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Abstract

This note gives sufficient conditions of cross risk vulnerability introduced by Malevergne and Rey (2005), which is the equivalent condition to guarantee that an unfair non–monetary background risk makes decision makers more risk averse. The sufficient conditions determined by this note expand the results for univariate utility function into bivariate utility functions.

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1 Introduction

In recent two decades, a considerable number of studies has been focused on the effects of background risks on optimal decisions under uncertainty. Since changes in risk aversion have unambiguous comparative statics predections in most models with uncertainty, many authors derived conditions to guarantee that background risks make decision mekers more risk averse. A landmark concerning this topic is the "risk vulnerability" introduced by Gollier and Pratt (1996), which is the weakest condition among them. As the risk vulnerability is a technical condition, several authors determined sufficient conditions of risk vulenrability which have some intuitive interpretations. The following two conditions are well known: (i) the standard risk aversion derived by Eeckhoudt and Kimball (1992), (ii) the decreasing and convex risk aversion derived by Gollier and Pratt (1996). An excellent survey on this topic can be found in Part IV in Gollier (2001), the interestend readers can see it.

Most studies concerning background risks in optimal choices under uncertainty, consider decision makers with univariate von–Neumann Morgenstern utility functions. However, there are many situations in which decision makers have multi–dimensional utility functions. For exapmle, utilities of decision makers are not only depend on finacial wealth, but also health status in the context of health economics. We can list many examples of decision making with multi–dimension, like environmental status, social status, and so on. Hence, we need to understand the effects of background risks on optimal choices with multi–dimension under uncertainty. In their recent intersting paper, Malevergne and Rey (2005) generalize risk vulnerability to the case that decision makers have bivariate utiity functions, and they call it "cross risk vulenerability". The purpose of this note gives the sufficient conditions of cross risk vulnerability, which are correspond to the well–known sufficient conditions of risk vulnerability above mentioned. A best way to understand the effects of background risks in multi–dimensional decisions under uncertainty is a generalization of results in one–dimensional as possible as we can, baecause we obtain many understaings regarding them in last two decades. One importance of our results comes from this point of view.

2 Preliminary Analysis

First of all, we give some preliminary analyses to derive sufficient conditions of cross risk vulnerability introduced by Malevergne and Rey (2005). Our setting is identical with them, and therefore we borrow their notaions. Let us consider a dicision maker (DM) with a bivariate (von Nuemann Morgenstern) utility function

$U : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We impose standard assumptions of the utility functions U: (i) $U_1,\ U_2 > 0$, (ii)$U_{11}, U_{22} \leq 0$, and (iii) $U_{12} \leq 0$. Here subscripts denote partial and cross derivatives of corresponding arguments, *i.e.* U_i denotes the partial derivative of the i–th argument, and U_{ij} denotes the cross derivative of the i– and j–th arguments. The assumption (ii) and (iii) mean risk and correlation aversion, respectively. The notion of correlation aversion was initiated by Epstein and Tanny (1980). In a recent paper, Eeckhoudt, *et al.* (2005) characterized some desiable properties of cross derivatives of bivariate utility functions using preferences over the fifty–fifty lottery suggested by Eeckhoudt and Schlensinger (2005). As Malevergne and Rey (2005), the first and second arguments stand for wealth and health, respectively.

We define the cross derived utility function toward health background risk $\tilde{\epsilon}$ as

$$V(x, y) = \mathbb{E}[U(x, y + \tilde{\epsilon})]. \tag{1}$$

The health background risk $\tilde{\epsilon}$ is a random variable followed by the distribution function $F : [\underline{\epsilon}, \bar{\epsilon}] \to [0, 1]$. For the sake of simplicity, we assume that the distibution function F is differentiable, that is the density function $f(\epsilon) = F'(\epsilon)$ exists. The cross derived utility function is a slight modification of derived utility function for univariate utility functions defined by Kihlstrom, *et al.* (1981) and Nachman (1982). The Arrow–Pratt (absolute) risk aversions toward a wealth risk of the utility function U and V are given as

$$A(x, y) := -\frac{U_{11}(x, y)}{U_1(x, y)}, \text{ and } A^V(x, y) := -\frac{V_{11}(x, y)}{V_1(x, y)}. \tag{2}$$

Following Malevergne and Rey (2005), we define the cross risk vulnerability:

Definition 2.1. The utility function U is cross risk vulnerable, if any unfair health background risks make the DM in a more risk averse manner:

$$\mathbb{E}[\tilde{\epsilon}] \leq 0 \Rightarrow A^V(x, y) \geq A(x, y),\ \forall (x, y). \tag{3}$$

As a slight generalization of the notion introduced by Kimball (1990), we define two precautionary premiums toward a zero–mean health background risk:

$$\mathbb{E}[U_1(x, y + \tilde{\epsilon})] = U_1(x, y - \Psi^H) = U_1(x - \Psi^W, y). \tag{4}$$

