Discussion Papers In Economics And Business

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June 2006

この研究は「大学院経済学研究科・経済学部記念事業」
基金より援助を受けた、記して感謝する。

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Comparative Risk Aversion under Background Risks Revisited*

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Abstract

This note determines a sufficient condition on (von Neumann–Morgenstern) utility functions to preserve (reserve) comparative risk aversion under general background risks. Our condition is weaker than the one determined by Nachman (1982, *Journal of Economic Theory*). Nachman's condition requires the monotonicity in the global sense, in other hand our condition only requires it in the local sense. And this generalization may make the condition on utility functions to hold the desirable property consisitent with the recent empirical observation.

JEL Classification: D81.
Keywords: Background risk, comparative risk aversion, single crossing condition.

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*The authors greatly acknowledge the financial support from the MEXT of the Japanese government (Ohnishi) and the JSPS Fellowships for Young Scientists (Osaki).

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1 Introduction

It is doubtless that seminal papers by Pratt (1964) and Arrow (1971) are most important contributions to decision analysis under expected utility theory. They introduced a notion of risk aversion and an associated order of (von Neumann-Morgenstern) utility functions. Comparative risk aversion, which is the order of risk aversion, has unambiguous qualitative properties for many decision making problems such as asset demand, insurance demand and other many problems. While comparative risk aversion has many intuitive properties for decision making under single source of a risk, it is well known that it has many counterintuitive ones for decision making under multiple sources of risks. Since most decision making under uncertainty involves multiple sources of risks, these properties have been taken special interests. Over the last two decades, a considerable number of studies have been devoted to the investigations of conditions on utility functions to obtain intuitive qualitative properties for decision making problems with multiple sources of risks.

Most decision making problems with multiple sources of risks deal with the situations in the presence of double sources of risks: One risks are endogenous and other risks are exogenous. Exogenous risks are usually called background risks. Two counterintuitive properties of risk aversion under background risks are well known: The first is that comparative risk aversion under background risks may be reversed, and the second is that risk aversion under background risks may decrease. We are concerned with the first property. Interseted readers can see Part IV in Gollier (2001) for an excellent survey about researches concerning the second property. Nachman (1982) derived a condition on utility functions for the preservation of comparative risk aversion under background risks with general payoff functions. Independently, Kihlstrom, Romer, and Williams (1981) derived the same condition for background risks with additive payoff functions. Hence their result is a special case of Nachman (1982). And then, Pratt (1988) obtained a necessary and sufficient condition on utility functions in the additive background risk case. Decreasing Absolute Risk Aversion (DARA) is a sufficient condition for background risks with additive payoff functions. DARA has been viewed as a relevant property for utility functions for a long time. However, Jackwerth (2000) observed U-shaped risk aversion implied by option prices written on index portfolios. DARA is not consistent with this empirical observation. On the other hand, we cannot imagine shapes of utility functions via the condition derived by Pratt (1988) because his condition is a technical one. The above things suggest that we need the further theoretical investigation of the conditions on utility functions to preserve comparative risk aversion under background risks, which have the possibility of the consistency with Jackwerth's empirical observations (2000). We consider the background risks with general payoff functions, and obtain a weaker sufficient condition than Nachman's one (1988). Our condition for background risk with additive payoff functions requires only the monotonicity of risk aversion in a local sense, hence it may be consistent with Jackwerth's empirical observation (2000).

The organization of the paper is as follows. In section 2, we give some preliminaries for the analysis. In section 3, we determine the condition on utility functions to preserve (reserve) comparative risk aversion under background risks and compare our result to Nachman's one. In section 4, we give the condition in the case that payoff functions have the additive form, and discuss the relation between our theoretical result and the recent empirical observation. In section 5, our results are also applied in the stochastic dominance changes in background risks. In last section, we give concluding remarks.

2 Preliminaries

Since our setting is identical with Nachman (1982), we borrow his notation. Let us consider a utility function $u : \mathbb{R} \to \mathbb{R}$ of a Decision Maker (DM). The utility function u is a strictly increasing function, and higher order derivatives required in the analysis assumed to exist. We note that the DM is not necessarily risk averter, that is concavity of the utility function is not required for the analysis. Let us consider a payoff function $g : X \times Y \to \mathbb{R}$. The payoff function g is a strictly increasing function of x. $x \in X$ is a realization of a decision variable and $y \in Y$ is that of an exogenous variable. The exogenous risk $\tilde{y}$ called background risk, is a random variable followed by a probability measure m defined over support Y. As considering financial markets with one risk–free asset and one risky asset as an example, endogenous risks are market portfolios, and exogenous risks stand for nontraded labor income risks (Weil; 1992).

