# Discussion Papers In Economics And Business

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# Holdup, Search and Inefficiency *

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**Abstract**

This paper investigates the holdup problem in the dynamic search market where buyers and sellers search for their trading partners and specific investments are made after match but before trade. We show that frictionless (competitive) market imposes severe limitations on attainable efficiencies: Markets with small friction make the holdup problem more serious than those with large friction because in any equilibrium, whether stationary or non–stationary, investment must be dropped down to the minimum level and trade must be delayed with positive probability.

Keywords: Delay of Trade, Holdup Problem, Search
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## 1 Introduction

In this paper we investigate the holdup problem in the dynamic search market where buyers and sellers search for their trading partners and specific investments are made after matching but before trade decisions. In particular we address the issues of whether or not frictionless (competitive) market helps mitigate the holdup problem. Our answer to this question is negative: In fact we show that markets with small friction make the holdup problem more serious and result in significant inefficiency, which never happens in the markets with large friction.

What is the holdup problem? Suppose that a seller delivers one indivisible good to a buyer and makes "specific" investment which has value only for the trade relationship with the buyer in question. For example the seller may need to make customized investment which meets the specific requirements of the buyer. Then, they are locked in "bilateral monopoly" situation and bargain over trade decision ex post after seller's specific investment was sunk. In the absence of complete contingent contracts the seller expects that some fraction of the return from his investment will be captured by the buyer through ex post negotiation, which in turn undermines the seller's ex ante investment incentives. This is called the "holdup problem."

The holdup problem has played the central role to understanding organizational design and boundaries of firms (See for example Willimason (1975, 1985), Klein, Crawford and Alchian (1978), Grossman and Hart (1986), Hart and Moore (1990) and Hart (1995)). These studies have discussed the solutions to the holdup problem through allocation of ownership rights (such as vertical integration). Other authors also proposed the mechanism design approach to resolve the holdup problem (See Aghion, Dewatripont and Rey (1994), Chung (1991) and Rogerson (1992)). However most of the existing studies have focused on bilateral trade games of particular parties by taking outside markets as exogenously given. Thus these studies have ignored the interactions between the holdup problem in bilateral trade and the market environments which govern the processes of trading opportunities such as how trading parties find alternative partners when current negotiation fails. In other words the *reservation values* of players, which determine the payoffs of players when they fail to trade with current partners, have been exogenously fixed in the models.

In this paper we will consider the holdup problem in the market environment with search friction which endogenously determines the players' reservation values. We will investigate the dynamic search market in which there are many buyers and sellers and each buyer (seller) searches a seller (buyer) for trading one indivisible good. Each buyer consumes at most one good and each seller can deliver at most one good. They are randomly matched with each other with search friction: Some of buyers (sellers) fail to meet sellers (buyers). After matching a buyer, seller makes "match specific" investment, which has the value only for the current match, before trade decision of the good. Since they can trade at most one good, all matched parties leave the market after they trade the goods. If matched parties do not agree on trading the good, they stay in the market and start searching for the next match again.

Alternatively our model can be interpreted as the "marriage market" with investment choices: [1] Male and female search for their partners and they make "match specific investment" for building trust relationship with matched partner, knowing each other well, cooperating in housework productions and so forth after matching. These investments have no values for all other matches than the current one. After agreeing on marriage, the parties leave the market.

In this dynamic search model the market friction is measured by the inverse of the discount factor of players and the probability of failing to meet a partner. Thus the market becomes frictionless as the discount factor and matching probability go to one. One might think that, since small friction makes the market more competitive, resulting market outcomes become more efficient as search friction tends to be smaller.

However, in contrast to this intuition, we show that frictionless markets become highly less efficient than those with large friction. This is because markets with small search friction make the holdup problem more serious than those with large friction. In fact we show that in any equilibrium, whether stationary or non–stationary, sellers' investments must be dropped down to the minimum level, zero, and trades must be significantly delayed with positive probability for infinitely many times, although these features never happen in the markets with large search friction.

This inefficiency result follows from the dynamic interactions between current and future sellers' investment incentives: Suppose that everyone expects the future sellers will make high investment. Then the "reservation values" for the current players become high because they expect high trade values will be realized in the future matches. This is more likely to be the case when search friction becomes small. However, if the reservation values of the current players are high, *ex post surplus* in the current match, which is defined as the current trade value minus the sum of the reservation values of buyer and seller for a given investment, becomes small. Each matched seller obtains his bargaining payoff from the negotiation over such ex post surplus after his investment was sunk. When the ex post surplus is expected to be small, the currently matched sellers will find it costly to make high investment. This will then undermine the sellers' investments. On the other hand, suppose that everyone expects the future sellers will make low investment. Then, the reservation values of the currently matched

[1] See for example Peters and Siow (2002) for the recent model of the marriage market with investment choices.

players become small because they expect low trade values will be realized in the future matches. However, then the ex post trade surplus in the current match becomes high and hence the current period sellers have more incentives to make high investment.

Thus the investment incentives of current sellers move in the opposite direction with those of the future sellers: If future sellers will have more incentives to make high investment, current sellers have less incentives to do so, and vice versa. This argument leads to the following results: First, if there exists a symmetric stationary equilibrium in which all matched sellers choose the same investment strategy in all periods, it must involve mixed strategy to randomize between high and low investments. Second, if equilibrium is not stationary, it must exhibit some cyclical feature that equilibrium investments fluctuate between high and low levels over time. In either case, equilibrium investment must be low with some positive probability. Also, whenever equilibrium investment goes down to zero, every match leads to no–trade and hence trade must be delayed. This dynamic mechanism makes the holdup problem more serious as search friction becomes smaller and hence imposes severe limitations on attainable efficiencies.

We will also investigate several extensions of the model: First, we introduce heterogeneity of buyers and sellers into the model and then show that our inefficiency result still holds even in such case. Second, we allow matched parties to write ex ante contracts contingent on trade outcome before investment choice of matched seller. Then we show that there exists some equilibrium when ex ante contracts are possible, which is less efficient than the equilibrium when ex ante contracts cannot be written at all, if search friction becomes small enough. Thus enlarging the set of possible contracts may make the market efficiency worse than the case of no ex ante contracts. This might lead to the implication that it may not be optimal to make markets competitive in developing countries with limited contract enforcement. Third, we examine the two–sided investment case that both sides of matched parties (buyer and seller) make specific investments before trade. Then we find that in the symmetric example in which both buyer and seller have identical preferences the two–sided investment helps recover efficiency even when it is lost in the one–sided investment case. However, inefficiency arises again when buyer and seller are heterogeneous.

