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Discussion Paper 06-14

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Abstract

This paper constructs a model that describes inflation cycles and prolonged depression as generated by the learning behavior of households who face a random liquidity shock in which money is needed. Households update the subjective probability of the shock based on the observation and change their liquidity preference accordingly. In this setting, we first derive a stationary cycles under perfect price adjustment, which is characterized by periods of gradual inflation and sudden sporadic falls of the price level. When the nominal stickiness is introduced, the liquidity shock is followed by a period of depression in which unemployment exists and deflation occurs gradually. Depression is deep and prolonged when the economy has experienced a long period of boom before encountering a liquidity shock.

JEL Classifications: D83, E41, E32

Keywords: Bayesian Learning in Continuous Time, Hamilton-Jacobi-Bellman Equations, Markov Modulated Poisson Processes, Partial Delay Differential Equations, Liquidity Preference.

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1 Introduction

This paper constructs a model that describes the cyclical movement of price level inflation and the possibility of prolonged depression as generated by the learning behavior of households. The representative household experiences a utility loss when a liquidity shock occurs, and the loss is smaller if she has a larger amount of money. The liquidity shock is randomly generated by a Poisson process, and the Poisson parameter, or the probability of the shock, changes unobservably following a Markov process (i.e., the shock occurs according to a certain Markov modulated Poisson process). The household updates the subjective probability of the shock based on the observation of the actual occurrences of the shock.[1]

In this setting, we first derive a stationary cycles under perfect price adjustment, which is characterized by the periods of gradual inflation and sudden sporadic deflations. The price level must jumps down when the liquidity shock occurs because the information delivered by the news of the liquidity shock is large enough to change the household's belief discretely. Such downward jumps are, however, unlikely to occur in the actual economies where various nominal frictions prevent the price level and/or the nominal wage to fall too rapidly; e.g., labor unions, moral issues, etc. Thus, we extend the benchmark model to incorporate the nominal stickiness that

[1] An example of such events is a bank run; even though people usually do business using checks, once a bank run occurs their transactions cannot be settled without money. This example suggests that the desire to hold money depends on the expectation about the possibility of such events in the future. When people observe a bank run, they guess that the underlying circumstance for banks is bad and therefore another bank may fail in near future. Thus, in this event they increase the demand for money to cope with another similar event. Given that the supply of nominal money cannot be changed instantly, the strong liquidity preference pushes up the value of money in terms of goods and therefore deflation occurs. Conversely, if banks operates successfully for a long time, people guess that the circumstance is stable and that it is unlikely for a bank run to occur in near future. Then, the money demand decreases, causing inflation.

puts an upper bound on the rate of deflation.

In the modified setting where price and thus the real money supply cannot jump, the representative household cannot increase the real money holding instantly when the subjective shock probability jumps up. The expected marginal utility of money holding stays high and the household decrease the amount of consumption to equalize the marginal rate of substitution between money and goods to the cost of holding money. This means that the liquidity shock causes a depression rather than a discrete fall in the price level. The drop in output is followed by a gradual recovery, in which price falls gradually. Once the price level falls to a certain threshold, the equilibrium in the goods and money markets is regained and then gradual inflation occurs.

The important finding is that the magnitude of depression is larger when the economy experiences a long period without the liquidity shock before encountering one because in that case price must fall significantly to regain the equilibrium in the money market. On the other hand, a economy with recurrent shocks harms little from an addition shock (e.g., a bank run) because the price level is already near the lowest level. The inflation rate immediately before the crush is not necessarily higher in the former case than the latter; specifically, the inflation rate converges to the rate of money growth (which should be the long-run average) when the economy luckily proceeds without a crush for a very long time. Thus, our result implies that an economy performing well for a long time is vulnerable to get into the long period of depression even when its contemporary inflation rate is low.

There exist a number of earlier studies that analyzed the macroeconomic behavior when an underlying state is only partially observable and information are revealed gradually (e.g., Andolfatto and Gomme 2003; Chalkley and Lee 1998; Horii and Ono 2005; Potter 2000; Sill and Wrase 1999). Those studies typically construct models in a way that the current belief regarding the underlying state determines macroeconomic variables at each date. This is also the case in our model when price adjustments are complete: i.e., the current belief on the risk of financial crises uniquely determines the current price level so that the demand and the supply of the real money balance equalize.

However, when nominal stickiness is introduced, the price level cannot completely follow the path that corresponds to the realized evolution of the belief. As a result, the current price level is determined not only by the current belief on the underlying state of the economy, but also by the history of beliefs that people held in the past. Accordingly, whether an economy experiences inflation and full employment, or deflation and unemployment, is not solely determined by how frequently financial shocks have occurred recently, but also by how the economy (i.e., the price level) has adapted to such occurrences.

The organization of the paper is as follows. In section 2, we describe the process of the liquidity shock and the evolution of the belief that is updated based on Bayes' law. Section 3 presents a benchmark model in which the price level is perfectly adjusted and shows the pattern of inflation cycles. The nominal stickiness is introduced in Section 4 to investigate the pattern of the crush and recovery. Section 5 concludes the paper. The proofs of all Lemmata are collated in Appendix.

2 Liquidity Shock and Bayesian Learning

We use a continuous-time model in which a representative household faces an aggregate liquidity shock that follows an exogenous Poisson process. Liquidity holding generates utility when the shock actually occurs, but does not while the shock does not occur. Since when the shock occurs cannot exactly be anticipated, even during the period without it the household holds liquidity so as to prepare for it.

There are two underlying states with different probabilities of the shock, called states H and L. In state $i \in \{H, L\}$ the shock occurs with probability θ^i per unit time, where $\theta^H > \theta^L > 0$. The household cannot directly observe the current state but knows that the state evolves according to a Markov process: state H changes to state L with Poisson probability p^H per unit time whereas state L changes to state H with probability p^L.

By observing whether the shock occurs or not she continuously revises her subjective shock probability in a Bayesian manner. Let θ_t denote the true shock probability at time t, which is unknown to her. Using information available up to time t, she forms a belief that $\theta_t = \theta^H$ with probability $f_t(\theta^H)$ and $\theta_t = \theta^L$ with probability $f_t(\theta^L)$. Obviously,

$$f_t(\theta^L) + f_t(\theta^H) = 1 \quad \text{for any } t. \tag{1}$$

In order to find how she updates $f_t(\theta^i)$ from t to $t + \Delta t$,[2] we first obtain the subjective probability that the shock does not occur between t and $t+\Delta t$ for given $f_t(\theta^i)$. It is denoted by $F_t\left[S_{(t,t+\Delta t]} = \phi\right]$, where $F_t[\cdot]$ is a probability operator based on information available at t, $S_{(a,b]}$ is the set of dates on which the shock actually occurs during $(a, b]$, and ϕ the empty set. Since the underlying state is either H or L at time $t+\Delta t$, this probability is divided into two components, $F_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^H\right]$ and $F_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^L\right]$.

Each of the two components is further divided into two probabilities. The former is the sum of the probability that 'the state is H at time t and neither the state change nor the shock occurs during the interval' and the probability that 'the present state is L and the state changes to H during the interval.' It is[3]

$$F_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^H\right] = \left(1 - (\theta^H + p^H)\Delta t\right) f_t(\theta^H) + p^L \Delta t f_t(\theta^L). \tag{2}$$

Similarly, the latter is

$$F_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^L\right] = \left(1 - (\theta^L + p^L)\Delta t\right) f_t(\theta^L) + p^H \Delta t f_t(\theta^H). \tag{3}$$

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[2] Time interval Δt is taken to be so short that the probability that the liquidity shock and a state change coexist in the interval is negligible.

[3] Throughout the paper we ignore the second-order term of Δt and higher because $\Delta t \to 0$.

