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The Monotonicity of Asset Prices with Changes in Risk*

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Abstract

The goal of this paper is the examination of the conditions on preferences to guarantee the monotonicity of asset prices, when their returns change in the sense of first– and second–order stochastic dominances.

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1 Introduction

Many studies have examined the effects of changes in risk on optimal risk–taking behavior. Rothschild and Stiglitz (1971) and Fishburn and Porter (1976) obtained two counterintuitive results in their seminal papers. Although risk averters improve their utilities for first– and second–order changes in risk, these stochastic dominances only lead to ambiguous comparative static results of optimal decisions for risk averters. In order to resolve these results, the development of this topic has followed two directions: restrictions on changes in risk and preferences.[1] An end of the research for the former direction is made by Gollier (1995) and that for the latter direction is made by Hadar and Seo (1990). It is natural to question how changes in risk affect asset prices. Gollier and Schlesinger (2002) gave the result for asset prices corresponding to the former direction. They determined the stochastic dominance that is an equivalent condition for asset prices to be monotone. This stochastic dominance cannot be compared with first– and second–order stochastic dominances. This means that asset prices do not necessarily have monotonicity when their returns change in first– and second–order stochastic dominance shifts. Hence the following question araises: What conditions on preferences must be imposed to guarantee monotone changes in asset prices for first– and secon–order changes in risk? This paper answers to this question. To obtain the answer, we only add a natural condition from empirical and theoretical viewpoints to the conditions on preferences for optimal risk–taking behavior: No additional condition is imposed for first–order stochastic changes in risk and only prudence is required for second–order stochastic changes in risk.[2]

2 Stochastic Dominance

As an introduction, we give the definition and properties of first– and second–order stochastic dominances, and denote a Cumulative Distribution Function (CDF) over

[1] Gollier and Eeckhoudt (2000) provided a survey of this topic along these directions.

[2] Kimball (1990) showed that prudence households have positive precautionary savings. From empirical observations, households save money for future uncertainty. This means that prudence is justified from the descriptive viewpoints. See Kimball (1990) for further discussions.

a bounded support $[a, b]$ as $F(x) := \mathbb{P}(\tilde{x} \leq x)$.

Definition 2.1.

- $F(2)$ dominates $F(1)$ in the sense of First–order Stochastic Dominance (FSD) if $F(x; 2) \leq F(x; 1)$ holds for all $x \in [a, b]$. We denote this as $F(2) \geq_{\text{FSD}} F(1)$;
- $F(2)$ dominates $F(1)$ in the sense of Second–order Stochastic Dominance (SSD) if $\int_a^x F(t; 2)dt \leq \int_a^x F(t; 1)dt$ holds for all $x \in [a, b]$ and $\int_a^b F(t; 2)dt = \int_a^b F(t; 1)dt$. We denote this as $F(2) \geq_{\text{SSD}} F(1)$.

The following properties are well known in the theory of stochastic dominance. Hence we give the following theorem without proofs. Readers may refer to Ch. 3 in Gollier (2001) for detailed discussions.

Theorem 2.1.

- $F(2) \geq_{\text{FSD}} F(1)$, if and only if $\mathbb{E}[g(\tilde{x}(2))] \geq \mathbb{E}[g(\tilde{x}(1))]$ for every increasing function g.
- $F(2) \geq_{\text{SSD}} F(1)$, if and only if $\mathbb{E}[g(\tilde{x}(2))] \geq \mathbb{E}[g(\tilde{x}(1))]$ for every concave function g.

3 Comparative Statics

3.1 Equilibrium Asset Price

Let us consider a static version of a Lucas (1978) economy. The economy is a two–date competitive and pure exchange economy with a representative investor. The representative investor has an expected utility representation with a strictly increasing, strictly concave and sufficiently smooth von Neumann–Morgenstern utility function (utility function) u, which means that all of required higher order derivatives are assumed to exist. The endowment of the investor is w units of a risk–free asset and one unit of a risky asset. The risk–free asset is the numeraire and the gross risk–free rate is normalized to one. The return on the risky asset is represented by a random variable $\tilde{x}$ with a CDF F defined over a bounded support $[a, b]$. The price of the risky asset is denoted as q.

Following Gollier and Schlesinger (2002), the equilibrium asset price is given as

$$q = \frac{\mathbb{E}[\tilde{x}u'(z(\tilde{x}))]}{\mathbb{E}[u'(z(\tilde{x}))]}, \tag{1}$$

where $z(x)$ is the final wealth in equilibrium defined by $z(x) := w + x$

3.2 First–order Stochastic Dominance

We consider an economy i $(= 1, 2)$ with a returen on a risky asset $\tilde{x}_i$ distributed according to a CDF $F(i)$, and suppose that $F(2)$ dominates $F(1)$ in the sense of FSD: $F(1) \leq_{\text{FSD}} F(2)$. In this subsection, we examine what conditions on utility functions guarantee $q_1 \leq q_2$. First, we show the following lemma.