**Related Literature**

Several recent papers have paid attentions to the holdup problem in market environments and its implications about the effects of competition (See Acemoglu and Shimer (1999), Cole, Mailath and Postlewaite (2001), Felli and Roberts (2002) and MacLeod and Malcomson (1993)). Some among these studies show that market competition may help recover the first best efficiency. In contrast to this, our result highlights on the negative effects of "competition" on the holdup problem where in our model the competitive environment can be interpreted as the market with small search friction.[2] Our inefficiency result is also in contrast to the recent paper by Che and Sákovics (2004) who investigate the holdup problem in a dynamic setting and show that there exists a Markov perfect equilibrium to attain the efficient outcome when discount factor goes to one.[3] De Mezza and Lockwood (2004) also reported that inefficiency may arise when search friction becomes small, although the inefficiency in their model comes from the overinvestment (not underinvestment) problem due to coordination failure in the frictional matching market.

Some papers have also focused on the issues of holdup problem in randomly matching markets (Acemoglu (1996, 2001) and Acemoglu and Shimer (1999)) where investing parties make *ex ante* investments *before* matching in contrast to our approach that investments are made after matching. Which approach of ex ante or ex post investment is more plausible depends on different economic circumstances we have in mind. Ex ante investments may fit to the cases of general human capital investment (education) by workers and physical capital installment by firms before employment. On the other hand, ex post investments may fit to the cases of the parties making customized investments (as is often observed in the productions of automobile parts) and building specific relationships with matched partners (as in the marriage market).

When we interpret our model as the marriage market as we have mentioned before, our result that competitive markets exaggerate the holdup inefficiency is also in contrast to the papers showing the efficiency property of competitive marriage markets (See for example Peters and Siow (2002)).

We can also relate our result to the literature about delay in bargaining games. Several papers have shown the phenomena of delay in bargaining games with complete information (See for example Busch and Wen (1995), Cai (2000), Jehiel and Moldovanu (1995a, 1995b) and Sákovics (1993)). [4] However, most of the existing studies have assumed that the trade value, which is a pie to be split between negotiating parties, is exogenously given. On the other hand, in our model the trade value is *endogenously* determined by players' strategies in equilibrium and this leads to significant delay of trade.[5] Frankel (1998) examines the bargaining game in which players exert

[2]Also, most of the papers cited above have focused on the classical holdup problem which assumes no contracts are possible before investment choices. In our paper we will examine the extension of the basic model to allow matched parties to write ex ante contracts before investments are made. Acemoglu and Shimer (1999) also consider a related issue but they assume wage posting by firms before ex ante investment and search decisions.

[3]In their model a buyer and a seller repeatedly interact with each other and there are no elements of switching partners in markets.

[4]See also Jehiel and Moldovanu (2004) for the phenomena of gradualism in contribution and bargaining games.

[5]Merlo and Wilson (1995) consider the situation in which a pie to be split between bar-

creative efforts to expand the set of proposals, although the main focus was on premature agreement which means that players agree on trade before total pie is larger. Also, most of the above papers have focused on repeated negotiations between particular players while we will consider the dynamic matching market in which a buyer and a seller interact each other only once after they match together and they will not meet again in the future.[6]

The remaining sections of the paper are organized as follows: In Section 2 we will set up the basic model of dynamic search market with the holdup problem. In Section 3 we will characterize the search market equilibria and in Section 4 we will show that small search friction makes the holdup problem worse and causes serious inefficiency. In Section 5 we will extend the basic model in several directions.

# 2 Model

## 2.1 Search Market

We consider the dynamic search market in which buyers and sellers are randomly matched with each other. Time is discrete and extends over infinity, $t = 0, 1, 2...$ There is a single consumption good. At initial period ($t = 0$) there are a unit mass of buyers and a unit mass of sellers in the market. Seller is indexed by $i \in [0, 1]$ and buyer by $j \in [0, 1]$. Each buyer can consume at most one good and each seller can produce at most one good. Each buyer (seller) matches a seller (buyer) with probability $\alpha \in (0, 1)$. If matched parties agree on trading the good, they leave the market because they can trade at most one good. There is no new entry into the market at all. Thus in every period the measure of buyers is same as that of sellers.[7] The underlying assumption is here that the matching technology exhibits the constant returns to scale.

gainers is changed over time, although such change follows exogenous stochastic process.

[6]Samuelson (1992) also considers the bargaining in a dynamic matching market and shows that delay of trade is observed as an equilibrium outcome. However, in Samuelson (1992) the asymmetric information between matched parties causes delay of trade while in our model there are no uncertainty but delay is observed.

[7]We can also consider alternative scenarios about matching process: First, we may assume that there are exogenous new inflows of players into the market at rate $\alpha \in (0, 1)$ in each period. Then the population sizes of buyers and sellers can be kept at constant equal to a unit measure over time in any stationary equilibrium. Second, we may assume that all the players who left the market are replaced by the same measure of new players. In that case the population sizes of buyers and sellers become constant over time even in non–stationary equilibrium. In any modeling choice the important point is that all players who traded leave the market. This makes the "outside option" relevant for trade decision. Many pagers in the literature about bargaining and contract choices in the search markets follow this approach. See for example Inderst (2001), Osborne and Rubinstein (1990) and Rubinstein and Wolinsky (1990) for the related models sharing the same approach.

After matching a buyer, matched seller makes "match specific" investment $a \in A \subset \Re_+$ which affects both the buyer's benefit from consuming the good $b(a)$ (cooperative investment) and the production cost of seller (selfish investment) $d(a)$. We will call $v(a) \equiv b(a) - d(a)$ *the trade value.* The seller's investment is "match specific" in the sense that each matched seller's investment has the value only for the trade with the currently matched buyer. Thus it has no values in all other matches. For example the seller needs to make customized investment which meets some specific requirements of the buyer. In fact customized inputs are often required for productions of automobiles. Also in the marriage market matched partners need to make "match specific" investments for building trust relationships with the matched partners, knowing each other well, cooperating in housework productions and so forth. These investments are however worthless for all other matches. [8]

Note here that we make no restrictions on the seller's investment $a \in A$. $a$ is allowed to be both discrete and continuous.

The seller personally incurs the cost of investment $c(a)$. We assume that both $v$ and $c$ are increasing and that $v(0) > 0$ and $\min_{a \in A} c(a) = c(0) = 0$. As in the traditional approach to the holdup problem, we assume that seller's investment $a \in A$ is observable to the buyer who matched the seller in question but not verifiable. We also assume that no contracts are possible before seller's investment choice $a \in A$. We will discuss how ex ante contracts affect the results in Section 5. Note also here that we make no restrictions on $v$ and $c$ except that they are increasing.

