac{\mathrm{d}L_{N+1}(t)}{L_{N+1}(t)} = \sigma_{N+1}(t)\mathrm{d}W^{T_{N+2}}(t), \quad t \in \mathbb{T}^* \tag{80}$$

$$\frac{\mathrm{d}L_N(t)}{L_N(t)} = \sigma_N(t)\mathrm{d}W^{T_{N+1}}(t), \quad t \in \mathbb{T}^*, \tag{81}$$

we have the expectation term in (78) as

$$\begin{aligned} & E^{T_{N+1}}\left[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right] \\ = \; & E^{T_{N+1}}\left[1+\delta L_N(T_N)\middle|S(T_{N-2})\right]E^{T_{N+1}}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right]. \end{aligned} \tag{82}$$

We evaluate the term $E^{T_{N+1}}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right]$ in (82). We consider the payoff $L_{N+1}(T_{N-1})$ at T_{N+1}. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N+1}$ and $\mathbb{P}^{T_{N+2}}$. We define R as the evaluated value at T_{N-2}.

$$\frac{R}{D(T_{N-2},T_{N+1})} = E^{T_{N+1}}\left[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+1})}\middle|S(T_{N-2})\right] \tag{83}$$

$$\frac{R}{D(T_{N-2},T_{N+2})} = E^{T_{N+2}}\left[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\middle|S(T_{N-2})\right] \tag{84}$$

So we have

$$
\begin{aligned}
& E^{T_{N+1}}\Big[L_{N+1}(T_{N-1})\Big|S(T_{N-2})\Big] \\
= & \frac{D(T_{N-2},T_{N+2})}{D(T_{N-2},T_{N+1})}E^{T_{N+2}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\Big|S(T_{N-2})\Big].
\end{aligned} \tag{85}
$$

In the same way as (74), we have the expectation term in (85) as

$$
E^{T_{N+2}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\Big|S(T_{N-2})\Big] \geq L_{N+1}(T_{N-2})(1+\delta L_{N+1}(T_{N-2})). \tag{86}
$$

Hence we evaluate the third term of the summation as

$$
E^{T_N}\Big[L_{N+1}(T_{N-1})\Big|S(T_{N-2})\Big] \geq L_{N+1}(T_{N-2}). \tag{87}
$$

In the same way we have the relations

$$
E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big] \geq L_{N+k-1}(T_{N-2})k=1,2,\cdots,n-1. \tag{88}
$$

Then A has the following relation.

$$
A \geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+ \tag{89}
$$

Comparing the first term of the left side on the equation (64) and the right side of equation (89), we have

$$
[S(T_{N-2})-K]_+ \leq \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+. \tag{90}
$$

$$
RHSof(90) \geq [\frac{1}{1+\delta L_{N-1}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+ \tag{91}
$$

Hence one of the sufficient conditions for non early exercise of the Bermudan swaption at $t=T_{N-2}$ derived from the comparison of the first terms is

$$
S(T_{N-2}) \leq \frac{1}{1+\delta L_{N-1}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}). \tag{92}
$$

We have the second term of the summation on the right hand side of (64), B, as

$$
\begin{aligned}
B &= D(T_{N-2},T_N)E^{T_N}\Big[\frac{D(T_{N-1},T_{N+1})\delta[S(T_{N-1})-K]_+}{D(T_{N-1},T_N)}\Big|S(T_{N-2})\Big] \\
&\approx \delta D(T_{N-2},T_N)E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-1})-K]_+\Big|S(T_{N-2})\Big] \\
&\geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]-K]_+.
\end{aligned} \tag{93}
$$

We evaluate the expectation terms of the equation (93). Under the following assumption that all Brownian motions are uncorrelated each other

$$\mathrm{d}W^{T_i}(t)\mathrm{d}W^{T_j}(t) = 0, \quad i \neq j, \tag{94}$$

we have

$$\begin{aligned} & E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big] \\ = \; & E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big]. \end{aligned} \tag{95}$$

Because the function

$$g(x) := \frac{1}{1+\delta x} \tag{96}$$

is convex in x for $x \geq 0$, we have

$$E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big] \geq \frac{1}{1+\delta L_N(T_{N-2})}. \tag{97}$$

Utilizing (97) and (88) we derive the relation

$$\begin{aligned} & E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big] \\ \geq \; & \frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})}. \end{aligned} \tag{98}$$

Hence B has the following relation.