Let us define the derived utility function as

$$v(x) := \int_Y u(g(x,s))m(ds). \tag{1}$$

The derivatives of the derived utility function are written as follows:

$$v'(x) = \int_Y g_x(x,y)u'(g(x,y))m(dy), \tag{2}$$

$$v''(x) = \int_Y \{g_{xx}(x,y)u'(g(x,y)) + g(x,y)^2 u'(g(x,y))\}m(dy), \tag{3}$$

where prime denotes derivatives, and g_x and g_{xx} denote the first and second partial derivatives of g with respect to x. The derived utility function v is an strictly increasing function of x by the above assumptions, $g_x > 0$ and $u' > 0$.

Let us define the function

$$h(x,y) := g_x(x,y)\mathcal{A}(g(x,y);u) + \mathcal{A}((x,y);g), \tag{4}$$

where $\mathcal{A}(u) := -u''/u'$, and $\mathcal{A}(g) := -g''/g'$. Recall that $\mathcal{A}(u)$ is the Arrow–Pratt absolute risk aversion of the utility function u (Pratt; 1964, and Arrow; 1971). Using the function

$$\hat{m}(y) := \frac{g_x(x,y)u(g(x,y))m(y)}{\int_Y g_x(x,s)u(g(x,s))m(ds)}, \tag{5}$$

the Arrow–Pratt absolute risk aversion of the utility function u under the background risk, or equivalently that of the utility function v, can be rewritten as

$$\mathcal{A}(x; v) = -\frac{v''(x)}{v'(x)} = \int_Y h(x, s)\hat{m}(ds). \tag{6}$$

We note that the function $\hat{M}(y) := \int^y \hat{m}(s)ds$ can be viewed as the cumulative distribution function defined over support Y, since $\hat{m}(y) > 0$ for all $y \in Y$ and $\int_Y \hat{m}(s)ds = 1$. In the case the function $h(x, y) = h(x)$ is a constant function for y, it is clear that the Arrow–Pratt absolute risk aversion under background risk is constant for all $\tilde{y}$, that is $\mathcal{A}(x, v) = h(x)$ for all $\tilde{y}$. This is a slight generalization of Proposition 1 in Gollier and Schlesinger (2003).

3 Main Result

Pratt (1964) and Arrow (1971) introduced the notion of comparative risk aversion defined as follows: u_1 is more risk–averse than u_2, if $\mathcal{A}(x; u_1) \geq \mathcal{A}(x; u_2)$. We denote this as $u_1 \geq_A u_2$. The goal of the paper is to give a sufficient condition to guarantee comparative risk aversion under background risks, $v_1 \geq_A v_2$. Our sufficient condition is weaker than Nachman's one (1982), hence our result is a generalization of his analysis.

3.1 Theorem

Before giving a theorem, we define the notion of the single crossing condition: a function $f : X \to \mathbb{R}$ satisfies the single crossing condition at x_0 from the above (below), if there exists $x_0 \in X$ such that $f(x) \geq (\leq)\ 0$ for all $x \leq x_0$ and $f(x) \leq (\geq)\ 0$ for all $x_0 \leq x$. And, we define $y_0 \in Y$ as follows: There exists y_0 such that $h(x, y_0) = \int_Y h(x, s)\hat{m}(ds) (= \mathcal{A}(x; v))$. The following theorem is our main result.

Theorem 3.1. Assume that $g(x, y)$ is an increasing function of y for all $x \in X$. If $u_1 \geq_A\ (\leq_A)\ u_2$ and either of functions $h_i(x, y) - h(x, y_0)$ satisfies the single crossing condition at y_0 from the above (below), then $v_1 \geq_{\mathrm{A}} v_2$.

3.2 Two Lemmas

For the preparation of the proof, we give the following two lemmas. A similar result to the first lemma was obtained such as Osaki (2005), Ohnishi and Osaki (2006), and others in various contexts. We note that Pratt (1988) obtained a similar result, and gave a different proof for the result of Kihlstrom, *et al.* (1981) using the relation of stochastic dominance. Before providing the first lemma, we define a notion of stochastic dominance, the Monotone Likelihood Ratio Dominance (MLRD). For the sake of simplicity, we consider that two random variables have a same support X.