Summing up (2) and (3) yields

$$F_t\left[S_{(t,t+\Delta t]} = \phi\right] = 1 - \theta_t^e \Delta t, \tag{4}$$

where θ_t^e represents the expected (or subjective) probability of the shock per unit time at time t:

$$\theta_t^e \equiv \theta^H f_t(\theta^H) + \theta^L f_t(\theta^L). \tag{5}$$

Let us consider how the representative household updates her belief if she eventually finds that the shock did not occur during $(t, t+\Delta t]$. In this case the information that $S_{(t,t+\Delta t]} = \phi$ is added to her knowledge. Thus, using Bayes' law we find updated subjective probability $f_{t+\Delta t}(\theta^i)$ to be

$$\begin{aligned} f_{t+\Delta t}(\theta^i) &\equiv F_{t+\Delta t}\left[\theta_{t+\Delta t} = \theta^i\right] = F_t\left[\theta_{t+\Delta t} = \theta^i | S_{(t,t+\Delta t]} = \phi\right] \\ &= \frac{F_t\left[S_{(t,t+\Delta t]} = \phi \cap \theta_{t+\Delta t} = \theta^i\right]}{F_t\left[S_{(t,t+\Delta t]} = \phi\right]}. \end{aligned}$$

Since the numerator is given by (2) or (3) and the denominator by (4), $f_{t+\Delta t}(\theta^H)$ equals[4]

$$f_{t+\Delta t}(\theta^H) = \frac{\left(1 - (\theta^H + p^H)\Delta t\right) f_t(\theta^H) + p^L \Delta t f_t(\theta^L)}{1 - \theta_t^e \Delta t}.$$

From this equation we derive the time derivative of $f_t(\theta^H)$:

$$\begin{aligned} \frac{df_t(\theta^H)}{dt} &= \lim_{\Delta t \to 0} \frac{f_{t+\Delta t}(\theta^H) - f_t(\theta^H)}{\Delta t} \\ &= (\theta_t^e - \theta^H - p^H) f_t(\theta^H) + p^L f_t(\theta^L). \end{aligned} \tag{6}$$

We next consider the case where the shock occurs during $(t, t+\Delta t]$. Since

$$F_t\left[S_{(t,t+\Delta t]} \neq \phi \cap \theta_{t+\Delta t} = \theta^i\right] = \theta^i f_t(\theta^i)\Delta t \quad \text{for} \quad i \in \{L, H\}, \tag{7}$$

the probability that the shock occurs is

$$F_t\left[S_{(t,t+\Delta t]} \neq \phi\right] = \left(\theta^H f_t(\theta^H) + \theta^L f_t(\theta^L)\right) \Delta t = \theta_t^e \Delta t, \tag{8}$$

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[4] $f_{t+\Delta t}(\theta^L)$ is analogously obtained. From (1) it equals $1 - f_{t+\Delta t}(\theta^H)$.

which is consistent with (4). From Bayes' law dividing (7) by (8) gives the updated subjective probability that $\theta_{t+\Delta t} = \theta^i$ under the condition that the shock occurs during $(t, t+\Delta t]$. It is

$$f_t(\theta^i) = \lim_{t' \to t-0} \frac{\theta^i f_{t'}(\theta^i)}{\theta^e_{t'}} \equiv \frac{\theta^i f_{t-0}(\theta^i)}{\theta^e_{t-0}}, \tag{9}$$

where subscript $t-0$ represents the state just before t.

Finally, we obtain the dynamics of subjective probability θ^e_t. From (1) and (5),

$$f_t(\theta^H) = \frac{\theta^e_t - \theta^L}{\theta^H - \theta^L}, \quad f_t(\theta^L) = \frac{\theta^H - \theta^e_t}{\theta^H - \theta^L}. \tag{10}$$

Substituting (6) and (10) into the time derivative of (5) yields the time derivative of θ^e_t in the case where the shock does not occur at time t:

$$\dot{\theta}^e_t = (\theta^e_t - \theta^L - p^L)(\theta^e_t - \theta^H - p^H) - p^L p^H \equiv g(\theta^e_t) \quad \text{for } t \notin S_{(0,\infty)}, \tag{11}$$

which satisfies

$$g(\theta) \lesseqgtr 0 \Longleftrightarrow \theta \gtreqless \theta^* \quad \text{for any } \theta \in \left[\theta^L, \theta^H\right], \quad \text{where}$$
$$\theta^* \equiv \frac{\theta^L + \theta^H + p^L + p^H - \sqrt{(\theta^H + p^H - \theta^L - p^L)^2 + 4p^L p^H}}{2} \in (\theta^L, \theta^H). \tag{12}$$

Similarly, by substituting (9) and (10) into (5) we obtain the value of θ^e_t as a function of θ^e_{t-0} in the case where the shock does occur at time t.

$$\theta^e_t = \theta^L + \theta^H - \frac{\theta^L \theta^H}{\theta^e_{t-0}} \equiv h(\theta^e_{t-0}) \quad \text{for } t \in S_{(0,\infty)}, \tag{13}$$

which satisfies

$$h(\theta^H) = \theta^H, \text{ and } \theta^e < h(\theta^e) < \theta^H \text{ for all } \theta^e \in (\theta^L, \theta^H).$$

Equations (11) and (13) describe the dynamics of θ^e_t with and without the shock respectively. It continuously declines as long as the shock does not occur, but discretely jumps upward when it occurs.[5] Intuitively, in the absence of the shock people

[5]From (11) and (13), we find that θ^e_t is trapped within interval $\left(\theta^*, \theta^H\right]$ in the long run. Since we are interested in the long-term behavior of the economy, it is assumed throughout this paper that θ^e_t is always within $\left(\theta^*, \theta^H\right]$. Under this assumption, θ^e_t always declines while the shock does not occur.

gradually become more and more optimistic and confident that the economy is in state L. Thus, their subjective probability of the shock gradually declines and converges to θ^*.[6] Due to the U-shape of function $g(\theta_t^e)$, the speed of adjusting their belief is slower when θ_t^e is near either θ^* or θ^H than when it is in the middle.

Conversely, when the shock is observed, people discretely change their expectation about the present state. Since $h(\theta^e) > \theta^e$, the more often people observe the shock, the more strongly people believe that they are in state H, and hence θ_t^e becomes closer to θ^H. In this way θ_t^e fluctuates between θ^* and θ^H.

3 Inflation Cycles under Perfect Price Adjustment

The economy is inhabited by a continuum of infinitely lived homogeneous households with measure one. Each household is endowed a unit labor at each point time and supplies labor inelastically to the labor market. A representative firm produces goods from labor. The output is $y\ell_t$, where $y > 0$ is a constant and ℓ_t is labor input. Since total labor supply is one, the aggregate output would be y each time as long as full employment obtains. The monetary authority issues a constant amount of nominal money stock, the size of which is normalized to one.[7] Goods are perishable and thus cannot be stored. The households will not borrow or lend among themselves because the are identical. The firm has no value because of its linear production technology and perfect competition. Therefore, money is the only asset in this economy.

At each date, the representative household gains utility $u(c_t)$ from consumption,

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[6] θ_t^e never becomes lower than $\theta^*(> \theta^L)$ since people take into account the possibility that state L might have changed to state H even though the shock does not occur.

[7] This assumption is made only for the simplicity of the description of the model and notations. The results to follow are essentially the same even when the nominal money growth rate is positive and constant. In that case the price level would not be stationary, and therefore we need to normalize the price level by by dividing by the nominal money supply.

where instantaneous felicity function $u(\cdot)$ is twice differentiable, $u'(\cdot) > 0$, and satisfies the Inada conditions. In addition, she experiences utility loss $v(m_t) < 0$ according to her real money holding m_t at the dates at which the liquidity shock occurs. We assume $v'(m) > 0$, which means that the size of utility loss is small when her real money holding is large. Function $v(\cdot)$ also satisfies $v''(m) < 0$, $\lim_{m\to 0} mv'(m) > 0$, and $\lim_{m\to\infty} v'(m) = 0$. Her expected utility EU_t is therefore given by

$$EU_t = E_t \left[\int_t^\infty u(c_\tau)e^{-\rho(\tau-t)}d\tau + \sum_{\tau\in S_{(t,\infty)}} v(m_\tau)e^{-\rho(\tau-t)} \right], \tag{14}$$

where ρ is her subjective discount rate and $S_{(t,\infty)}$ is the set of future dates at which the shock occurs.

