Discussion Papers In Economics And Business

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December 2005

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

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December 1, 2005

[*]We are grateful to Shigehiro Serizawa, Koji Takamiya, and seminar participants at the 2005 RIMS Workshop "Mathematics of Optimization: Methods and Practical Solutions" for their valuable comments.

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Abstract

We explore the relationships between non-bossiness and Nash implementability. We provide a new domain-richness condition, *weak monotonic closedness*, and prove that on weakly monotonically closed domains, non-bossiness together with individual monotonicity is equivalent to monotonicity, a necessary condition for Nash implementation. The result shows an impossibility of Nash implementation in all economies except pure public goods economies, in the sense that it indicates that in all economies except pure public goods economies, it is impossible to implement bossy social choice functions in Nash equilibria, which embody the characteristics inherent in those economies.

Keywords: Non-Bossiness, Individual Monotonicity, Monotonicity, Weak Monotonic Closedness.

JEL Classification Numbers: D51, D71, D78.

1 Introduction

The mechanism design literature has dealt with a very large number of allocation rules (or direct revelation mechanisms). The following is an example of the allocation rules in a private goods economy: an allocation rule where there is an agent, called a *boss*, who can change another agent's consumption bundle by changing her preferences without changing her own bundle. This type of allocation rule was called *bossy* by Satterthwaite and Sonnenschein (1981); but the idea of the bossy allocation rules had already been known, since the well-known Vickrey–Clarke–Groves type of allocation rules (Vickrey (1961), Clarke (1971), and Groves (1973)) were bossy. So, bossy allocation rules can be regarded as acceptable if the Vickrey–Clarke–Groves type of allocation rules seem attractive.

The standard economic theory tells us that agents are assumed to be selfish. This means that bosses do not care about consumption bundles of the other agents; so bosses will not deliberately change another agent's consumption bundle by changing her preferences even if her own bundle is kept unchanged. This is a key to making the Vickrey–Clarke–Groves type of allocation rules work well.

Nevertheless, Satterthwaite and Sonnenschein (1981) thought of bossy allocation rules as undesirable at least in terms of simplicity (see Satterthwaite and Sonnenschein (1981) for details). So, they introduced the notion of *non-bossiness*, which requires that there should be no boss. Non-bossiness has since been widely used in the literature on strategy-proofness.[1]

But, almost all of the literature has not explained non-bossiness's reasonableness and desirability. Non-bossiness has often been imposed for technical convenience with the following exceptions: Barberà and Jackson (1995) showed that non-bossiness plus strategy-proofness implies weak coalitional strategy-proofness in pure exchange economies, and Pàpai (2000a) and Takamiya (2001) showed that non-bossiness together with strategy-proofness is equivalent to coalitional strategy-proofness in the house allocation problem and in the Shapley–Scarf housing market with strict preferences, respectively. These tell us that non-bossiness is desirable in the sense that when combined with strategy-proofness, it prevents manipulation by coalitions of agents.

However, coalitional strategy-proofness is too demanding in general,

[1] A partial list of such literature includes Barberà and Jackson (1995), Ju (2004), and Serizawa (2005) for pure exchange economies; Svensson (1999), Takamiya (2001), Miyagawa (2002), and Pàpai (2003) for the housing market of Shapley and Scarf (1974); Miyagawa (2001) and Svensson and Larsson (2002) for the Shapley–Scarf housing market with compensation; Svensson (1999), Pàpai (2000a), Pàpai (2000b), and Pàpai (2001) for the house allocation problem; Schummer (2000a) and Svensson and Larsson (2002) for the house allocation problem with compensation; Klaus (2001) for allotment economies with single-dipped preferences; Serizawa (1996) for public good economies; and Dearden and Einolf (2003) for excludable public good economies.

because it prevents not only self-enforcing coalitional manipulations but also non-self-enforcing coalitional manipulations. The standard economic theory also tells us that in non-cooperative environments, agents can freely discuss their actions but cannot make binding commitments. This indicates that there is no need to rule out coalitional manipulations that are not self-enforcing unless an additional assumption that agents can sign binding agreements is imposed. So, when coupled with strategy-proofness, non-bossiness appears too strong without the additional assumption.[2]

Thus, the issue concerning reasonableness and desirability of non-bossiness seems still open. This paper examines the desirability of non-bossiness by exploring the relationships between non-bossiness and Nash implementability.[3]

This paper is organized as follows. Section 2 gives notation and definitions. We explore the relationships between non-bossiness and Nash implementability in Section 3. Section 4 contains a concluding remark.