Lemma 3.1. Consider a random variable $\tilde{x}_i$ $(i = 1, 2)$ with a CDF $F(i)$, and suppose that $F(2)$ dominates $F(1)$ in the sense of FSD: $F(1) \leq_{\text{FSD}} F(2)$. If relative risk aversion defined by $\mathcal{R}(x) := -xu''(x)/u'(x)$ is less than unity, then $\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))] \leq \mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]$

Proof. Since the proof is similar to those of Lemma 1 in Hadar and Seo (1990) and Proposition 9 in Gollier (2001), we provide only an intuition of the proof. We find a condition on preference to guarantee that the function $xu'(x)$ is increasing in x. □

We obtain the next proposition.

Proposition 3.1. Consider an economy i $(= 1, 2)$ with a risky asset return $\tilde{x}_i$ distributed according to a CDF $F(i)$, and suppose that $F(2)$ dominates $F(1)$ in the sense of FSD. If relative risk aversion is less than unity, then $q_1 \leq q_2$.

Proof. By the above lemma, we have that

$$\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))] \leq \mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]. \tag{2}$$

Since $u'(z(x))$ is a decreasing function of x, we have that

$$\mathbb{E}[u'(z(\tilde{x}_1))] \geq \mathbb{E}[u'(z(\tilde{x}_2))]. \tag{3}$$

Combining Eqs. (2) and (3), we obtain

$$q_1 = \frac{\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))]}{\mathbb{E}[u'(z(\tilde{x}_1))]} \leq \frac{\mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]}{\mathbb{E}[u'(z(\tilde{x}_2))]} = q_2. \tag{4}$$

□

3.3 Second–order Stochastic Dominance

We consider an economy i $(= 1, 2)$ with a return on a risky asset $\tilde{x}_i$ distributed according to a CDF $F(i)$, and suppose that $F(2)$ dominates $F(1)$ in the sense of SSD: $F(1) \leq_{\text{SSD}} F(2)$. In this subsection, we examine what conditions on utility functions guarantee $q_1 \leq q_2$. The analysis is parallel to the previous subsection.

Lemma 3.2. Consider a random variable $\tilde{x}_i$ $(i = 1, 2)$ with a CDF $F(i)$ and suppose that $F(2)$ dominates $F(1)$ in the sense of SSD: $F(1) \leq_{\text{SSD}} F(2)$. If

- absolute risk aversion defined by $\mathcal{A}(x) := -u''(x)/u'(x)$ is a decreasing function of x, and relative risk aversion is less then unity and increasing function of x; and/or

- relative prudence defined by $x\mathcal{P}(x) := -xu'''(x)/u''(x)$ is positive and less than 2,

then $\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))] \leq \mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]$. □

Proof. Since the proof is similar to those of Lemma 1 in Hadar and Seo (1990) and Proposition 9 in Gollier (2001), we provide only an intuition of the proof. We find a condition on preferences to guarantee that the function $xu'(x)$ is a concave function of x. □

We obtain the next proposition.

Proposition 3.2. Consider an economy i $(= 1, 2)$ with a risky asset return $\tilde{x}_i$ distributed according to a CDF $F(i)$. Suppose that $F(2)$ dominates $F(1)$ in the sense of SSD and that the representative investor is prudent, i.e. $u'''(x) \geq 0$. If

- absolute risk aversion is a decreasing function, and relative risk aversion is less then unity and an increasing function; and/or

- relative prudence is positive and less than 2,

then $q_1 \leq q_2$. □

Proof. By the above lemma, we have that

$$\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))] \leq \mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]. \tag{5}$$

Since $u'(z(x))$ is a convex function of x,

$$\mathbb{E}[u'(z(\tilde{x}_1))] \geq \mathbb{E}[u'(z(\tilde{x}_2))]. \tag{6}$$

Combining Eqs. (5) and (6), we obtain

$$q_1 = \frac{\mathbb{E}[\tilde{x}_1 u'(z(\tilde{x}_1))]}{\mathbb{E}[u'(z(\tilde{x}_1))]} \leq \frac{\mathbb{E}[\tilde{x}_2 u'(z(\tilde{x}_2))]}{\mathbb{E}[u'(z(\tilde{x}_2))]} = q_2. \tag{7}$$

□

4 Concluding Remarks

We examine the conditions on preferences to guarantee the monotonicity of asset prices, when their returns change in the sense of FSD and SSD. Our motivation stems from the counterintuitive results obtained by Gollier and Schlesinger (2002): the FSD and SSD changes in risk only yield ambiguous comparative static results of asset prices. Whereas their approach to this result is the restrictions on changes in risk, our approach introduces the restrictions on preferences. Compared with the conditions on preferences to guarantee the unambiguous comparative static results of optimal risk–taking behavior, an additional condition on preferences for asset prices is prevalent from both empirical and theoretical viewpoints: risk aversion for the FSD changes in risk and prudence for the SSD changes in risk.[3]

[3]Since risk aversion is necessary for the existence of optimal portfolio and equilibrium, it is not explicitly appeared in conditions on preferences.

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