Let p_t denote the price of consumption good, and W_t the nominal wage at t. The household maximize (14) under the budget constraint

$$\dot{M}_t = W_t - p_t c_t. \tag{15}$$

Since there is no friction in the market, full employment always obtains and thus $W_t = p_t y$ for all t. Thus, in the real terms, (15) becomes

$$\dot{m}_t = y - \pi_t m_t - c_t \quad \text{when } p_t \text{ changes continuously,} \tag{16}$$

$$m_t = m_{t-0}/\Pi_t \quad \text{when } p_t \text{ jumps,} \tag{17}$$

where $\pi_t \equiv \dot{p}_t/p_t$ and $\Pi_t \equiv p_t/p_{t-0}$.

There is no steady state in equilibrium at which the price level stays constant for all t because the expectation about the probability of the liquidity shock changes forever according to (11) and (13) and the decisions of household depend on θ_t^e. We instead search for a stationary equilibrium dynamics in which p_t evolves as a function of θ_t^e. Specifically, we search for a function $p(\cdot)$ that satisfies[8]

$$p_t = p(\theta_t^e) \quad \text{for all } t. \tag{18}$$

[8]This approach is similar to Lucas (1978).

Then, the inflation rate can also be written as a function of θ_t^e,

$$\pi_t = \frac{p'(\theta_t^e)}{p(\theta_t^e)} g(\theta_t^e) \equiv \pi(\theta_t^e), \quad \Pi_t = \frac{p(h(\theta_t^e))}{p(\theta_t^e)} \equiv \Pi(\theta_t^e). \tag{19}$$

We are interested in a monetary equilibrium path in which money has a positive value. Suppose that $p_{t_0} = \infty$ for some date t_0, which means that money has no value at t_0. Then, it follows that $p_t = \infty$ for all $t \geq t_0$ since otherwise an arbitrage opportunity arises: consumers can obtain an arbitrary amount of money at date t_0 at no cost and then sell money (i.e., purchase goods) at a date in which p_t is finite to increase their expected utility. Since θ_t^e evolves within (θ^*, θ^H) recurrently, (18) implies that if $p(\theta^\infty) = \infty$ for some $\theta^\infty \in (\theta^*, \theta^H)$ then $p(\theta^e) = \infty$ for all $\theta^e \in (\theta^*, \theta^H)$. That is, if there is such θ^∞, then $p_t = \infty$ for all t and therefore money is never demanded. Since the nonmonetary equilibrium dynamics is obvious and is of no interest here, we assume that[9]

Assumption 1 *$p(\theta^e) \in (0, \infty)$ for all $\theta^e \in (\theta^*, \theta^H)$.*

Let $U(\theta^e, m)$ denote the value function of a household with belief θ^e and real money holding m. Then, the bellman equation for the problem of (11), (13), (14), (16) and (17), combined with (18) and (19), is

$$\begin{aligned} U(\theta^e, m) = \max_c \Big[& u(c)\Delta t + (\theta^e \Delta t) v(m'') \\ & + \frac{1}{1+\rho\Delta t} \big\{ (1 - \theta^e \Delta t) U(\theta^{e\prime}, m') + (\theta^e \Delta t) U(h(\theta), m'') \big\} \Big], \end{aligned} \tag{20}$$

where $\theta^{e\prime} = \theta^e + g(\theta^e)\Delta t$, $m' = m + (y - \pi(\theta^e)m - c)\Delta t$, and $m'' = m/\Pi(\theta^e)$. Taking the limit $\Delta t \to 0$ in (20) yields the Hamilton-Jacobi-Bellman (HJB) equation for the problem:

$$\begin{aligned} \rho U(\theta^e, m) = \max_c \Big[& u(c) + \theta^e \big(v(m/\Pi(\theta^e)) + U(h(\theta^e), m/\Pi(\theta^e)) - U(\theta^e, m) \big) \\ & + g(\theta^e) U_\theta(\theta^e, m) + \big(y - \pi(\theta^e) m - c \big) U_m(\theta^e, m) \Big]. \end{aligned} \tag{21}$$

[9] We also rule out the possibility that $p(\theta^e) = 0$ for some $\theta^e \in (\theta^*, \theta^H)$ because the value of consumption good never becomes zero from $u'(\cdot) > 0$.

Differentiating the right hand side of (21) with respect to c gives the first order condition

$$u'(\widetilde{c}) = U_m(\theta^e, m), \tag{22}$$

where $\widetilde{c}$ denotes the optimal amount of consumption. Since θ^e_t and m_t evolves according to (11) and (16), equation (22) shows that the movement of consumption is characterized by

$$\frac{d}{dt}u'(\widetilde{c}_t) = g(\theta^e_t)U_{m\theta} + \big(y - \pi(\theta^e)m - \widetilde{c}\big)U_{mm} \quad \text{for } t \notin S_{(0,\infty)}, \tag{23}$$

abbreviating the arguments for $U(\cdot,\cdot)$ functions when they are (θ^e, m). From the envelope theorem, (21) can be differentiated with respect to m at $c = \widetilde{c}$ to give

$$\begin{aligned}(\rho + \pi(\theta^e) + \theta^e)U_m =& g(\theta^e)U_{\theta m} + \big(y - \pi(\theta^e)m - \widetilde{c}\big)U_{mm} \\ &+ \theta^e\Pi(\theta^e)^{-1}\big(v'(m/\Pi(\theta^e) + U_m(h(\theta^e), m/\Pi(\theta^e))\big).\end{aligned} \tag{24}$$

By substituting (22) and (23) for (24), we can eliminate the value function from it to obtain the Euler equation,

$$\frac{d}{dt}u'(\widetilde{c}_t) = (\rho + \pi(\theta^e) + \theta^e)u'(\widetilde{c}_t) - \theta^e\frac{v'(m_t/\Pi(\theta^e)) + u'(\widetilde{c}''_t)}{\Pi(\theta^e)} \quad \text{for } t \notin S_{(0,\infty)}, \tag{25}$$

where $\widetilde{c}''_t$ represents the optimal amount of consumption when the state changes to $(h(\theta^e_t), m/\Pi(\theta^e_t))$.

Since all households are symmetric, the goods and money markets clear when

$$\widetilde{c}_t = y,\ m_t = p(\theta^e_t)^{-1} \quad \text{for all } t. \tag{26}$$

Function $p(\cdot)$ is determined so that the household's demand for goods and money satisfies (26) given the path of price, (18). Substituting (26) into (25) yields a condition that must be satisfied for all possible values of θ^e,

$$\rho + \pi(\theta^e) = \theta^e\Pi(\theta^e)^{-1}v'(p(h(\theta^e))^{-1}) + \theta^e\left(\Pi(\theta^e)^{-1} - 1\right). \tag{27}$$

The left hand side represents the cost of holding money: the utility loss from postponing consumption plus the capital loss caused by inflation. In the other side are the expected benefits of holding money: the first term is the expected utility from holding money, whereas the second term represents the expected capital gain by the downward jump in the price level (the upward jump in the value of money) when the liquidity shock occurs. Thus, (27) shows that function $p(\cdot)$ is determined so that the cost and the benefit of holding money are equalized with each other.

From (19) and (27), we obtain a (delay) differential equation for $p(\cdot)$.

$$\begin{aligned} p'(\theta^e) &= \frac{p(\theta^e)}{g(\theta^e)}\gamma_p(\theta^e), \text{ where} \\ \gamma_p(\theta^e) &\equiv -(\rho+\theta^e) + \theta^e \frac{p(\theta^e)}{p(h(\theta^e))} \frac{v'(p(h(\theta^e))^{-1}) + u'(y)}{u'(y)}. \end{aligned} \tag{28}$$

Note that $\gamma_p(\theta^e)$ gives the growth rate of p_t (i.e., the inflation rate) that must hold in absence of the liquidity shock as a function of θ_t^e. Since functions u, v, g, h are already known, (28) is an autonomous differential equation with respect to function $p(\cdot)$. The following lemma gives a boundary condition with which function $p(\cdot)$ is pinned down.