After seller's investment choice, the matched parties decide whether or not to trade the good. We here assume the simple bargaining process: With probability $\beta \in (0, 1)$ the seller makes a take–it–or–leave–it offer to the buyer about trade decision and payment while with probability $1 - \beta$ the buyer makes a take–it–or–leave–it offer to the seller about trade decision and payment. This is equivalent to assuming the Nash bargaining solution which has been commonly used in the literature.

All players are risk neutral and discount their future payoffs at the common discount factor $\delta \in (0, 1)$.

The outside payoff of each player is normalized to zero. Thus each player obtains the payoff of zero if he or she does not trade the good in the current period.

[8] We can explicitly introduce some "specificity" aspects of trade into the model. In each match some random state $\eta$ is independently drawn from $[0, 1]$. Then the buyer who has drawn $\eta$ needs the good with specification $\eta$. Then trade value is realized as $v(a)$ when the matched seller delivers the good with specification $\eta$ and chooses the investment $a$ to enhance such value. On the other hand, when the seller does not deliver the good meeting the buyer's required specification $\eta$ the trade value is given by $\underline{v} \equiv v(0) > 0$ irrespective of his investment. Here we can interpret the minimum investment level $a = 0$ as the "general" investment which attains the minimum trade value $\underline{v}$ whatever buyer's required specification.

Each period is divided into three dates. The timing of events in a period is as follows:

date 0 Buyers and sellers randomly match. The matched parties make a lump–sum transfer, which is used to distribute the ex ante total surplus between them as we will explain below. A matched seller makes a take–it–or–leave–it offer of lump–sum transfer with probability $\gamma \in (0, 1)$ while a matched buyer does so with probability $1 - \gamma$.

date 1 The matched seller makes match specific investment $a \in A$.

date 2 The matched parties bargain to decide whether or not they trade the good and how much they make payments. If they agree on trading the good, the seller produces and delivers the good to the buyer who consumes it. Then they leave the market. Otherwise they stay in the market and search for the next match.

The friction in the search market described above is inversely related to the discount factor $\delta$ and the matching probability $\alpha \in (0, 1)$: If these values are large, the market is said to have small friction because the trades of the goods are quickly made and many matches can form. Then our main concern is to investigate how equilibrium outcomes in the search market change when the market friction goes to zero.

As we have said, our model can be also interpreted as the marriage market where male and female search for each other and make "match specific" investment after matching. Then they leave the market after they agree on marriage.

## 2.2 The Static Benchmark

First we consider the static benchmark which corresponds to the case that the market friction is extremely high ($\delta = 0$) in the above dynamic search model.

In this static case all players obtain their outside payoff, zero, when they do not trade the goods. Thus ex post surplus, which is defined as the trade value $v(a)$ minus the reservation values the parties will obtain under no trade agreement, is simply given by the trade value $v(a)$ itself.

Since seller can extract $\beta$ fraction of ex post surplus $v(a)$, he obtains the following payoff:

$$\beta v(a) - c(a). \tag{1}$$

The seller chooses the specific investment $a \in A$ to maximize this payoff. Let $a^s \in A$ be the static equilibrium investment:

$$a^s \in \arg\max_{a \in A} \beta v(a) - c(a). \tag{2}$$

We define the ex ante surplus as

$$S(a) \equiv v(a) - c(a), \tag{3}$$

and make the following assumption:

**Assumption 1**. *$a^s$ is unique and $a^s \neq 0$.*

Assumption 1 ensures that $\beta v(a^s) - c(a^s) > \beta v(0)$.

# 3 Search Market Equilibria

Next we turn to the dynamic search market model. Since matching is random and the set of players is continuum, it is the measure zero event that each player can meet the same partner again as he or she has previously matched. Also, the market is anonymous in the sense that any player cannot observe the past history about what decisions the matched partner has made. Thus each matched seller simply chooses his current period investment in order to maximize his payoff by taking his own and the buyer's reservation values as given. These assumptions are all standard in the literature of dynamic matching markets.

The main difference of this dynamic model from the static game is that the reservation values of players are endogenous. Here *the reservation value* of a player means the discounted present value of his or her future payoffs to be obtained when he or she does not trade the good in the current period.

Consider a matched pair of a buyer and a seller in period $t$. Then let $U_t^b$ and $U_t^s$ denote the reservation values of the buyer and the seller, which they will obtain when they do not trade the good in the current period $t$.

Then the *ex post surplus* from this match is defined as the trade value $v(a)$ minus these reservation values:

$$v(a) - U_t^b - U_t^s. \tag{4}$$

After the investment has been sunk, as long as this surplus is non–negative, the parties have the incentive to trade the good in the current period $t$. Then, with probability $\beta$ the seller makes a take–it–or–leave–it offer of payment and obtains the payoff $v(a) - U_t^b$. On the other hand, with probability $1 - \beta$ the buyer makes a take–it–or–leave–it offer of payment and hence the seller obtains the payoff $U_t^s$. If the ex post surplus (4) is negative, they will not trade the good and will stay in the market for searching the next matches, in which case they simply obtain their reservation values.

Thus the seller's payoff from ex post negotiation is given by $\beta \max\{v(a) - U_t^b - U_t^s, 0\} + U_t^s$. Then the seller's net payoff subtracting his investment cost $c(a)$ can be written by

$$\beta \max\{v(a) - U_t^b - U_t^s, 0\} + U_t^s - c(a). \tag{5}$$

The seller will choose investment $a \in A$ to maximize this payoff. More specifically, the mixed strategy of matched seller $i$ in period $t$ is represented by a probability measure $f_t^i \in \Delta(A)$ where $\Delta(A)$ is the set of all probability measures on $A$. Let also $E_{f_t^i}[\,\cdot]$ denote the expectation operator according to $f_t^i$.