$$B \geq \delta D(T_{N-2}, T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})} - K]_+. \tag{99}$$

Comparing the second terms of the left side on the equation (64) and the right side of equation (99), we have

$$[S(T_{N-2}) - K]_+ \leq \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_N)}[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})} - K]_+. \tag{100}$$

Hence one of the sufficient conditions for non early exercise of the Bermudan swaption at $t = T_{N-2}$ derived from the comparison of the second terms is

$$S(T_{N-2}) \leq \frac{1}{1+\delta L_N(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}). \tag{101}$$

We have the third term of the summation on the right hand side of (64), C, as

$$
\begin{aligned}
C &= D(T_{N-2},T_N)E^{T_N}\Big[\frac{D(T_{N-1},T_{N+2})\delta[S(T_{N-1})-K]_+}{D(T_{N-1},T_N)}\Big|S(T_{N-2})\Big]\\
&\approx \delta D(T_{N-2},T_N)E^{T_N}\Big[\frac{1}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}\\
&[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-1})-K]_+\Big|S(T_{N-2})\Big]\\
&\geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)\\
&E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}\Big|S(T_{N-2})\Big]-K]_+.
\end{aligned}
\tag{102}
$$

In the same way we evaluate the expectation terms of the equation (102) under the assumption all Brownian motions are uncorrelated each other.

$$
\begin{aligned}
&E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}\Big|S(T_{N-2})\Big]\\
\geq\ &\frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))}
\end{aligned}
\tag{103}
$$

Hence C has the following relation.

$$
C \geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))}-K]_+
\tag{104}
$$

Comparing the third terms of the left side on the equation (64) and the right side of equation (104), we have

$$
\begin{aligned}
&[S(T_{N-2})-K]_+ \leq \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N+1})}\\
&[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))}-K]_+.
\end{aligned}
\tag{105}
$$

Because $\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N+1})} > 1$ one of the sufficient conditions for non early exercise of the Bermudan swaption at $t=T_{N-2}$ derived from the comparison of the third terms is

$$
S(T_{N-2}) \leq \prod_{s=3}^{4}\frac{1}{1+\delta L_{N+s-3}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}).
\tag{106}
$$

In the same way we can derive that the sufficient conditions to satisfy this proposition are summarized as

$$
S(T_{N-2}) < \frac{1}{1+\delta L_{N+l-2}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}), l=1,2;
\tag{107}
$$

$$
S(T_{N-2}) < \prod_{s=3}^{l+1}\frac{1}{1+\delta L_{N+s-3}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}), l=3,4,\cdots,n.
\tag{108}
$$

□

C Proof of the Proposition 2

Proof. At $t = T_{N-2}$ from the result of Proposition 1 we prove that we do not exercise the option under the conditions (107) and (108). At $t = T_{i+1}$ we suppose that we do not exercise the option under the conditions

$$S(T_{i+1}) < \frac{1}{1+\delta L_{i+l+1}(T_{i+1})}\sum_{k=0}^{n-1} w_{i+k+3}(0)L_{i+k+2}(T_{i+1}), l = 1, 2; \tag{109}$$

$$S(T_{i+1}) < \prod_{s=3}^{l+1}\frac{1}{1+\delta L_{i+s}(T_{i+1})}\sum_{k=0}^{n-1} w_{i+k+3}(0)L_{i+k+2}(T_{i+1}), l = 3, 4, \cdots, n, \tag{110}$$

that is

$$\begin{aligned} W(S(T_{i+1})) &= D(T_{i+1}, T_{i+3})E^{T_{i+3}}\Big[\frac{W(S(T_{i+2}))}{D(T_{i+2}, T_{i+3})}\Big|S(T_{i+1})\Big] \\ &> \delta[S(T_{i+1}) - K]_+\sum_{k=0}^{n-1} D(T_{i+1}, T_{i+k+2}). \end{aligned} \tag{111}$$

At $t = T_i$ we would like to show that under the conditions

$$S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i), l = 1, 2; \tag{112}$$

$$S(T_i) < \prod_{s=3}^{l+1}\frac{1}{1+\delta L_{i+s-1}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i), l = 3, 4, \cdots, n, \tag{113}$$

we do not exercise the option by the induction, that is

$$\begin{aligned} &\delta[S(T_i) - K]_+\sum_{k=0}^{n-1} D(T_i, T_{i+k+1}) \\ &< D(T_i, T_{i+2})E^{T_{i+2}}\Big[\frac{W(S(T_{i+1}))}{D(T_{i+1}, T_{i+2})}\Big|S(T_i)\Big]. \end{aligned} \tag{114}$$