Lemma 1 *Under Assumption 1 and transversality condition*[10]

$$\lim_{T\to\infty} E_t e^{-\rho(T-t)} u'(c_T) m_T = 0 \quad \textit{for all } t, \tag{29}$$

function $\gamma_p(\cdot)$ *must satisfy* $\lim_{\theta^e \to \theta^*} \gamma_p(\theta^e) = 0$.

proof: in appendix

Note that p_t converges to zero as $\theta_t^e \to \theta^*$ if $\gamma_p(\theta^e)$ is negative at the limit. Then the real money holding of a household, $m_t = p_t^{-1}$, increases unboundedly in a way that the transversality condition is violated. Conversely, the positive limiting value of $\gamma_p(\theta^e)$ means that p_t becomes arbitrarily large. In this case, the value of money vanishes in a finite time period, which violates the assumption of the monetary equilibrium.

[10] Operator E_t represents the expectation based on the information available to agents at date t.

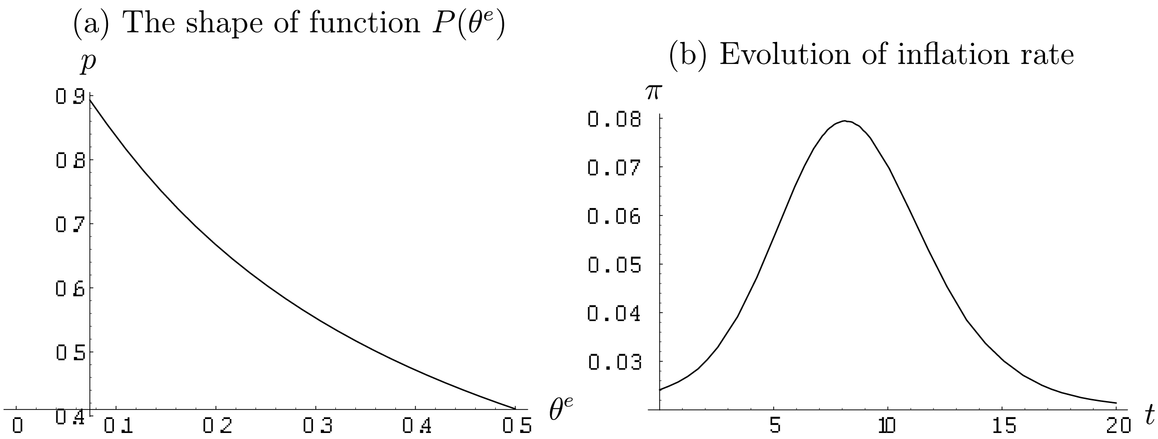

Figure 1: Inflation cycles without nominal frictions

The stationary dynamics of a monetary equilibrium can be calculated from (28) and the boundary condition given by Lemma 1. Panel (a) of Figure 1 shows the representative shape of function $p(\cdot)$ against θ^e, which is downward sloping.[11] A large value of θ_t^e means that people anticipates that the liquidity shock occurs with a high probability. In that situation, their liquidity preference is high. Thus, to clear the market for money, the value of money must be sufficiently high in relative to the value of good, which means a low price level.

During the period without the liquidity shock, θ_t^e gradually declines and p_t increases. Panel (b) of Figure 1 shows the evolution of inflation rate against time as θ_t^e moves from θ^H to θ^*. Inflation accelerates temporarily when the households adjusts their belief responding to observing no shock for a certain time length, but it gradually falls to the rate of nominal money growth, which is zero in this case, as the economy converges to the most optimistic state. When the liquidity shock occurs, θ_t^e jumps up. Then p_t jumps down so that the (θ_t^e, p_t) pair is always on the curve depicted in panel (a). Thus, the dynamics of the economy is characterized by

[11] In all examples presented in this paper, we specify $u(c) = \ln c$, $v(m) = -m^{-1}$, $y = 1$, $\rho = .05$, $\theta^H = .5$, $\theta^L = .05$, $p^H = .025$ and $p^L = .0025$. We have confirmed that our results are robust to changes in parameter values.

gradual inflation with sporadic and discrete falls in the price level.

At each event of the liquidity shock, price level must jump down in order to clear the increased liquidity demand induced by the change in people's belief. However, we rarely observe such a discrete fall in the price level in the aggregate economy; although we do sometimes observe a discrete fall in the prices of certain goods, the aggregated general price level tends to fall only slowly. One explanation for this is the existence of a (downward) nominal stickiness in the price level caused by labor unions, menu costs, moral issues, and the all other factors discussed in the literature. If the price cannot jump downward, our model predicts that the demand for money exceeds for the supply, and, by Warlas' law, a demand shortage occurs in the goods and labor market. The next section investigates this possibility.

4 Depressions with Nominal Stickiness

Consider an economy similar to the one introduced in the previous section, with a only difference in that the price level cannot fall faster than a certain rate,[12]

$$\dot{p}_t/p_t \geq -\delta, \quad \delta \in (0, \infty). \tag{30}$$

The discrete fall in the price level derived in the previous section implies that the instantaneous rate of inflation is minus infinity. Condition (30) rules out this possibility by assuming that δ is finite. The immediate consequence of this restriction is as follows. Even when the liquidity preference jumps up following the liquidity shock, the real money supply p_t^{-1} cannot increase instantly. The representative household then feels that money at their hands is more valuable than before, and consequently reduces the demand for goods. It creates a shortage in the goods market, which

[12]Condition (30) is equivalent to assuming that $\dot{W}_t/W_t \geq -\delta$ because the competition of firms ensures $p_t = yW_t$ for all t. We could also assume a symmetric restriction such as $\dot{p}_t/p_t \in [-\delta, \delta]$. This would make the analysis a little complicated without changing the final results.

cannot be cleared instantly because of the nominal stickiness in p_t.

This consideration implies that there are several differences in the pattern of economic dynamics from the case analyzed in the previous section. Now there is no one-for-one relationship between the price level and the belief because p_t cannot jump when θ^e_t changes discretely. Thus, the state of the economy is described not only by θ^e_t but by the pair of (θ^e_t, p_t). Consumption does not necessarily coincide with the full employment output y. We can imagine two possibilities at each point in time. First, constraint (30) is not binding and full employment obtains ($w_t = y$). The second possibility is that (30) is binding, i.e., $\dot{p}_t/p_t = -\delta$, and unemployment exists ($w_t < y$).

Which one of these possibilities occurs depends on the state of the economy, summarized by (θ^e_t, p_t). It is natural to guess that, for a given level of θ^e_t, there is a level of p_t at which the money market clears and full employment obtains. Let us denote this critical level by $\overline{p}(\theta^e_t)$. Price level p_t cannot be below the threshold $\overline{p}(\theta^e)$ since there is no upward stickiness in the price level and thus can be adjusted instantly if $p_t < \overline{p}(\theta^e)$. Similarly to the previous section, if there is some $\theta^\infty \in (\theta^*, \theta^H)$ such that $\overline{p}(\theta^\infty) = \infty$ then $p_t = \infty$ for all t. We rule out this trivial path by assuming that

Assumption 2 $\overline{p}(\theta^e) \in (0, \infty)$ *for all* $\theta^e \in (\theta^*, \theta^H)$.