We give a definition of equilibrium in the search market as follows:

**Definition of Equilibrium**. A sequence $\{f_t, U_t^s, U_t^b\}_{t=0}^{\infty}$ is said to be an equilibrium in the search market if the following conditions are satisfied:

(i) Symmetry and Optimality of Seller's Investment: In period $t$ each matched seller follows the symmetric strategy $f_t$ to maximize his payoff:

$$\int_A \{\beta \max\{v(a) - U_t^s - U_t^b, 0\} + U_t^s - c(a)\} df_t$$
$$\geq \int_A \{\beta \max\{v(a) - U_t^s - U_t^b, 0\} + U_t^s - c(a)\} df' \;\; \forall f' \in \Delta(A).$$

(ii) The Reservation Values: The buyer's and seller's reservation values $U_t^b$ and $U_t^s$ are determined as follows:

$$U_t^b = \delta\{\alpha u_{t+1}^b + (1-\alpha) U_{t+1}^b\},$$

and

$$U_t^s = \delta\{\alpha u_{t+1}^s + (1-\alpha) U_{t+1}^s\}$$

where $u_{t+1}^b$ and $u_{t+1}^s$ are the payoffs of a matched buyer and a matched seller respectively:

$$u_{t+1}^s \equiv \gamma \max\{E_{f_t}[S(a)] - U_{t+1}^b - U_{t+1}^s, 0\} + U_{t+1}^s,$$

and

$$u_{t+1}^b \equiv (1-\gamma) \max\{E_{f_t}[S(a)] - U_{t+1}^b - U_{t+1}^s, 0\} + U_{t+1}^b.$$

Condition (i) says that in period $t$ each matched seller optimally chooses symmetric strategy $f_t$ to maximize his payoff, given the reservation values $U_t^b$ and $U_t^s$. Since all sellers are identical, it is natural to focus on the symmetric equilibrium in which all matched sellers follow the same strategy in each period. However, note that this does not necessarily mean that the sellers' equilibrium strategy is stationary because it may depend on time index $t$.

Condition (ii) says that the reservation value of a buyer (resp. seller) $U_t^b$ (resp. $U_t^s$) in period $t$ is the discounted present value of his or her expected future payoffs which come from the payoff when matching $u_{t+1}^b$ (resp. $u_{t+1}^s$)

and the reservation value $U^b_{t+1}$ (resp. $U^s_{t+1}$) when not matching respectively. The former payoff is realized with probability $\alpha \in (0, 1)$, while the latter payoff is realized with probability $1 - \alpha$. The buyer's (resp. seller's) payoff $u^b_{t+1}$ ($u^s_{t+1}$) when matching is equal to $\gamma$ fraction of *ex ante total surplus*

$$\max\{E_{f_{t+1}}[S(a)] - U^b_{t+1} - U^s_{t+1}, 0\}$$

plus her (his) reservation value $U^b_{t+1}$ (resp. $U^s_{t+1}$). This is because the seller has the chance of making a lump–sum transfer offer to the buyer with probability $\gamma$ at date 0 in each period and hence he can extract $\gamma$ fraction of the ex ante surplus. Note that we are taking $\max\{\cdot, \cdot\}$ operator to express these payoffs because the ex ante surplus may be negative, in which case both parties simply obtain their reservation values $U^b_t$ and $U^s_t$.

The reservation values of players $U^b_t$ and $U^s_t$ depend on what investments and trade decisions the future matched players will do. Then, since the investment choices by the currently matched sellers are affected by the reservation values, the current sellers' investments are indirectly linked with those of the future sellers. This intertemporal linkage between the current and future sellers' investments choices plays the central role in the following analysis.

Note also that the *sum* of the reservation values of a buyer and a seller $U^b_t + U^s_t$ matters for determining the equilibrium investment of matched sellers (how to distribute the ex ante surplus does not affect seller's investment incentives). Once the total value $U^b_t + U^s_t$ is determined, we can decompose it into the separate values $U^b_t$ and $U^s_t$ by splitting it appropriately according to ex ante bargaining power $\gamma$. Thus in what follows we will mostly focus on the determination of the total value $U^b_t + U^s_t$ but not each of it.

We first show the following two lemmas both of which make the characterization of equilibria much simpler.

**Lemma 1**. *In any equilibrium any matched seller never chooses* $a \notin \{0, a^s\}$ *in any period* $t$, *i.e.,* $f_t(A \setminus \{0, a^s\}) = 0 \ \forall \ t$.

**Proof**. See Appendix.

Lemma 1 follows from Assumption 1 and the fact that the payoff function of matched seller (5) attains its maximum at $a = 0$ or $a = a^s$ (or both). In particular we can readily show that each seller optimally chooses the static investment $a^s$ if and only if

$$v(a^s) - c(a^s)/\beta \geq U^b_t + U^s_t. \tag{IC}$$

Otherwise, seller chooses the minimum investment $a = 0$. The reason for this is as follows: Suppose that each matched seller chooses $a^s$. Then, since any seller can guarantee at least $U_t^s$ by choosing $a = 0$, we must have $\beta \max\{v(a^s) - U_t^b - U_t^s, 0\} \geq c(a^s)$ which then implies (IC). On the other hand, each matched seller prefers $a = a^s$ to $a = 0$ if $\beta \max\{v(a^s) - U_t^b - U_t^s, 0\} - c(a^s) \geq \beta \max\{v(0) - U_t^b - U_t^s, 0\}$. This inequality holds by (IC) and definition of $a^s$.

By Lemma 1 without loss of generality we can write the equilibrium strategy of each matched seller by a probability $x_t \in [0,1]$ to choose $a = a^s$ and the remaining probability $1 - x_t$ to choose $a = 0$ in period $t$. Thus in what follows we will refer to $x_t$ for denoting the equilibrium strategy of matched sellers instead of using $f_t$.

**Lemma 2** *In any equilibrium trade is (resp. not) realized in any period in which $a = a^s$ (resp. $a = 0$) is chosen.*

**Proof**. See Appendix.

Lemma 2 follows from Lemma 1 and the fact that trade occurs if and only if the ex post surplus (4) is non–negative.

The equilibrium features crucially depend on the degree of search friction which is represented by the discount factor $\delta \in (0,1)$. We define the cutoff value of the discount factor, denoted $\delta^* \in (0,1)$, to satisfy the following equation:

$$\beta \left\{ v(a^s) - \frac{\delta^* \alpha S(a^s)}{1 - \delta^* \alpha} \right\} = c(a^s). \tag{6}$$

By Assumption 1, $c(a^s) > 0$ and $\beta < 1$, such $\delta^* \in (0,1)$ uniquely exists.

Then we first show the following result:

**Proposition 1**. *Suppose that $\delta \in (0, \delta^*)$. Then equilibrium is unique and has stationary property such that every matched seller chooses the static equilibrium investment $a^s$ with probability one in every period.*

**Proof**. See Appendix.

Proposition 1 states that when search friction is large the equilibrium outcome coincides with the static one.

# 4 Small Search Friction and Inefficiency

## 4.1 Low Investment and Delay of Trade

Next we will turn to the case of small search friction (large discount factor $\delta$). For any equilibrium profile $e \equiv \{x_t, U_t^b, U_t^s\}_{t=0}^{\infty}$, we define the following set:

$$L(e) \equiv \{t \in \{0,1,2...\} \mid x_t < 1\}. \tag{7}$$

$L(e)$ is the set of the periods in which matched sellers choose the minimum investment, zero, with some positive probability.