2 Notation and Definitions

Let $N := \{1, 2, \ldots, n\}$ be the set of *agents*, where $2 \leq n < +\infty$. Let X_i be the *consumption space* for agent $i \in N$, where X_i is an arbitrary non-empty set. Let $A \subseteq X_1 \times X_2 \times \cdots \times X_n$ be the set of *feasible allocations*. Given $a \in A$, let a_i denote agent i's *consumption bundle*.

Each agent $i \in N$ has *preferences* over X_i, which are represented by a complete and transitive binary relation R_i. The strict preference relation associated with R_i is denoted by P_i. Let $\mathscr{R}_i$ denote the set of possible preferences for agent $i \in N$. A *domain* is denoted by $\mathscr{R} := \mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_n$. A *preference profile* is a list $R = (R_1, R_2, \ldots, R_n) \in \mathscr{R}$.

Let $LC_i(a; R_i) := \{b \in A \mid a_i R_i b_i\}$ be agent i's *lower contour set* of $a \in A$ at $R_i \in \mathscr{R}_i$. Let $UC_i(a; R_i) := \{b \in A \mid b_i R_i a_i\}$ be agent i's *upper contour set* of $a \in A$ at $R_i \in \mathscr{R}_i$.

A *social choice function* is a single-valued function $f \colon \mathscr{R} \to A$ that assigns a feasible allocation $a \in A$ to each preference profile $R \in \mathscr{R}$. Let f_i denote agent i's consumption bundle assigned by f.

Now we introduce a domain-richness condition. A domain $\mathscr{R}$ is *weakly monotonically closed* if, for all $i \in N$, all $R_i, R'_i \in \mathscr{R}_i$, and all $a, b \in A$ with $a_i = b_i$, there exists $\bar{R}_i \in \mathscr{R}_i$ such that $LC_i(a; R_i) \subseteq LC_i(a; \bar{R}_i)$ and $LC_i(b; R'_i) \subseteq LC_i(b; \bar{R}_i)$.

[2] It might not be necessary to worry about manipulation by very large coalitions, because it is difficult to coordinate actions of agents in such coalitions, as pointed out by Schummer (2000b) and Serizawa (2005). So, non-bossiness together with strategy-proofness might still seem strong, even if the additional assumption is imposed.

[3] In the Shapley–Scarf housing market with strict preferences, Takamiya (2001) has already shown that non-bossiness has relationships to Nash implementability. In other environments, however, the relationship of non-bossiness to Nash implementability is not yet known.

Note that every rich domain in the sense of Dasgupta et al. (1979) is weakly monotonically closed, but not vice versa. As shown by Dasgupta et al. (1979), for example, the domain of all quasi-linear preferences over public goods and transfers is not rich, but is weakly monotonically closed.

Next we introduce two properties of social choice functions. *Non-bossiness* (Satterthwaite and Sonnenschein (1981)) requires that if an agent changes her preferences but her consumption bundle is unchanged, then the bundle of each agent should be unchanged.

Definition 1 (Non-Bossiness). A social choice function f satisfies *non-bossiness* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$, if $f_i(R) = f_i(R'_i, R_{-i})$, then $f(R) = f(R'_i, R_{-i})$.

Monotonicity (Maskin (1999)) is a necessary condition for Nash implementation.

Definition 2 (Monotonicity). A social choice function f satisfies *monotonicity* if, for all $R, R' \in \mathscr{R}$, if $LC_i(f(R); R_i) \subseteq LC_i(f(R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$.