Definition 3.1. $M(2)$ dominates $M(1)$ in the sense of MLRD, if $m(x; 2)/m(x; 1)$ is increasing in $x \in X$. We denote this as $M(1) \leq_{\mathrm{MLRD}} M(2)$.

Lemma 3.1. $u_1 \geq_A (\leq_A)\ u_2$, if and only if $\hat{M}(1) \leq_{\mathrm{MLRD}} (\geq_{\mathrm{MLRD}})\hat{M}(2)$.

Proof. Since the proof is similar, we give only the first case: $u_1 \geq_A u_2$, if and only if $\hat{M}(1) \leq_{\text{MLRD}} \hat{M}(1)$.

It follows a straightforward calculation that

$$\frac{\partial}{\partial y}\left(\frac{u_2'(g(x,y))}{u_1'(g(x,y))}\right) = \frac{g_y(x,y)}{\{u_1'(g(x,y)\}^2}(u_1'(g(x,y))u_2''(g(x,y)) - u_1''(g(x,y))u_2'(g(x,y))). \quad (7)$$

Since $g_y(x,y)/\{u_1'(g(x,y)\}^2 \geq 0$,

$$\text{sgn}\left\{\frac{\partial}{\partial y}\left(\frac{u_2'(g(x,y))}{u_1'(g(x,y))}\right)\right\} = \text{sgn}\{\mathcal{A}(g(x,y);u_1) - \mathcal{A}(g(x,y);u_2)\}. \quad (8)$$

Therefore, $\mathcal{A}(g(x,y);u_1) \geq \mathcal{A}(g(x,y);u_2)$, if and only if $u_2'(g(x,y))/u_1'(g(x,y))$ is an increasing function of y, or equivalently

$$\frac{u_2'(g(x,y))}{u_1'(g(x,y))} \leq \frac{u_2'(g(x,z))}{u_1'(g(x,z))} \quad (9)$$

for all $y, z \in Y$ with $y \leq z$ because of $g(x,y)$ being an increasing function of y. On the other hand, we have that

$$\frac{\hat{m}(y;2)}{\hat{m}(y;1)} \leq \frac{\hat{m}(z;2)}{\hat{m}(z;1)} \Leftrightarrow \frac{g_x(x,y)u_2'(g(x,y))m(y)}{g_x(x,y)u_1'(g(x,y))m(y)} \leq \frac{g_x(x,z)u_2'(g(x,z))m(z)}{g_x(x,z)u_1'(g(x,z))m(z)} \quad (10)$$

$$\Leftrightarrow \frac{u_2'(g(x,y))}{u_1'(g(x,y))} \leq \frac{u_2'(g(x,z))}{u_1'(g(x,z))}. \quad (11)$$

Combining the above two discussions, we obtain the following:

$$\mathcal{A}(g(x,y);u_1) \geq \mathcal{A}(g(x,y);u_2) \Leftrightarrow \frac{\hat{m}(y;2)}{\hat{m}(y;1)} \leq \frac{\hat{m}(z;2)}{\hat{m}(z;1)} \ \forall\ y, z \in Y \text{ with } y \leq z. \quad (12)$$

Hence we complete the proof. □

The following lemma is known as the variation diminishing property. Hence we give the following lemma without a proof. It was given by Karlin and Novikoff (1963), and Karlin (1968). Jewitt (1987) and Athey (2002) discussed some economic applications. Before giving the lemma, we give the definition of TP_2 function as follows: $h : X \times Y \to \mathbb{R}$ is a TP_2 function of x and y, if for $x_1 \leq x_2$ and $y_1 \leq y_2$, $h(x_1,y_2)h(x_2,y_1) \leq h(x_1,y_1)h(x_2,y_2)$.[1)]

Lemma 3.2. Let us consider that a function $g : \mathbb{R} \to \mathbb{R}$ satisfies the single crossing condition at x_0 from above (below), that is, there exists $x_0 \in X$ such that $g(x) \geq (\leq)\ 0$ for $x \leq x_0$ and $g(x) \leq (\geq)\ 0$ for $x_0 \leq x$. If a function $h(x;i) : X \to \mathbb{R}$ is a TP_2 fuction for x and i, then $\int_X g(s)h(s;1)dt = 0$ implies $\int_X g(s)h(s;2)dt \leq (\geq)\ 0$.