Then we show the following result which has the different feature from the case of large search friction.

**Proposition 2**. *Suppose that $\delta \in (\delta^*, 1)$. Then in any equilibrium $e$ the minimum investment level, zero, must be chosen with positive probability for infinitely many times, i.e.,*

$$\sup L(e) = +\infty.$$

**Proof**. See Appendix.

Given an equilibrium $e$, we also define the following set:

$$D(e) \equiv \{t \in \{0,1,2,...\} \mid \text{trade does not occur with positive probability in period } t.\}.$$

Then the direct corollary from Proposition 2 is the following:

**Corollary**. *Suppose that $\delta \in (\delta^*, 1)$. Then in any equilibrium $e$ trade must be delayed with positive probability for infinitely many times, i.e.,*

$$\sup D(e) = +\infty.$$

**Proof**. By Proposition 2 we know that $x_t < 1$ must happen for infinitely many times when $\delta \in (\delta^*, 1)$. Then, by Lemma 2, since trade does not occur when $a = 0$ was realized, no–trade outcome must happen with positive probability for infinitely many times as well. Q.E.D.

The intuition behind these results can be explained as follows: If there exists some equilibrium in which the minimum investment $a = 0$ is chosen with positive probability only for finite times, then by Lemma 1 the equilibrium investment must coincide with the static equilibrium one $a^s$ with probability one from some period $T$ onward. However, if $a_t = a^s$ for all $t \geq T$, the sum of the reservation values of a buyer and a seller at period $T$ must be stationary and equal to

$$U_T^b + U_T^s = \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}.$$

This is because the total surplus $S(a^s)$ will be realized in every match from period $T$ onward. Since each matched seller can obtain at least $U_T^s$ by choosing $a = 0$, the following inequality must be satisfied for each seller to choose $a^s$:

$$\beta\{v(a^s) - U_T^b - U_T^s\} + U_T^s - c(a^s) \geq U_T^s,$$

which is reduced to

$$\beta\left\{v(a^s) - \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}\right\} - c(a^s) \geq 0.$$

However this inequality is not satisfied when $\delta > \delta^*$. Thus each matched seller deviates and chooses $a = 0$ with certainty, which gives him the higher payoff $U_T^s$ than the equilibrium payoff obtained by choosing $a^s$. Hence the minimum investment $a = 0$ must be chosen with positive probability for infinitely many times in any equilibrium. Then by Lemma 2 trade must be delayed with positive probability for infinitely many times as well.

We can also relate this result to the literature about delay in bargaining games. Our above result sheds a new light on equilibrium delay in the view point of the holdup problem and search friction. The key factor to cause delay in our model is that the trade value $v(a)$ is *endogenous* and it is affected by matched seller's investment $a \in A$. This is the main difference from the standard bargaining games which have mostly assumed that trade surplus is exogenous.

When trade value is endogenous, players' expectation about the future investment decisions plays the important role: If they expect the future sellers will invest much and hence the future trade values will be high, the reservation values of buyer and seller $U_t^b$ and $U_t^s$ become high as well. Then the current negotiation surplus $v(a) - U_t^b - U_t^s$ becomes small so that current sellers invest little, which then leads to no–trade outcome in the current match. On the other hand, if players expect the future sellers will invest little, then the current net negotiation surplus becomes high and hence the current sellers invest much, which leads to trade. This argument shows that in any equilibrium there must exist *both* the event in which investment is low and trade is not realized and the event in which investment is high and trade occurs with non–trivial probabilities. By this reason, delay of trade occurs with some positive probability in our model.

Next we will focus on some particular equilibrium which possesses the feature of low investment and delayed trade as identified in Proposition 2. We can show that there are many equilibria which exhibit this feature for some large discount factors.

**Proposition 3**. *Suppose that* $\delta \in (\delta^*, 1)$. *Then there exists a unique stationary equilibrium in which every matched seller chooses the static investment* $a^s$ *and the minimum one, zero, with probabilities* $x^* \in (0, 1)$ *and* $1 - x^*$ *respectively in every period where* $x^*$ *satisfies*

$$v(a^s) - c(a^s)/\beta = \frac{\alpha\delta x^* S(a^s)}{1-\delta(1-\alpha x^*)}.$$

**Proof**. Suppose that the sum of the reservation values $U^b$ and $U^s$ is given by

$$U^b + U^s = \frac{\alpha\delta x^* S(a^s)}{1-\delta(1-\alpha x^*)}.$$

Note here that $v(a^s) > U^b + U^s > v(0)$ by Assumption 1. Also, such $x^* \in (0,1)$ is uniquely determined.

Then every matched seller is indifferent for choosing $a^s$ and zero because

$$\beta\{v(a^s) - U^b - U^s\} + U^s - c(a^s) = U^s.$$

Thus we can suppose that each matched seller randomizes his investment choice by putting the probability $x^* \in (0,1)$ on $a = a^s$ and the probability $1-x^*$ on $a = 0$ respectively. Also, such sellers's behavior can be consistent with the sum of the reservation values $U^b$ and $U^s$ we have defined above. Q.E.D.

When equilibrium is stationary in that every matched seller follows the same investment strategy in every period, it must be unique and involve the mixed strategy of seller's investment choice randomizing between $a^s$ and zero.

We can also show that there exist non–stationary equilibria as well.

**Proposition 4**. *For any positive integer $K \geq 1$, there exist some $\underline{\delta}_K$ and $\overline{\delta}_K$ such that for all $\delta \in (\underline{\delta}_K, \overline{\delta}_K)$ a non–stationary equilibrium arises where each matched seller chooses equilibrium investment $a_t$ in period $t$ as follows:*

$$a_t = \begin{cases} a^s & \text{for } t = \tau K + \tau - 1 \ (\tau = 1, 2, ...), \\ \\ 0 & \text{otherwise} \end{cases}$$

**Proof**. See Appendix.

The equilibrium shown in Proposition 4 is non–stationary in the sense that equilibrium investments of matched sellers cyclically change over time. Following the matched sellers' investment $a^s$ in some period, equilibrium investments become zero and hence trades are not realized in all subsequent $K$ periods. In the next period sellers' investment will be returned back to the static equilibrium one $a^s$ again, which leads to trade agreements for all matches in that period. Thus equilibrium investment and trade have cyclical patterns with period $K$.