We call Ψ^H a precautionary premium in health, and Ψ^W a precautionary premium in wealth, respectively. It is obvious that both the precoutionary premium Ψ^H and Ψ^W are the functions of x, y, $\tilde{\epsilon}$, and U, but we suppress them in order to simplify the notation. Using the Arrow–Pratt approximation, we obtain the precautionary premium in health and wealth as follows:

$$\Psi^H \simeq \frac{1}{2}\left(-\frac{U_{122}(x, y)}{U_{12}(x, y)}\right)\mathbb{E}[\tilde{\epsilon}^2] = \frac{1}{2}P^H(x, y)\mathbb{E}[\tilde{\epsilon}^2], \tag{5}$$

$$\Psi^W \simeq \frac{1}{2}\left(-\frac{U_{122}(x,y)}{U_{11}(x,y)}\right)\mathbb{E}[\tilde{\epsilon}^2] = \frac{1}{2}P^W(x,y)\mathbb{E}[\tilde{\epsilon}^2]. \tag{6}$$

P^H and P^W can be interpreted as Kimball (absolute) cross prudence in health and wealth toward a health risk. As recalling that U_{12} and U_{11} is positive, this (approximation) equality imlies that

$$\Psi^{H(W)} \geq (\leq)\ 0 \Leftrightarrow U_{122}(x,y) \geq (\leq)\ 0. \tag{7}$$

Eeckhoudt, *et al.* (2005) justified the nonnegativity of U_{122}, and they call it cross prudence in wealth.

3 Main Result

In the remainder of the paper, we give three sufficient conditions of cross risk vulnerabulity, thier remarks, and proofs.

3.1 Propositions and their Remaks

Proposition 3.1. If the Arrow–Pratt risk averison is decreasing and convex in the second argument, *i.e.* $A_2(x,y) \leq 0$ and $A_{22}(x,y) \geq 0$, then a utility function U is cross risk vulenrable.

Proposition 3.2. If the Arrow–Pratt risk aversion is decreasing in the second argument, and the Kimball prudence in halth is decreasing in the first argument, *i.e.* $A_2(x,y) \leq 0$ and $P_1^H(x,y) \leq 0$, then a utility function U is cross risk vulenrable.

Proposition 3.3. If the Arrow–Pratt risk aversion is decreasing in the first and second argument, and the Kimball prudence in wealth is decreasing in the first argument, *i.e.* $A_1(x,y) \leq 0$, $A_2(x,y) \leq 0$ and $P_1^W(x,y) \leq 0$, then a utility function U is cross risk vulenrable.

Proposition 3.1 was already obtained by Malevaergne and Rey (2005) in a complete different way. They obtained it as a corollary of proposition 1 and 3 in their paper. In contrast to them, we prove it in a direct way. While they did not give a comment, Proposition 3.1 is a natural generalization of the result that is a sufficient condition of risk vulenerablity in the case of univariate utility functions, which is determined by Gollier and Pratt (1996). Indeed, our proof provides for a different proof of their result. As our proof is similar to Eeckhoudt and Kimball (1992) which derived a sufficient condition of risk vulenerablity in the case of univariate utility functions, Proposition 3.2 and 3.3 can be viwed as a natural generalization of the result derived by Eeckhoudt and Kimball (1992). This implies that the conditions of the proposition 3.2 and 3.3 correspond to the standard risk aversion in the case of bivariate utility functions, which is introduced by Kimball (1993).

3.2 Proofs

The proofs of proposition 3.2 and 3.3 are same except for some minor points, hence we omit the proof of proposition 3.3.

Proof of propsition 3.1: Let us define the following function for $\epsilon \in [\underline{\epsilon}, \overline{\epsilon}]$

$$\hat{f}(\epsilon) := \frac{U_1(x, y+\epsilon) f(\epsilon)}{\int_{\underline{\epsilon}}^{\overline{\epsilon}} U_1(x, y+\epsilon) f(\epsilon) d\epsilon}. \tag{8}$$

Since $\hat{f}(\epsilon)$ is positive for all $\epsilon \in [\underline{\epsilon}, \overline{\epsilon}]$ and $\hat{F}(\overline{\epsilon}) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \hat{f}(\epsilon) d\epsilon = 1$, $\hat{F} : [\underline{\epsilon}, \overline{\epsilon}] \to [0,1]$ can be viewed as a distribution function defined over the compact support $[\underline{\epsilon}, \overline{\epsilon}]$. The correlation aversion implies that $U_1(x, y+\epsilon) \geq U_1(x, y+\epsilon')$ for all $\epsilon, \epsilon' \in [\underline{\epsilon}, \overline{\epsilon}]$ with $\epsilon \leq \epsilon'$. Hence, we have that

$$\frac{f(\epsilon')}{f(\epsilon)} \geq \frac{U_1(x, y+\epsilon') f(\epsilon')}{U_1(x, y+\epsilon) f(\epsilon)} \Leftrightarrow \frac{f(\epsilon')}{f(\epsilon)} \geq \frac{\hat{f}(\epsilon')}{\hat{f}(\epsilon)}. \tag{9}$$