[1)] TP_2 is also called log–supermodularity.

3.3 Proof

In this subsection, we give a proof of the Theorem 3.1. The proof is a similar way, hence we give only the proof for the first statement: If $u_1 \geq_A u_2$ and either of functions $h_i(x,y) - h(x,y_0)$ satisfies the single crossing condition at x_0 from the above, then $v_1 \geq_{\text{A}} v_2$.

First, let us consider the case: $h_1(x,y) - h(x,y_0)$ satisfies the single crossing condition at x_0 from the above. By Lemma 3.1,

$$u_1 \geq_{\text{A}} u_2 \Leftrightarrow \hat{M}(1) \leq_{\text{MLRD}} \hat{M}(2). \tag{13}$$

Or equivalently, the probability measure $\hat{m}(y,i)$ is a TP_2 function for y and i. We have the following inequality:

$$\begin{aligned} 0 &= \int_Y \{h_1(x,y) - h_1(x,y_0)\}\hat{m}(dy;1) & (14) \\ &\geq \int_Y \{h_1(x,y) - h_1(x,y_0)\}\hat{m}(dy;2) & (15) \\ &\geq \int_Y \{h_2(x,y) - h_1(x,y_0)\}\hat{m}(dy;2). & (16) \end{aligned}$$

The first inequality comes from the variation diminishing property (Lemma 3.2). It follows that by a direct calculation $h_1(x,y) - h_2(x,y) = g_y(x,y)\{\mathcal{A}(g(x,y);u_1) - \mathcal{A}(g(x,y);u_2)$, hence $\mathcal{A}(g(x,y);u_1) \geq \mathcal{A}(g(x,y);u_2)$ is equivalent to $h_1(x,y) \geq h_2(x,y)$. Therefore, we have the second inequality. The above inequality can be rewritten as

$$0 \geq \int_Y \{h_2(x,y) - h_1(x,y_0)\}\hat{m}(dy;2) \Leftrightarrow \mathcal{A}(x;v_1) \geq \mathcal{A}(x,v_2). \tag{17}$$

Since the proof for the case of h_2 is similar, we omit the proof. Hence we complete the proof. □

3.4 Remark

Giving a comment on the relation between our and Nachman's analysis (1982), we close this section. He determined another sufficient condition to guarantee the comparative risk aversion under background risks. First, we review his analysis. Nachman (1979) proved that if $u_1 \geq_A u_2$, then $\hat{M}(2)$ dominates $\hat{M}(1)$ in the sense of First–order Stochastic Dominance (FSD), $\hat{M}(y;2) \leq \hat{M}(y;1)$ for all $y \in Y$. It is well known that the following two conditions are equivalent:

- $F(2)$ dominates $F(1)$ in the sense of FSD;
- $\mathbb{E}[g(\tilde{x}_1)] \leq \mathbb{E}[g(\tilde{x}_2)]$ for every increasing function g

(See *e.g.* Gollier; 2001, and Müller and Stoyan; 2002.). Applying the above condition, we have that if $u_1 \geq_{\text{A}} u_2$ and either of $h_i(x,y)$ is a decreasing function of y, then $v_1 \geq_{\text{A}} v_2$.

Since functions satisfying the monotonicity condition imply ones satisfying the single crossing condition, the condition determined in this paper is weaker than that detrmined by Nachman (1982). The reason why our condition is weaker than Nachman's one, is that our analysis is based on the stronger stochastic dominance than his analysis, that is the MLRD is a stronger stochastic dominance than the FSD. We summarize the above discussion into the table.

Table 1: A summary of the relation our and Nachman's analysis

	stochastic dominance	condition of h
Nachman's paper	FSD	monotonicity
relation	$\Uparrow$	$\Downarrow$
our paper	MLRD	single crossing

4 Additive form

Over the past two decades, many studies have been concerned with the effects of background risks with additive payoff functions, $g(x, y) := x + y$. Hence we investigate that which conditions on utility functions guarantee $v_1 \geq_{\mathrm{A}} v_2$ under background risks which have the additive form. When payoff functions have the additive form, $h(x, y) = A(x + y; u)$. Hence the Arrow–Pratt absolute risk aversion of the utility function u under additive background risks, that is, that of the utility function v is given by

$$\mathcal{A}(x; v) = \int_Y \mathcal{A}(x + s; u)\hat{m}(ds). \tag{18}$$

We define $y_0 \in Y$ as a similar way in the previous section: There exists $y_0 = \int_Y \mathcal{A}(x+ s; u)\hat{m}(ds)(= \mathcal{A}(x; v))$. We obtain the following corollary applied by Theorem 3.1.