Also, if there exists a non–stationary equilibrium with period $K$ cycle for some range of discount factors, then there also exist non–stationary equilibria with cycle $k$ ($k \leq K-1$) for the same range of the parameters of the model as well. This is because we have $(\underline{\delta}_K, \overline{\delta}_K) \subset (\underline{\delta}_{K-1}, \overline{\delta}_{K-1})$: If we have $\delta \in (\underline{\delta}_K, \overline{\delta}_K)$ such that a period $K$ cycle equilibrium exists, then

$\delta \in (\underline{\delta}_{K-1}, \overline{\delta}_{K-1})$ and hence period $K-1$ cycle equilibrium also exists as well.

Proposition 3 and 4 show that there exist multiple equilibria even for the same levels of market friction ($\delta \in (0,1)$). The next natural question is what equilibrium can attain the highest efficiency in the market among all possible equilibria. We will now explore this issue.

## 4.2 Welfare Analysis

In this subsection we will conduct the welfare analysis about how the change of search friction affects the equilibrium outcomes and hence the market welfare.

We define the social welfare of the search market as the discounted present value of the sum of buyers' and sellers' payoffs evaluated at initial period $t = 0$. Then, given an equilibrium path $e \equiv \{f_t, U_t^b, U_t^s\}_{t=0}^{\infty}$, we can derive the social welfare under this equilibrium $e$ for a given $\delta \in [0, 1)$ as follows:

$$W(\delta|e) \equiv \alpha \max\{E_{f_0}[S(a)], U_0^b + U_0^s\} + (1-\alpha)(U_0^b + U_0^s) \tag{8}$$

where

$$U_t^b + U_t^s = \delta\{\alpha \max\{E_{f_{t+1}}[S(a)], U_{t+1}^b + U_{t+1}^s\} + (1-\alpha)(U_{t+1}^b + U_{t+1}^s)\}, \quad t = 0, 1, 2, ... \tag{9}$$

Note here that the ex ante surplus in period $t$ is given by $\max\{E_{f_t}[S(a)], U_t^b + U_t^s\}$ because every matched pair will be dissolved and will obtain their reservation values when the net ex ante surplus $E_{f_t}[S(a)] - U_t^b - U_t^s$ is negative.

By using Lemma 1 and 2, we know that in any equilibrium each matched seller chooses $a^s$ with probability $x_t \in [0, 1]$ only in which case trade occurs and the trade value $S(a^s)$ is realized. In all other cases trade values are never realized. By using this fact, we can rewrite the social welfare as follows:

$$W(\delta|e) = \alpha x_0 S(a^s) + (1 - \alpha x_0)(U_0^b + U_0^s), \tag{10}$$

where

$$U_t^b + U_t^s = \delta\{\alpha x_{t+1} S(a^s) + (1 - \alpha x_{t+1})(U_{t+1}^b + U_{t+1}^s)\}, \quad t = 0, 1, 2, ... \tag{11}$$

Then we give the following definition.

**Definition**. An equilibrium $e = \{x_t, U_t^b, U_t^s\}_{t=0}^{\infty}$ is said to be *constrained efficient* if it attains the highest social welfare among all possible equilibria. Otherwise, it is said to be *inefficient*.

We first show the following lemma.

**Lemma 3**. *Consider a sequence* $\hat{e} = \{\hat{x}_t, \hat{U}_t^b, \hat{U}_t^s\}_{t=0}^{\infty}$ *satisfying the following:*

(i) $t = 0$: $\hat{x}_0 = 1$.

(ii) $\forall\ t \geq 1$: $\hat{x}_t = x^* \in [0,1]$ *where* $x^* = 1$ *if* $\delta \in (0, \delta^*)$ *and* $x^* \in [0,1]$ *satisfies*

$$\beta \left\{ v(a^s) - \frac{\delta \alpha x^* S(a^s)}{1 - \delta(1 - \alpha x^*)} \right\} = c(a^s)$$

*if* $\delta \in (\delta^*, 1)$.

(iii) *The reservation values:*

$$\hat{U}_t^b + \hat{U}_t^s = \frac{\alpha x^* S(a^s)}{1 - \delta(1 - \alpha x^*)}, \quad t = 0, 1, 2, ...$$

*Then* $\hat{e}$ *can be an equilibrium.*

**Proof**. Since $x^* = 1$ for all $\delta \in (0, \delta^*)$, $\hat{e}$ coincides with the unique static equilibrium shown in Proposition 1 for any $\delta \in (0, \delta^*)$.

Next suppose that $\delta \in (\delta^*, 1)$. Then, given $\hat{U}_t^b$ and $\hat{U}_t^s$ above, each matched seller faces the following payoff function in period $t$:

$$u_S(a) = \beta \max \left\{ v(a) - \frac{\delta \alpha x^* S(a^s)}{1 - \delta(1 - \alpha x^*)}, 0 \right\} + \hat{U}_t^s - c(a).$$

Then, by definition of $x^*$, $\max_{a \in A} u_S(a) = u_S(0) = u_S(a^s)$. Thus each matched seller is willing to randomize between $a = a^s$ and $a = 0$ with probabilities $x^*$ and $1 - x^*$ respectively in any period $t \geq 1$. Also it is optimal for each matched seller to choose $a^s$ with certainty ($\hat{x}_0 = 1$) in initial period $t = 0$.

Note that when $a^s$ was realized trade occurs because $v(a^s) > \hat{U}_t^b + \hat{U}_t^s$ for any $t$ while when $a = 0$ was realized trade never happens because $v(0) < v(a^s) - c(a^s)/\beta = \hat{U}_t^b + \hat{U}_t^s$ for any $t$. Thus trade occurs with probability $x^*$ per match and hence the surplus $S(a^s)$ is realized only with probability $\alpha x^*$. Thus the above seller's strategy leads to the reservation values $\hat{U}_t^b + \hat{U}_t^s$ defined above. Q.E.D.

Then we show the following result.

**Proposition 5**. *The equilibrium* $\hat{e}$ *shown in Lemma 3 is constrained efficient.*

**Proof**. See Appendix.

The intuition behind Proposition 5 can be explained as follows: First, as we have shown, every matched seller chooses either the static equilibrium investment $a^s$ or the minimum one, zero, in any equilibrium. In particular each matched seller chooses $a^s$ in period $t$ if and only if (IC) is satisfied as we have noted. Then (IC) shows that the discounted present value of the total future payoffs of a buyer and a seller in the market from period $t+1$ onward (which is evaluated at period $t$), i.e., $U_t^b + U_t^s$, is bounded from above by $v(a^s) - c(a^s)/\beta$. The equilibrium identified in Lemma 3, $\hat{e}$, can attain this upper bound and implement the highest investment incentive of matched sellers while keeping (IC)s hold as equalities in any period $t$.