The equation (9) means that F dominates $\hat{F}$ in the sense of monotone likelihood ration dominance. The monotone likelihood ratio dominance is a strong stochastic dominance of the first–order stochastic dominance. Since the expectation of the health background risk is non–positive, we have $\hat{\mathbb{E}}[\tilde{\epsilon}] \leq \mathbb{E}[\tilde{\epsilon}] \leq 0$. For details of stochastic dominance, the intersted readers are refereed to *e.g.* Shaked and Shanthikumar (1994) or Müller and Stoyan (2002). Using this distirubtion function, we have

$$\begin{aligned} A^V(x, y) &= \int_{\underline{\epsilon}}^{\overline{\epsilon}} A(x, y+\epsilon) \hat{f}(\epsilon) d\epsilon & (10) \\ &= \hat{\mathbb{E}}[A(x, y+\tilde{\epsilon})], & (11) \end{aligned}$$

where $\hat{\mathbb{E}}$ denotes the expectation operator with respect to the distribution function $\hat{F}$. By the equation (11), we obtain that

$$\begin{aligned} A^V(x, y) &= \hat{\mathbb{E}}[A(x, y+\tilde{\epsilon})] & (12) \\ &\geq A(x, y+\hat{E}[\tilde{\epsilon}]) & (13) \\ &\geq A(x, y). & (14) \end{aligned}$$

The first inequaltiy follows from the Jensen's inequality, the second inequaliy follows from $\hat{\mathbb{E}}[\tilde{\epsilon}] \leq 0$ and $A_2 \leq 0$. □

Proof of propsition 3.2: The health background risk $\tilde{\epsilon}$ decomposes to ϵ_0 and $\tilde{\epsilon}_1$, *i.e.* $\tilde{\epsilon} \stackrel{d}{=} \epsilon_0 + \tilde{\epsilon}_1$, where $\epsilon_0 = \mathbb{E}[\tilde{\epsilon}] \leq 0$ and $\mathbb{E}[\tilde{\epsilon}_1] = 0$. By the definition of the precautionary premium in health Ψ^H, we have that

$$V_1(x, y) = U_1(x, y+\epsilon_0 - \Psi^H), \tag{15}$$

$$V_{11}(x,y) = U_{11}(x, y+\epsilon_0 - \Psi^H) - U_{12}(x, y+\epsilon_0 - \Psi^H)\Psi^H_x, \tag{16}$$

where Ψ^H_x denotes the partial derivative of Ψ^H with respect to x, *i.e.* $\Psi^H_x = \partial\Psi^H/\partial x$. Using the equation (15) and (16), we obtain that

$$\begin{aligned}
-\frac{V_{11}(x,y)}{V_1(x,y)} &= -\frac{U_{11}(x,y+\epsilon_0-\Psi^H) - U_{12}(x,y+\epsilon_0-\Psi^H)\Psi^H_x}{U_1(x,y+\epsilon_0-\Psi^H)} && (17)\\
&= -\frac{U_{11}(x,y+\epsilon_0-\Psi^H)}{U_1(x,y+\epsilon_0-\Psi^H)} + \frac{U_{12}(x,y+\epsilon_0-\Psi^H)}{U_1(x,y+\epsilon_0-\Psi^H)}\Psi^H_x && (18)\\
&\geq -\frac{U_{11}(x,y+\epsilon_0-\Psi^H)}{U_1(x,y+\epsilon_0-\Psi^H)} && (19)\\
&\geq -\frac{U_{11}(x,y)}{U_1(x,y)} = A(x,y). && (20)
\end{aligned}$$

As $U_1 > 0$, $U_{12} < 0$, and $P^H_1 \leq 0$ is equivalent to $\Psi^H_x \leq 0$, we have that the second term in the equation (18) is non–negative. Hence, we have the first inequality. The second inequality follows from $A_2 \leq 0$ and $y > y + \epsilon_0 - \Psi^H$. □

4 Concluding Remarks

In this note, we give sufficient conditions of cross risk vulnerability. These conditions are a generealization of the well–known sufficient conditions of risk vulnerability: standard risk aversion and decreasing and convex risk aversion. Since most decisions under uncertainty are multi–dimensional, we need to understand the effects of background risks in multi–dimensional decisions under uncertainty. A best way to understand them generalize those of one–dimesional decisions under uncertainty. This paper can be regarded as one of a series of studies. We have many future reserches to understand the effects of background risks in multi–dimensional optimal choices under uncertainty, since we have a great number of understandings regarding them in one–dimension.

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