Unemployment occurs when constraint (30) is binding, i.e., when $p > \overline{p}(\theta^e)$. In this case, the economy experience deflation at the rate of δ. If (30) is not binding, full employment obtains and the price level evolves so that equilibrium condition $p = \overline{p}(\theta^e)$ is maintained. Let us denote by $C(\theta^e, p)$ aggregate demand for goods at state (θ^e, p). Then,

$$C(\theta^e, p) \begin{cases} = y & \text{if } p = \overline{p}(\theta^e), \\ < y & \text{if } p > \overline{p}(\theta^e). \end{cases} \tag{31}$$

The inflation rate for a given state can be summarized as

$$\pi(\theta^e, p) = \begin{cases} \dfrac{\overline{p}'(\theta^e)}{\overline{p}(\theta^e)} g(\theta^e) \quad \text{if } p = \overline{p}(\theta^e), \\ \\ -\delta \quad \text{if } p > \overline{p}(\theta^e). \end{cases} \tag{32}$$

The representative household maximize (14) under budget constraint (15). Since the final good production is competitive and it uses only labor as input, the nominal income of the representative household coincides with the nominal aggregate demand for goods, $W_t = p_t C(\theta_t^e, p_t)$ for all t. The budget constraint can thus be written as

$$\dot{m}_t = C(\theta_t^e, p_t) - \pi(\theta_t^e, p_t) m_t + c_t \tag{33}$$

as long as p_t evolves continuously, and (17) if p_t jumps. From (30), price level p_t never jumps down, and it jumps up only when θ_t^e jumps up to $h(\theta_t^e)$ *and* p_t is smaller than the new market clearing price level, $\overline{p}(h(\theta_t^e))$. Let us denote the value function of the household by $U(\theta^e, p, m)$, which now depends on the current value of p because it affects the aggregate demand and thus her income. The Bellman equation for this problem is

$$\begin{aligned} U(\theta^e, p, m) = \max_c \Big[& u(c)\Delta t + (\theta^e \Delta t) v(m'') \\ & + \frac{1}{1 + \rho \Delta t} \big\{ (1 - \theta^e \Delta t) U(\theta^{e\prime}, p', m') + (\theta^e \Delta t) U(h(\theta^e), p'', m'') \big\} \Big], \end{aligned} \tag{34}$$

where $\theta^{e\prime} = \theta^e + g(\theta^e)\Delta t$, $p' = p + \pi(\theta^e, p) p \Delta t$, $m' = m + (C(\theta^e, p) - \pi(\theta^e, p) m + c)\Delta t$, $p'' = \max\{\overline{p}(h(\theta)), p\}$, and $m'' = (p/p'')m$. Taking the limit of $\Delta t \to 0$ in (34) yields the HJB equation,

$$\begin{aligned} \rho U = \max_c \Big[& u(c) + \theta^e \big(v(m'') + U(h(\theta^e), p'', m'') - U \big) + g(\theta^e) U_\theta \\ & + \pi(\theta^e, p) p U_p + \big(C(\theta^e, p) - \pi(\theta^e, p) m - c \big) U_m \Big], \end{aligned} \tag{35}$$

where the arguments of function $U(\cdot, \cdot, \cdot)$ and its partial derivatives are abbreviated when they are (θ^e, p, m). The first order condition for (35) is $u'(\widetilde{c}) = U_m(\theta^e, p, m)$, where $\widetilde{c}$ is the optimal amount of consumption. Then, the envelope condition is

$$\begin{aligned} (\rho + \pi(\theta^e, p) + \theta^e) U_m = & \theta^e (p/p'') \big(v'(m'') + U_m(h(\theta^e), p'', m'') \big) + g(\theta^e) U_{\theta m} \\ & + \pi(\theta^e, p) p U_{pm} + \big(C(\theta, p) - \pi(\theta^e, p) m - \widetilde{c} \big) U_{mm}. \end{aligned} \tag{36}$$

Note that the RHS of (36) depends on whether p jumps in the event of the liquidity shock. As long as function $\overline{p}(\cdot)$ is weakly downward sloping, $p \geq \overline{p}(\theta^e) \geq \overline{p}(h(\theta^e))$ and therefore $p'' = p$ and $m'' = m$. The following analysis focuses on this case and we leave for Appendix the analysis of the case of $p < \overline{p}(h(\theta^e))$.

Substituting the first order condition, its time derivative, and the conditions for the representative household, $\widetilde{c} = C(\theta^e, p)$ and $m = p^{-1}$, into (36) yields the Euler equation,

$$\left(\rho - \frac{\frac{d}{dt}u'(C(\theta^e, p))}{u'(C(\theta^e, p))}\right) + \pi(\theta^e, p) = \theta^e \frac{v'(p^{-1})}{u'(C(\theta^e, p))} + \theta^e \left(\frac{u'(C(h(\theta^e), p))}{u'(C(\theta^e, p))} - 1\right) \quad (37)$$

for all $t \notin S_{(0,\infty)}$. Equation (37) has an interpretation similar to (27). The cost of holding money, given by the LHS, is the sum of time preference and inflation. The benefit is the sum of the direct utility gain and the expected capital gain measured in terms of utility when a shock occurs and consumption jumps down.

Functions $\overline{p}(\cdot)$ and $C(\cdot, \cdot)$ are determined so that equation (37) holds for all possible pairs of (θ^e, p). Let us first consider the case of full employment, where $p = \overline{p}(\theta^e)$, $C(\theta^e, p) = y$ and $\pi(\theta^e, p) = \overline{p}'(\theta^e)g(\theta^e)/\overline{p}(\theta^e)$ from (31) and (32). Substituting these for (37) gives a differential equation that determines the form of function $\overline{p}(\cdot)$:

$$\begin{aligned} \overline{p}'(\theta^e) &= \frac{\overline{p}(\theta^e)}{g(\theta)}\gamma_{\overline{p}}(\theta^e), \quad \text{where} \\ \gamma_{\overline{p}}(\theta^e) &= -(\rho + \theta^e) + \theta^e \frac{v'\left(\overline{p}(\theta^e)^{-1}\right) + u'(C(h(\theta^e), \overline{p}(\theta^e)))}{u'(y)}. \end{aligned} \quad (38)$$

Function $\gamma_{\overline{p}}(\theta^e)$ in (38) represents the growth rate of $\overline{p}_t \equiv \overline{p}(\theta^e_t)$ as a function of θ^e_t assuming that $p \geq \overline{p}(\theta^e)$. The correct expression for $\gamma_{\overline{p}}(\theta^e)$ when $p < \overline{p}(\theta^e)$ is given by (43) in Appendix A. The difference between (28) and (38) lies in the fact that consumption is adjusted in the occurrence of the liquidity shock when nominal stickiness exists, while adjustment is done fully by the price level when price is completely flexible. A boundary condition for function $\overline{p}(\cdot)$ is given by the following lemma.

Lemma 2 *Under Assumption 2 and transversality condition (29), function* $\gamma_{\overline{p}}(\cdot)$ *must satisfy* $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) = 0$

proof: in appendix

Suppose that $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) < 0$, which implies that $\overline{p}(\theta^e)$ approaches 0 as θ^e converges to θ^*. This means that the level of p must also converges to 0 since it evolves according either to $\dot{p}/p = \overline{p}(\theta^e) < 0$ or to $\dot{p}/p = -\delta < 0$ as $\theta^e \to 0$. Lemma 1 shows that the transversality condition is violated in this case. In addition, the limiting value of $\gamma_{\overline{p}}(\theta^e)$ as $\theta^e \to \theta^*$ should not be positive since otherwise Assumption 2 is violated. Thus, function $\overline{p}(\cdot)$ must satisfy differential equation (38) and boundary condition $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) = 0$.

Next, consider the case of unemployment, where $\pi(\theta^e, p) = -\delta$. Substituting this for (37) gives a partial differential equation of function $C(\cdot, \cdot)$,

$$\begin{aligned} g(\theta^e) C_\theta(\theta^e, p) - p\delta C_p(\theta^e, p) &= \frac{u'(C(\theta^e, p))}{u''(C(\theta^e, p))} \gamma_{u'}(\theta^e, p), \text{ where} \\ \gamma_{u'}(\theta^e, p) &= \rho - \delta + \theta^e - \theta^e \frac{v'(p^{-1}) + u'(C(h(\theta^e), p))}{u'(C(\theta^e, p))}. \end{aligned} \tag{39}$$

In (39), $\gamma_{u'}(\theta^e, p)$ is the rate of change in marginal utility. Combined with the boundary condition $C(\theta^e, \overline{p}(\theta^e)) = y$ for all θ^e, this partial differential equation determines the shape of function $C(\cdot, \cdot)$ for all $(\theta^e, p) \in \{(\theta^e, p) | p > \overline{p}(\theta^e))\}$.