3 Results

In this section, we examine the relationships between non-bossiness and Nash implementability. We begin by looking at the relationships between non-bossiness and monotonicity.

Lemma 1. *Suppose that $\mathscr{R}$ is weakly monotonically closed. Then, if a social choice function f satisfies monotonicity, then it satisfies non-bossiness.*

Proof. Pick any $R \in \mathscr{R}$, any $i \in N$, and any $R'_i \in \mathscr{R}_i$ such that $f_i(R) = f_i(R'_i, R_{-i})$. We want to show $f(R) = f(R'_i, R_{-i})$.

Since $\mathscr{R}$ is weakly monotonically closed, we can choose $\bar{R}_i \in \mathscr{R}_i$ such that $LC_i(f(R); R_i) \subseteq LC_i(f(R); \bar{R}_i)$ and $LC_i(f(R'_i, R_{-i}); R'_i) \subseteq LC_i(f(R'_i, R_{-i}); \bar{R}_i)$. Since $LC_j(f(R); R_j) \subseteq LC_j(f(R); R_j)$ and $LC_j(f(R'_i, R_{-i}); R_j) \subseteq LC_j(f(R'_i, R_{-i}); R_j)$ for all $j \neq i$, monotonicity implies $f(\bar{R}_i, R_{-i}) = f(R)$ and $f(\bar{R}_i, R_{-i}) = f(R'_i, R_{-i})$, respectively. Thus, $f(R) = f(\bar{R}_i, R_{-i}) = f(R'_i, R_{-i})$. □

Lemma 1 says that monotonicity implies non-bossiness. However, non-bossiness alone cannot imply monotonicity; but, as shown in Lemma 2 below, non-bossiness together with *individual monotonicity* (Takamiya (2001)) can imply monotonicity.

Definition 3 (Individual Monotonicity). A social choice function f satisfies *individual monotonicity* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$, if $LC_i(f(R); R_i) \subseteq LC_i(f(R); R'_i)$, then $f_i(R'_i, R_{-i}) = f_i(R)$.

Remark 1. Individual monotonicity is weaker than monotonicity by definition.

Lemma 2. *If a social choice function f satisfies both non-bossiness and individual monotonicity, then it satisfies monotonicity.*

Proof. Pick any $R, R' \in \mathscr{R}$ such that $LC_i(f(R); R_i) \subseteq LC_i(f(R); R'_i)$ for all $i \in N$. We want to show $f(R') = f(R)$.

Step 1: $f(R'_i, R_{-i}) = f(R)$.
Since $LC_i(f(R); R_i) \subseteq LC_i(f(R); R'_i)$, individual monotonicity implies $f_i(R'_i, R_{-i}) = f_i(R)$. So, we have $f(R'_i, R_{-i}) = f(R)$ by non-bossiness.

Step 2: $f(R'_i, R'_j, R_{-i,j}) = f(R'_i, R_{-i})$.
Since $f(R) = f(R'_i, R_{-i})$ by Step 1, we have $LC_j(f(R'_i, R_{-i}); R_j) \subseteq LC_j(f(R'_i, R_{-i}); R'_j)$. So, individual monotonicity implies $f_j(R'_i, R'_j, R_{-i,j}) = f_j(R'_i, R_{-i})$. Therefore, we obtain $f(R'_i, R'_j, R_{-i,j}) = f(R'_i, R_{-i})$ by non-bossiness.

Iteration of these steps for remaining agents in N yields $f(R') = f(R)$. □

We are now ready to provide the result concerning the relationships between non-bossiness and monotonicity, which follows directly from Lemmas 1 and 2 and Remark 1.