From Proposition 5 we can derive the interesting implication about the efficiency property of the search market. Proposition 5 shows that the social welfare in the constrained efficient equilibrium $\hat{e}$ is given by

$$W(\delta|\hat{e}) \equiv \alpha S(a^s) + (1-\alpha)\frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)}, \tag{12}$$

where $x^* = 1$ for all $\delta \in (0, \delta^*)$ and $x^*$ satisfies $v(a^s) - c(a^s)/\beta = \frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)}$ for $\delta \in [\delta^*, 1)$ respectively. Note here that $W(\delta|\hat{e})$ is increasing in $\delta \in (0, \delta^*)$ but constant at $W(\delta|\hat{e}) = \alpha S(a^s) + (1-\alpha)[v(a^s) - c(a^s)/\beta]$ over all $\delta \in [\delta^*, 1)$.

We take any inefficient equilibrium $e \neq \hat{e}$ and consider its welfare, denoted by $W(\delta|e)$. Then, since $W(\delta|\hat{e})$ is constant over all $\delta \in (\delta^*, 1)$ and $W(\delta|e) = W(\delta|\hat{e})$ for all $\delta \in (0, \delta^*)$ due to Proposition 1, the social welfare in any inefficient equilibrium $W(\delta|e)$ must be non–monotonic with respect to the discount factor $\delta$. This is because $W(\delta|e)$ must be strictly less than $W(\delta|\hat{e})$ for some higher discount factors than $\delta^*$.

For example, we first consider the stationary equilibrium identified in Proposition 3. Let $e^s = \{x^*, U^b, U^s\}$ denote this unique stationary equilibrium. Then the social welfare in the stationary equilibrium is given by

$$W(\delta|e^s) \equiv \alpha x^* S(a^s) + (1-\alpha x^*)\frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)}$$

where note that trade will occur and hence the total surplus $S(a^s)$ will be realized only when matched sellers choose the static equilibrium investment $a^s$ which occurs with probability $x^* \in [0, 1]$. By Proposition 1 we know that $x^* = 1$ when $\delta \in [0, \delta^*]$. However, by Proposition 3 we have $x^* < 1$ when $\delta > \delta^*$. Note that in such case $x^*$ is decreasing in $\delta$. Also, since $\frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)} = v(a^s) - c(a^s)/\beta$ for all $\delta \in (\delta^*, 1)$, we can rewrite $W(\delta|e^s)$ as

$$W(\delta|e^s) = \alpha x^* S(a^s) + (1-\alpha x^*)[v(a^s) - c(a^s)/\beta]$$

for $\delta \in (\delta^*, 1)$. Thus, as shown in Figure 1, the social welfare $W(\delta|e^s)$ in the stationary equilibrium is non–monotonic with respect to the discount factor $\delta$: Smaller market friction reduces the social efficiency.

Second, we consider the non–stationary equilibria characterized in Proposition 4: We have the non–stationary equilibrium with period $K$ cycle, denoted $e^{ns}$, for $\delta \in (\underline{\delta}_K, \overline{\delta}_K)$. Then the social welfare in the non–stationary equilibrium $e^{ns}$, denoted $W(\delta|e^{ns})$, must be *strictly* less than that in the constrained efficient equilibrium $W(\delta|\hat{e})$. This is because

$$\begin{aligned} W(\delta|e^{ns}) &= \delta^{K-1}\{U^b_K + U^s_K\} \\ &\leq \delta^{K-1}\{v(a^s) - c(a^s)/\beta\} \\ &< v(a^s) - c(a^s)/\beta \\ &< \alpha S(a^s) + (1-\alpha)\{v(a^s) - c(a^s)/\beta\} \\ &= W(\delta|\hat{e}) \end{aligned}$$

where the first inequality follows from the fact that all matched sellers choose $a = 0$ and trades are not realized until period $K$ in which $v(a^s) - c(a^s)/\beta \geq U^b_K + U^s_K$ holds so that the matched sellers choose $a^s$, the second inequality from $\delta < 1$ and $K \geq 2$ and the final inequality from $S(a^s) > v(a^s) - c(a^s)/\beta$ respectively. Thus, as shown in Figure 2, non–stationary equilibrium $e^{ns}$ may arise for $\delta \in (\underline{\delta}_K, \overline{\delta}_K)$ even when we have the constrained efficient equilibrium $\hat{e}$ for all other discount factors. In such case the social welfare is not monotonic with respect to the discount factor $\delta$. Again, small search cost has the adverse effect on the market efficiency.

Next we will extend the basic model in the several directions.

# 5 Extensions

## 5.1 Heterogeneous Agents

In this subsection we will allow buyers and sellers to be heterogeneous. Let $\theta^i_t \in \Theta$ and $\psi^j_t \in \Psi$ be the random variables, called "types," which affect the valuation of buyer $i$ and investment cost of seller $j$ in period $t$. We denote by $v(a, \theta)$ the trade value which depends on the buyer's current type $\theta$. We also denote by $c(a, \psi)$ the seller's investment cost which depends on the seller's current type $\psi$. At the beginning of each period every buyer and seller do not know their own types in that period. After matching, the types of matched buyer and seller, $\theta_t$ and $\psi_t$, are realized according to some probability distributions and are then known to them. We assume that any type pair $(\theta, \psi)$ has positive measure. The realized types are common knowledge for matched players but the types of future players will be changed. We also assume that the probability distributions from which $\theta_t$ and $\psi_t$ are drawn are independent of time index $t$ and that any type of any player is replaced by a new entrant with the same type when he or she leaves the market. These assumptions can ensure that the distributions of types can be kept constant over time.

Suppose that in period $t$ buyer $i$ and seller $j$ matched and observed their realized types as $\theta$ and $\psi$ respectively. Given these types, in period $t$ the seller will obtain the following payoff:

$$\beta \max\{v(a,\theta) - U_t^b - U_t^s, 0\} + U_t^s - c(a,\psi) \tag{13}$$

where the sum of the reservation values $U_t^b$ and $U_t^s$ are given by

$$U_t^b + U_t^s = \delta\{\alpha E_{\theta,\psi}[v(a_{t+1},\theta) - c(a_{t+1},\psi)] + (1-\alpha)(U_{t+1}^b + U_{t+1}^s)\}. \tag{14}$$

Here $E_{\theta,\psi}[\cdot]$ denotes expectation operator with respect to the types of buyers and sellers $(\theta, \psi)$.