Since $\overline{p}(\cdot)$ and $C(\cdot, \cdot)$ are interrelated as described above, they are determined simultaneously so that they satisfy the system of partial differential equations, (38) and (39), along with two boundary conditions specified above. Figure 2 shows a representative shape of function $C(\cdot, \cdot)$ in (θ^e, p) space, where the solid curve on the edge represents function $\overline{p}(\cdot)$. Observe that function $\overline{p}(\cdot)$ is downward sloping in θ^e. The reason behind it is the same as the reason for the property of $p(\cdot)$ in the previous section: when θ^e is large, people's liquidity preference is high and thus a low price level (a high relative price of money to goods) is required to equalize the money demand to the money supply. The height of curved surface indicates the value of function $C(\theta^e, p)$ at each state in region $p > \overline{p}(\theta^e)$. $C(\theta^e, p)$ is equal to y

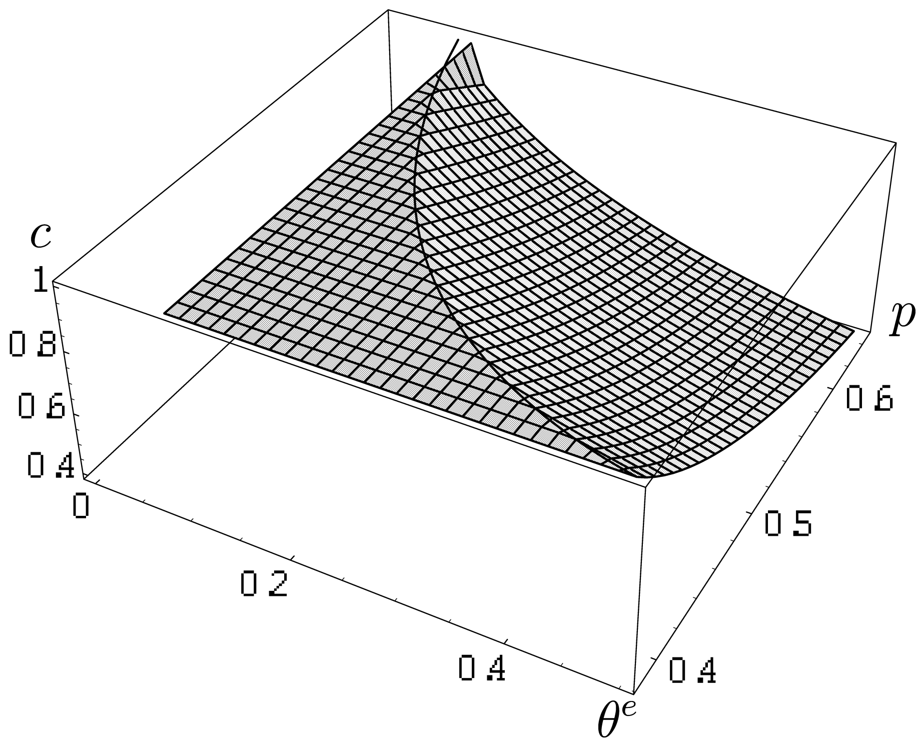

Figure 2: Representative shapes of function $C(\theta^e, p)$ and function $\overline{p}(\theta^e)$

on the curve of $p = \overline{p}(\theta^e)$ and gets smaller as the pair (θ^e, p) moves to the direction of north-east. That is, a pair of high θ_t^e and high p_t implies a combination of high liquidity preference, a low relative price of money to goods, and a small supply of real money stock. In that case, the excess demand for money is huge and therefore the aggregate demand for goods (and thus employment) is small.

The pattern of the dynamics of the economy is described as follows. If the current price level p_t is higher than the market clearing price level $\overline{p}(\theta_t^e)$, then price gradually falls and consumption grows according to

$$\frac{\dot{p}_t}{p_t} = -\delta, \quad \dot{c}_t = \frac{u'(c_t)}{u''(c_t)} \left[\rho - \delta + \theta^e - \theta_t^e \frac{v'\left(p_t^{-1}\right) + u'(c_t'')}{u'(c_t)} \right], \tag{40}$$

where $c_t'' \equiv C(h(\theta^e), p_t)$. As long as no shock occurs, the pair (θ^e, p_t) follows (40) until it reaches the market clearing line $p_t = \overline{p}(\theta_t^e)$ in a finite time. From that time on, consumption stays constant and the price level rises so that the pair traces the

(a) phase diagram in (θ^e, p) space (b) phase diagram in (c, p) space

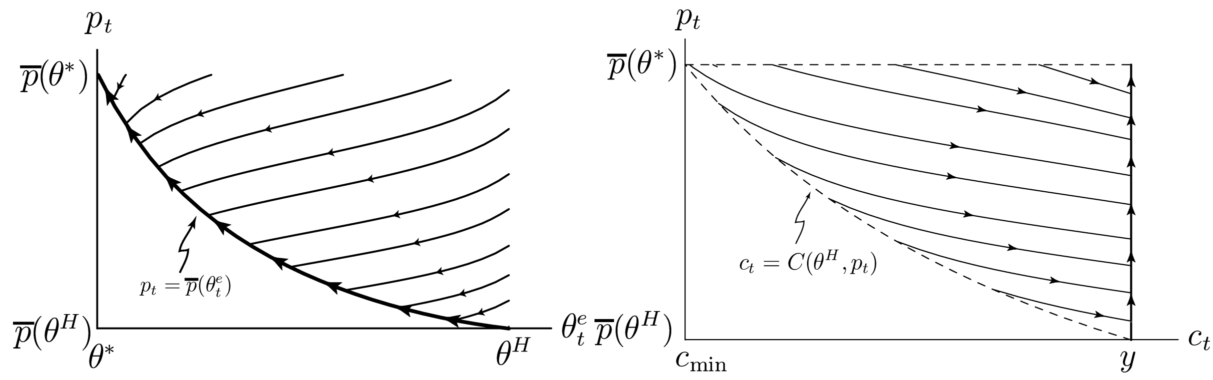

Figure 3: Inflation cycles with nominal stickiness

market clearing line,

$$\frac{\dot{p}_t}{p_t} = -(\rho + \theta_t^e) + \theta_t^e \frac{v'(p_t^{-1}) + u'(c_t'')}{u'(y)}, \quad c_t = y. \tag{41}$$

As θ^e approaches θ^*, the price level converges to $p^* \equiv \overline{p}(\theta^*)$, and inflation rate converges to zero.

Figure 3 shows the pattern of movement of (θ^e, p) pair. Once a liquidity shock occurs, the pair (θ^e, p_t) jumps toward east. Since p_t cannot jump immediately, the economy follows (40) until it reaches the market clearing line and then again follows (41). The liquidity shock may occur even before the economy goes back to the full employment phase. If the shock occurs many times in a short while, θ_t^e changes considerably while giving little time for the price level to adjust through deflation. Then, consumption must decrease by large and, moreover, it takes long time for the economy to regain full employment.

This problem becomes more serious if there had long been no shocks observed: in that case, the price level have gone up to near the highest level $p^* \equiv \overline{p}(\theta^*)$ before the shock occurs and thus the process of adjustment becomes a long way. Conversely, when an economy is experiencing the liquidity shock regularly, then p_t should always be near the lowest level $p^H \equiv \overline{p}(\theta^H)$ and thus it will take a short time for the price to adjust even after an avalanche of shocks. This mechanism explains why once a