From the hypothesis, (111), substituting RHS of (111) for the RHS of (114) we obtain

$$\begin{aligned} RHSof(114) &> D(T_i, T_{i+2})E^{T_{i+2}}\Big[\frac{\delta[S(T_{i+1}) - K]_+\sum_{m=0}^{n-1} D(T_{i+1}, T_{i+m+2})}{D(T_{i+1}, T_{i+2})}\Big|S(T_i)\Big] \\ &= D(T_i, T_{i+2})E^{T_{i+2}}\Big[\delta[\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_{i+1}) - K]_+ \\ &\quad \frac{\sum_{m=0}^{n-1} D(T_{i+1}, T_{i+m+2})}{D(T_{i+1}, T_{i+2})}\Big|S(T_i)\Big]. \end{aligned} \tag{115}$$

Comparing the first terms of RHS of (115) and LHS of (114), we obtain one of the non early exercise conditions as

$$\begin{aligned} &[S(T_i) - K]_+ < \\ &\frac{D(T_i, T_{i+2})}{D(T_i, T_{i+1})}E^{T_{i+2}}\Big[\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_{i+1}) - K]_+\Big|S(T_i)\Big]. \end{aligned} \tag{116}$$

Utilizing the following relation

$$
\begin{aligned}
RHSof(116) &\geq \frac{1}{1+\delta L_{i+1}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)E^{T_{i+2}}\Big[L_{i+k+1}(T_{i+1})\Big|S(T_i)\Big] - K]_+ \\
&\geq [\frac{1}{1+\delta L_{i+1}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i) - K]_+, \qquad (117)
\end{aligned}
$$

(116) is satisfied under the condition (112). Comparing the second terms of RHS of (115) and LHS of (114), we obtain one of the non early exercise conditions as

$$
[S(T_i) - K]_+ < E^{T_{i+2}}\Big[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})}[\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_{i+1}) - K]_+\Big|S(T_i)\Big]. \qquad (118)
$$

Utilizing the following relation

$$
\begin{aligned}
RHSof(118) &\geq [\frac{1}{1+\delta L_{i+2}(T_{i+1})}\sum_{k=0}^{n-1} w_{i+k+2}(0)E^{T_{i+2}}\Big[L_{i+k+1}(T_{i+1})\Big|S(T_i)\Big] - K]_+ \\
&\geq [\frac{1}{1+\delta L_{i+2}(T_{i+1})}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i) - K]_+, \qquad (119)
\end{aligned}
$$

(118) is satisfied under the condition (112). Comparing the l-th($l \geq 3$) terms of RHS of (115) and LHS of (114), we obtain the one of the non early exercise conditions as

$$
\begin{aligned}
&[S(T_i) - K]_+ < \\
&\frac{D(T_i, T_{i+2})}{D(T_i, T_{i+l})}E^{T_{i+2}}\Big[\frac{D(T_{i+1}, T_{i+l+1})}{D(T_{i+1}, T_{i+2})}[\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_{i+1}) - K]_+\Big|S(T_i)\Big]. \qquad (120)
\end{aligned}
$$

Utilizing the following relation

$$
\begin{aligned}
RHSof(120) &\geq \prod_{u=0}^{l-3}(1+\delta L_{i+u+2}(T_i))E^{T_{i+2}}\Big[[\prod_{s=3}^{l+1}\frac{1}{1+\delta L_{i+s+2}(T_{i+1})} \\
&\quad \sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_{i+1}) - K]_+ \\
&\geq [\prod_{s=3}^{l+1}\frac{1}{1+\delta L_{i+s+2}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i) - K]_+, \qquad (121)
\end{aligned}
$$

(121) is satisfied under the condition (113). Then we prove that the holder of the Bermudan swaption does not exercise the option at $t = T_i$ under the conditions

$$
S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i), l = 1, 2; \qquad (122)
$$

$$
S(T_i) < \prod_{s=3}^{l+1}\frac{1}{1+\delta L_{i+s-1}(T_i)}\sum_{k=0}^{n-1} w_{i+k+2}(0)L_{i+k+1}(T_i), l = 3, 4, \cdots, n. \qquad (123)
$$

□

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