Theorem 1. *Suppose that $\mathscr{R}$ is weakly monotonically closed. Then, a social choice function satisfies monotonicity if and only if it satisfies both non-bossiness and individual monotonicity.*

Remark 2. In classical pure exchange economies with continuous, strictly monotone, and strictly convex preferences, it is easy to show that Theorem 1 remains true if we replace monotonicity and individual monotonicity with *weak monotonicity*[4] and *individual weak monotonicity*,[5] respectively. So, non-bossiness plus individual weak monotonicity is equivalent to weak monotonicity in pure exchange economies. This improves upon the well-known result of Barberà and Jackson (1995), which states that non-bossiness together with *strategy-proofness*[6] implies weak monotonicity in pure exchange economies, because individual weak monotonicity is weaker than strategy-proofness in pure exchange economies.

The following is a direct corollary of Theorem 1.

[4]A social choice function f satisfies *weak monotonicity* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$, if (i) $UC_i(f(R); R'_i) \subseteq UC_i(f(R); R_i)$ and (ii) $a_i P_i f_i(R)$ for all $a \in UC_i(f(R); R'_i)$ with $a_i \neq f_i(R)$, then $f(R'_i, R_{-i}) = f(R)$.

[5]A social choice function f satisfies *individual weak monotonicity* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$, if (i) $UC_i(f(R); R'_i) \subseteq UC_i(f(R); R_i)$ and (ii) $a_i P_i f_i(R)$ for all $a \in UC_i(f(R); R'_i)$ with $a_i \neq f_i(R)$, then $f_i(R'_i, R_{-i}) = f_i(R)$.

[6]A social choice function f satisfies *strategy-proofness* if, for all $R \in \mathscr{R}$ and all $i \in N$, there is no $R'_i \in \mathscr{R}_i$ such that $f_i(R'_i, R_{-i}) P_i f_i(R)$.

Corollary 1 (Takamiya (2001)). *In the Shapley–Scarf housing market with strict preferences (Shapley and Scarf (1974)), a social choice function satisfies monotonicity if and only if it satisfies both non-bossiness and individual monotonicity.*

When coupled with the well-known results of Maskin (1999), Theorem 1 leads to the following corollaries, too.

Corollary 2. *Suppose that $\mathscr{R}$ is weakly monotonically closed. Then, if a social choice function is Nash implementable, then it satisfies both non-bossiness and individual monotonicity.*

Corollary 3. *Suppose that $n \geq 3$. Then, if a social choice function satisfies non-bossiness, individual monotonicity, and no veto power,[7] then it is Nash implementable.*

Corollary 2 implies that every social choice function that violates non-bossiness or individual monotonicity is never Nash implementable, provided that $\mathscr{R}$ is weakly monotonically closed. So, bossy social choice functions (e.g., the second-price auction (Vickrey (1961)), the pivotal mechanism (Clarke (1971)), the inversely dictatorial rule (Zhou (1991)), etc.) are never Nash implementable.

Corollaries 2 and 3 together indicate that non-bossiness has close relationships to Nash implementability, in the sense that non-bossiness is a necessary condition for Nash implementation and is part of the sufficient condition for Nash implementation. These relationships tell us that non-bossiness is desirable from the point of view of Nash implementability. The desirability of non-bossiness seems important in terms of requiring no additional assumption, which is in contrast to the desirability mentioned in the Introduction.

4 Conclusion

As pointed out by Satterthwaite and Sonnenschein (1981), non-bossiness is automatically satisfied in pure public goods economies, i.e., in economies with non-excludability and non-rivalness. It might show the universality of imposing non-bossiness. At the same time, however, it means that bossiness is characteristic of economies with excludability or rivalness, such as private goods economies, excludable public goods economies, and the commons. So, a negative aspect of imposing non-bossiness is that it rules out social choice functions inherent in economies with excludability or rivalness to identify such economies with pure public goods economies. In taking account of these, Corollary 2 shows an impossibility of Nash implementation in economies with excludability or rivalness, in the sense that

[7] A social choice function f satisfies *no veto power* if, for all $R \in \mathscr{R}$, all $a \in A$, and all $i \in N$, if $a_j R_j b_j$ for all $b \in A$ and all $j \neq i$, then $f(R) = a$.

the corollary indicates that in economies with excludability or rivalness, it is impossible to implement bossy social choice functions in Nash equilibria, which embody the characteristics inherent in those economies.

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