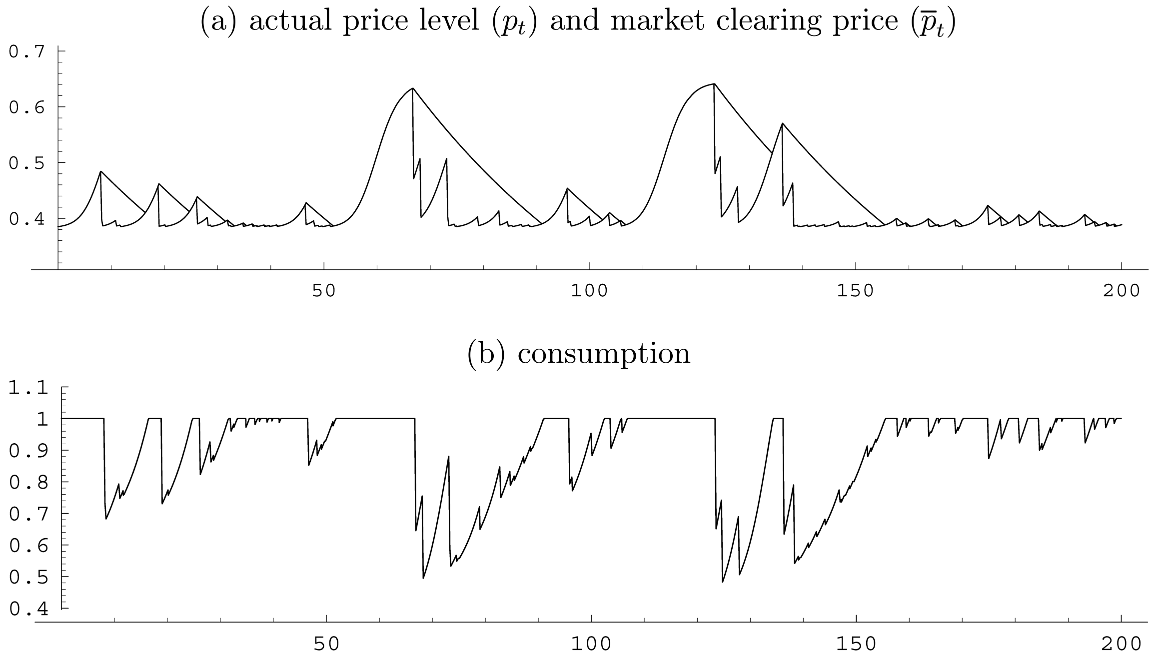

Figure 4: Simulated time paths of p_t, $\overline{p}_t \equiv \overline{p}(\theta_t^e)$, and c_t

long-time booming economy experiences large negative shocks it has to go through a long and deflationary period of depression, while recovery seems not so difficult when the boom period preceding that was a short one.

5 Conclusion

This paper presents a theory of inflation cycles and prolonged depression based on households' learning behavior. The model is constructed in a number of steps. First, we analyzed the process of Bayesian learning in continuous time by the representative household who observes a shock that follows a certain Markov modulated Poisson process. Second, using Hamilton-Jacobi-Bellman equations, we investigated the evolutions of consumption and money holding of the household who behaves rationally based on his belief about the state of the economy. Third, we derived a stationary cycle under perfect price adjustments in terms of a delay differential equation (or, sometimes called a difference-differential equation), and demonstrated that the price level would experience sporadic downward jumps in such a setting. Forth, we extended the model to incorporate nominal stickiness and derived stationary dynamics. In this case, since the belief and the slow-moving real variables cannot correspond one-to-one, we obtained a system of partial delay differential equations, by which we have shown that the economy experience a prolonged depression if and only if it have enjoyed a long tranquil periods before it experiences successive occurrence of shocks.

Appendix

A Analysis of the case of $p < \overline{p}(h(\theta^e))$

If $p_t < \overline{p}(h(\theta_t^e))$, $p'' = \overline{p}(h(\theta_T^e))$ and $m'' = mp/\overline{p}(h(\theta_t^e))$ in (34), (35) and (36). Substituting the first order condition, its time derivative, and the equilibrium conditions, $\widetilde{c} = C(\theta^e, p)$ and $m = p^{-1}$, into (36) yields the Euler equation,

$$\begin{aligned}\left(\rho - \frac{\frac{d}{dt}u'(C(\theta^e,p))}{u'(C(\theta^e,p))}\right) &+ \pi(\theta^e,p) \\ &= \theta^e \frac{p}{\overline{p}(h(\theta^e))}\frac{v'(\overline{p}(h(\theta^e))^{-1})}{u'(C(\theta^e,p))} + \theta^e\left(\frac{p}{\overline{p}(h(\theta^e))}\frac{u'(y)}{u'(C(\theta^e,p))} - 1\right)\end{aligned} \tag{42}$$

for all $t \notin S_{(0,\infty)}$. Substituting $p = \overline{p}(\theta^e)$, $C(\theta^e,p) = y$ and $\pi(\theta^e,p) = \overline{p}'(\theta^e)g(\theta^e)/\overline{p}(\theta^e)$, from (31) and (32), for (42) gives the growth rate of p_t during the period of full employment:

$$\gamma_{\overline{p}}(\theta^e) = -(\rho + \theta^e) + \theta^e \frac{\overline{p}(\theta^e)}{\overline{p}(h(\theta^e))}\frac{v'\left(\overline{p}(h(\theta^e))^{-1}\right) + u'(y)}{u'(y)}. \tag{43}$$

When unemployment exists (i.e., $p \geq \overline{p}(\theta^e)$), the rate of change in marginal utility is obtained by substituting $\pi(\theta^e,p) = -\delta$ for (42),

$$\gamma_{u'}(\theta^e,p) = \rho - \delta + \theta^e + \theta^e \frac{p}{\overline{p}(h(\theta^e))}\frac{v'\left(p^{-1}\right) + u'(y)}{u'(C(\theta^e,p))}. \tag{44}$$

B Proof of Lemmata

Let $\theta^{ns}_{t,T}$, $c^{ns}_{t,T}$, $p^{ns}_{t,T}$ and $m^{ns}_{t,T}$ denote respectively the values of θ_T, c_T, p_T and m_T conditional on that no shock occurs between t and T. Then, the probability that no shock occurs between t and T is given by $\exp\Big(-\int_t^T \theta^{ns}_t \, d\tau\Big)$.

The transversality condition (TVC) can be written as $\lim_{T\to\infty} E_t V_{t,T} = 0$, where $V_{t,T} \equiv e^{-\rho(T-t)} u'(c_T) m_T$ and E_t denotes the expectation taken upon the information available at t. Since $u'(c_T)m_T \geq 0$ for all T,

$$E_t V_{t,T} \geq \exp\left(-\int_t^T \theta^{ns}_t \, d\tau\right) e^{-\rho(T-t)} u'(c^{ns}_{t,T}) m_{t,T} \equiv V^{ns}_{t,T}. \tag{45}$$

Note that while $V_{t,T}$ is a random variable, $V^{ns}_{t,T}$ is a deterministic variable given the information available at t. From (45), a necessary condition for the TVC is

$$\lim_{T\to\infty} V^{ns}_{t,T} \leq 0. \tag{46}$$

Differentiating (45) with respect to T and using equilibrium condition $m^{ns}_{t,T} = 1/p^{ns}_{t,T}$ yield

$$\frac{dV^{ns}_{t,T}/dT}{V^{ns}_{t,T}} = -\theta^{ns}_{t,T} - \rho + \frac{du'(c^{ns}_{t,T})/dT}{u'(c^{ns}_{t,T})} - \frac{dp^{ns}_{t,T}/dT}{p^{ns}_{t,T}}. \tag{47}$$

Lemma 1

Without nominal stickiness, $c^{ns}_{t,T} = y$ for all T and thus $du'(c^{ns}_{t,T})/dT = 0$. From (28), $(dp^{ns}_{t,T}/dT)/p^{ns}_{t,T} = \gamma_p(\theta^{ns}_T)$. Substituting these into (47) yields

$$\frac{dV^{ns}_{t,T}}{dT} = -\theta^{ns}_{t,T} \frac{p(\theta^{ns}_{t,T})}{p(h(\theta^{ns}_{t,T}))} \frac{v'(p(h(\theta^{ns}_{t,T}))^{-1}) + u'(y)}{u'(y)} V^{ns}_{t,T}. \tag{48}$$

Using $p(\theta^{ns}_{t,T}) = p^{ns}_{t,T} = 1/m^{ns}_{t,T}$ and the definition of $V^{ns}_{t,T}$ in (45), equation (48) reduces to $dV^{ns}_{t,T}/dT = -\exp(-\int_t^T \rho + \theta^{ns}_{t,v} \, dv)\theta^{ns}_{t,T} Z(h(\theta^{ns}_{t,T}))$, where $Z(\theta) \equiv \left(v'(p(\theta)^{-1}) + u'(y)\right)/p(\theta)$. Integrating this differential equation with respect to T from t to ∞ and using the fact that $V^{ns}_{t,t} = u'(y)m_t$ give

$$\lim_{T\to\infty} V^{ns}_{t,T} = u'(y)m_t - \int_t^\infty \exp\left(-\int_t^T \rho + \theta^{ns}_{t,v} \, dv\right) \theta^{ns}_{t,T} Z(h(\theta^{ns}_{t,T})) \, dT. \tag{49}$$