Corollary 4.1. Assume that $g(x, y)$ is an increasing function of y for all $x \in X$. If $u_1 \geq_A \ (\leq_A)\ u_2$ and either of functions $\mathcal{A}(x + y; u_i) - \mathcal{A}(x + y_0, u_i)$ satisfies the single crossing condition at y_0 from the above (below), then $v_1 \geq_{\mathrm{A}} v_2$.

Any functions satisfying the monotonicity condition imply also ones satisfying the single crossing condition. Hence Corollary 4.1 holds under the monotonicity condition, and this result was obtained by Kihlstrom, *et al.* (1981). Corollary 4.1 can be viewed as a generalization of Kihlstrom, *et al.* (1981). And this generalization is not only important from a technical viewpoint with respect to previous studies, but also from a empirical viewpoint. Jackwerth (2000) observed U-shaped risk aversion. This empirical observation means that DARA has not any predictions about conditions of utility functions to guarantee the desirable property, $u_1 \leq_{\mathrm{A}} u_2 \Rightarrow v_1 \leq_{\mathrm{A}} v_2$. On the other hand, when absolute risk aversion satisfies the single crossing condition from the above, we do not require that absolute risk aversion is decreasing in the global sense, in other words absolute risk aversion is only decreasing in the neighborhood of the single crossing point. Corollary 4.1 may imply the existence of utility functions such that

- they preserve comparative risk aversion under additive background risks;
- they are consistent with recent empirical observations, *e.g.* DARA for low wealth and IARA for high wealth.

5 Stochastic Dominance

As another direction of studies concerning comparative risk aversion under background risks, Eeckhoudt, Gollier, and Schlesinger (1996, hereafter EGS) determined conditions on utility functions to guarantee that first– and second–order deteriorations in background risks increase risk aversion. As we note that MLRD and comparative risk aversion have the same property, that is both can be represented by TP_2 function, we can obtain the parallel result in comparative risk aversion with MLRD changes in risk. Hence we only offer the result without a proof.

Theorem 5.1. Assume that $g(x, y)$ is an increasing function of y for all $x \in X$. If $M(1) \leq_{\text{MLRD}} \ (\geq_{\text{MLRD}})\ M(2)$ and $h(x, y) - h(x, y_0)$ satisfies the single crossing condition at y_0 from the above (below), then $v_1 \geq_{\text{A}} v_2$.

In the payoff functions being the additive form $g(x, y) = x + y$, EGS (1996) claimed that DMs with DARA utility functions in the sense of Ross (1981) behave more risk averse manner when background risks change in the sense of FSD. Since comparative risk aversion in the sense of Ross is stronger than that of Arrow–Pratt, DARA in the Ross means DARA in the Arrow–Pratt. When payoff functions are additive, a sufficient condition to preserve comparative risk aversion is that Arrow–Pratt absolute risk aversion satisfies the single crossing condition from the above by Theorem 5.1. And it is clear that this condition is a weaker condition than DARA in the sense of Ross determined in EGS (1996). This observation is consistent with the fact that the MLRD is a stronger notion of the FSD. We summarize the above discussion into the table.

Table 2: A summary of the relation our and EGS analysis

	stochastic dominance	condition of h
EGS paper	FSD	Ross DARA
relation	$\Uparrow$	$\Downarrow$
our paper	MLRD	single crossing

6 Concluding Remarks

We determine a sufficient condition on utility functions to preserve (reserve) comparative risk aversion under background risks in this note. Nachman's condition

(1982) requires the monotonicity in the global sense, on the other hand our condition only requires it in the local sense. This generalization does not only comes from theoretical importance, but also empirical one, because Jackwerth (2000) observed U–shaped risk aversion in the asset market. Our condition has some predictions concernign comparative risk aversion under additive background risks, in contrast Nachman's condition does not have them. We also give some comments on the effect of risk aversion on changes in background risks.

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