Fix a small constant $a > 0$ and define a closed interval $A \equiv [h(\theta^*), h(\theta^* + a)] \in (\theta^*, \theta^H)$. Note that Assumption 1 implies that $Z(\theta) \in (0, \infty)$ for all $\theta \in (\theta^*, \theta^H)$. In addition, it is continuous in this interval because (28) implies that $p(\cdot)$ is differentiable. Thus, there exist finite constants $Z_{\min} \equiv \min_{\theta \in A} Z(\theta) \in (0, 1)$ and $Z_{\max} \equiv \max_{\theta \in A} Z(\theta) \in (0, 1)$. Whenever $\theta_t^e \in (\theta^*, \theta^* + a)$, $\theta_{t,T}^{ns} \in (\theta^*, \theta^* + a)$ for all $T \geq t$. and therefore there is upper and lower bounds for the second term in the RHS of (49), given by

$$\left(\frac{\theta^* Z_{\min}}{\rho + \theta^* + a}, \frac{(\theta^* + a) Z_{\max}}{\rho + \theta^*} \right) \equiv (I_{\min}, I_{\max}) \subset (0, \infty). \tag{50}$$

Now suppose that $\lim_{\theta^e \to \theta^*} \gamma_p(\theta^e) < 0$. Then as θ_t^e converges to θ^*, $p_t \to 0$ and therefore $m_t \to \infty$. However, this violates the TVC since conditions (46), (49) and (50) imply that the TVC requires $m_t \leq I_{\max}/u'(y)$ whenever $\theta_t^e \in (\theta^*, \theta^* + a)$.

Suppose conversely that $\lim_{\theta^e \to \theta^*} \gamma_p(\theta^e) < 0$. Then as θ_t^e converges to θ^*, $p_t \to \infty$ and therefore $m_t \to 0$. For sufficiently small m_t, (49) and (50) imply $\lim_{T \to \infty} V_{t,T}^{ns} < 0$. Since $V_{t,t}^{ns} = u'(c_t) m_t > 0$ and $V_{t,T}^{ns}$ is continuous in T, there should be a value of $T \geq t$ such that $V_{t,T}^{ns} = 0$. From the definition of $V_{t,T}^{ns}$ in (45) this implies that $m_{t,T}^{ns} = 0$ and therefore $p(\theta_{t,T}^{ns}) = p_{t,T}^{ns} = \infty$, violating Assumption 1.

Lemma 2

We first derive a contradiction under assumption $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) < 0$. Fix $a > 0$ and define $A = [h(\theta^*), h(\theta^* + a)]$. Then, from $\overline{p}(\theta) \in (0, \infty)$ and its continuity, there exists $\overline{p}_{\min} \equiv \min_{\theta \in A} \overline{p}(\theta) \in (0, \infty)$. The assumption $\lim_{\theta^e \to \theta^*} \gamma_{\overline{p}}(\theta^e) < 0$ implies that $\overline{p}(\theta) \to 0$ as $\theta \to \theta^*$. Recall, in addition, that p_t falls at the rate of δ whenever $p_t > \overline{p}(\theta^*)$. Thus, there is a positive probability that $(\theta_t^e,\ p_t)$ pair satisfies $\theta_t^e < \theta^* + a$ and $p_t \leq \overline{p}_{\min}$ when the liquidity shock does not occur for a sufficiently long while.

Suppose that the current $(\theta_t^e,\ p_t)$ pair satisfies the above inequalities. Then $p_{t,T}^{ns} < \overline{p}(h(\theta_{t,T}^{ns}))$ for all $T \geq t$, which means that the analysis in Appendix A applies.

Substituting the results obtained in Appendix A into (47) yields

$$\frac{dV_{t,T}^{ns}/dT}{V_{t,T}^{ns}} = \begin{cases} -\theta_{t,T}^{ns} - \rho + \gamma_{u'}(\theta_{t,T}^{ns}) + \delta & \text{if } p_{t,T}^{ns} > \overline{p}(\theta_{t,T}^{ns}), \\ -\theta_{t,T}^{ns} - \rho - \gamma_{\overline{p}}(\theta_{t,T}^{ns}) & \text{if } p_{t,T}^{ns} = \overline{p}(\theta_{t,T}^{ns}). \end{cases} \tag{51}$$

Substituting (43) and (44) into (51) and using $p_{t,T}^{ns} = 1/m_{t,T}^{ns}$ and the definition of $V_{t,T}^{ns}$ in it, equation (51) reduces to $dV_{t,T}^{ns}/dT = -\exp(-\int_t^T \rho + \theta_{t,v}^{ns}\, dv)\theta_{t,T}^{ns}\overline{Z}(h(\theta_{t,T}^{ns}))$, where $\overline{Z}(\theta) \equiv \left(v'(\overline{p}(\theta)^{-1}) + u'(y)\right)/\overline{p}(\theta)$. Integrating this differential equation with respect to T from t to ∞ and using the fact that $V_{t,t}^{ns} = u'(c_{t,T}^{ns})m_t$ give

$$\lim_{T\to\infty} V_{t,T}^{ns} = u'(c_{t,T}^{ns})m_t - \int_t^\infty \exp\left(-\int_t^T \rho + \theta_{t,v}^{ns}\, dv\right)\theta_{t,T}^{ns}\overline{Z}(h(\theta_{t,T}^{ns}))\, dT. \tag{52}$$

Note that $h(\theta_{t,T}^{ns}) \in A$ for all $T \geq t$ and that there exists a finite constant $\overline{Z}_{\max} \equiv \max_{\theta\in A}\overline{Z}(\theta)$. From $c_{t,T}^{ns} \leq y$, $u'(c_{t,T}^{ns}) \geq u'(y)$ for all T. Thus (46) and (52) jointly imply that

$$m_t \leq \frac{(\theta^* + a)\overline{Z}_{\max}}{(\rho + \theta^*)u'(y)}. \tag{53}$$

While assumption $\lim_{\theta^e\to\theta^*}\gamma_{\overline{p}}(\theta^e) < 0$ implies that an arbitrarily large $m_t = 1/p_t$ realizes with a positive probability, the RHS of (53) is constant. Thus (53) and hence the TVC will be violated with a positive probability.

Next, assume conversely that $\lim_{\theta^e\to\theta^*}\gamma_{\overline{p}}(\theta^e) > 0$, which means that $\overline{p}(\theta^e)$ become arbitrarily large as $\theta^e \to \theta^*$. Then, $\theta_{t,T}^{ns} \in (\theta^*, \theta^* + a)$ and $p_{t,T}^{ns} = \overline{p}(\theta_{t,T}^{ns}) > \overline{p}_{\max} \equiv \max_{\theta\in A}\overline{p}(\theta)$ for sufficiently large T. In this case, Analysis in Section 4 applies and full employment obtains. From $m_{t,T}^{ns} = 1/p_{t,T}^{ns}$ and (38),

$$\frac{dm_{t,T}^{ns}}{dT} = (\rho + \theta_{t,T}^{ns})m_{t,T}^{ns} - \theta_{t,T}^{ns}\frac{v'(m_{t,T}^{ns})m_{t,T}^{ns} + u'(C(h(\theta_{t,T}^{ns}), 1/m_{t,T}^{ns}))m_{t,T}^{ns}}{u'(y)} \tag{54}$$

for sufficiently large T. As $T \to \infty$, $\overline{p}(\theta_{t,T}^{ns}) \to \infty$ and therefore $m_{t,T}^{ns} \to 0$. In this case, (54) implies $\lim_{T\to\infty} dm_{t,T}^{ns}/dT < \theta_T^{ns}u'(y)^{-1}\lim_{m\to 0} v'(m)m < 0$, where the latter inequality follows from the definition of $v(\cdot)$. These properties jointly imply that there is a finite T such that $m_{t,T}^{ns} = 0$ and therefore $\overline{p}(\theta_{t,T}^{ns}) = p_{t,T}^{ns} = \infty$, violating Assumption 2.

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