Matched seller of type $\psi$ chooses his investment level $a_t$ to maximize his expected payoff (13), given the reservation values $U_t^b$ and $U_t^s$. By using a similar argument to Lemma 1, we can show that the optimal investment choice by each matched seller becomes $a = 0$ or $a = a^s(\theta,\psi)$ where

$$a^s(\theta,\psi) \equiv \arg\max_{a \in A} \beta v(a,\theta) - c(a,\psi).$$

Also, we can show that trade will occur when the seller chooses $a^s(\theta,\psi)$ again. Then let $x_t(\theta,\psi) \in [0,1]$ denote the equilibrium probability that seller of type $\psi$ chooses $a^s(\theta,\psi)$ when he matches with buyer of type $\theta$ in period $t$. $x_t(\theta,\psi)$ also means the probability of trade in period $t$ when buyer of type $\theta$ and seller of type $s$ match with each other. Let $x_t \equiv (x_t(\theta,\psi))_{(\theta,\psi)\in\Theta\times\Psi}$ be a collection of possible trade probabilities in period $t$. Then we can define an equilibrium as a path $e \equiv \{x_t, U_t^b, U_t^s\}_{t=0}^{\infty}$ as before.

For a given equilibrium $e$, we define the following set:

$$\tilde{L}(e) \equiv \{t = 0, 1, 2... \mid x_t(\theta,\psi) < 1 \text{ for some } (\theta,\psi) \in \Theta \times \Psi\}.$$

$\tilde{L}(e)$ is the set of the periods in which trades do not occur with some positive probabilities according to equilibrium $e$. Then we can extend our inefficiency result (Proposition 2) to the case of heterogeneous agents:

**Proposition 6**. *There exists some* $\overline{\delta} \in (0,1)$ *such that for all* $\delta \in (\overline{\delta}, 1)$ *we have* $\sup\tilde{L}(e) = +\infty$ *in any equilibrium* $e$.

**Proof**. See Appendix.

## 5.2 Ex Ante Contracts

We have so far assumed that matched parties cannot write ex ante contracts at all before investment choice because trade decision is assumed to be not contractible ex ante. In this subsection we will assume that it is ex ante verifiable whether or not trade is realized. Thus matched parties can write ex ante contract contingent on trade outcome before sellers make specific investments.

Since ex ante contract is introduced, one might think that the inefficiency identified in the previous sections can be reduced. However, we will show that this is not always the case and rather the market outcome may be less efficient when ex ante contracts are possible than when they are not at all. This implies that enlarging the set of contract space may not improve the market efficiency when the market is more competitive in the sense that search cost is so small.[9]

To keep consistency of all our assumptions, we also assume that each matched pair of a buyer and a seller can verify whether or not they actually traded the good in their current match but each of them cannot know the past history of his or her matched partner such as whether he or she has matched but has not traded the goods in past periods. [10]

We also assume away the message games which use the messages sent by the parties and make the trade decision and payment contingent on them. The message game is more general mechanism than the simple non–contingent contract which specifies only fixed payments. However, since our main purpose here is to show the possibility that ex ante contracting may make the market efficiency worse, it suffices to focus on the simple case that message games are ruled out: Our result shows at least that expanding the set of possible contracts does not always improve the market efficiency.

Now, since at the beginning of each period ex ante contracts are possible contingent on trade outcome, in period $t$ matched parties can write an ex ante contract $\{P_t^1, P_t^0\}$ where $P_t^1$ ($P_t^0$) denotes the payment made from the buyer to the seller when the good is (not) traded.

Recall that the buyer's valuation of the good is given by $b(a)$ and the seller's production cost by $d(a)$. Here we assume that $d(a)$ represents the personal disutility seller incurs for delivering the good to buyer as well as $c(a)$ is the personal disutility of investment. We also assume that both $b$ and $d$ are increasing and positive for all $a \in A$. In the previous sections the total trade value $v(a) = b(a) - d(a)$ matters for the equilibrium outcome. On the other hand, once we handle the case of ex ante contracts, we need to make these notations $b(a)$ and $d(e)$ explicitly.

We make the following assumptions:

**Assumption 2**. *(i) Each matched buyer has the full bargaining power to make an offer of ex ante contract to matched seller. (ii) Each seller has*

[9]We may be able to connect this result to the literature about price discrimination among competing firms, which has discussed the effects of enlarging the instruments of firms' pricing policies on industry profits, consumer surplus and social welfare. In some cases welfare may become lower when more instruments are available for firms. See Armstrong (2006) for the survey on related topics.

[10]If matched parties can observe their past history, they can may make their current strategies contingent on the observed past history of trade outcomes. This is not consistent with the assumption of anonymous market.

*no wealth. (iii)* $A = [0, \overline{a}]$. *(iv)* $b$, $d$ *and* $c$ *are all continuous. (v)* $S(a)$ *is increasing in* $a \in [0, a^s]$.

Assumption 2(i) says that ex ante contract is offered by a matched buyer to a matched seller in the take–it–or–leave–it fashion. One reason for why buyer has full bargaining power ex ante but not ex post might be that, since seller's investment is match specific and indispensable for trade, the seller has some bargaining power ex post after investment was made. Assumption 2(ii) requires that any payment made from buyer to seller must be non–negative, $P_t^i \geq 0$ for $i = 0, 1$ and all $t$. We can readily show that any equilibrium attains the first best outcome when sellers are not wealth constrained (See the Remark below). Thus the limited liability of sellers will play the role of making equilibrium comparison meaningful. Assumption 2(iii) and (iv) say that seller chooses continuous investment which continuously affects both the buyer's value $b$ and seller's disutility of delivery $d$. Finally Assumption 2(v) is satisfied when $S$ is a concave function because $a^s$ is not higher than the first best investment which maximizes $S(a)$ over $A$.

Since contracts are incomplete and sellers are subject to limited liability, by Assumption 2 we are concerned with the situations in which contract enforcement is limited and severe as in emerging markets.

The timing of events in a period is changed as follows:

- date 0 Matched buyer offers an ex ante contract $\{P_t^1, P_t^0\}$ to a matched seller where $P_t^i \geq 0$ for $i = 0, 1$. Then the seller decides whether or not to accept it. If he rejects it, the parties obtain their reservation values $U_t^b$ and $U_t^s$.
- date 1 The seller makes specific investment $a \in A$. This is observed to both parties.
- date 1.5 With probability $\beta \in (0, 1)$, the seller makes a renegotiation contract offer to the buyer while with probability $1 - \beta$ the buyer makes a renegotiation contract offer to the seller. If the parties agree on renegotiation contract, this will be in force. Otherwise, the ex ante contract will be in force.
- date 2 Trade decision and payments are made according to binding contract (ex ante or renegotiated one).

Given an ex ante contract $\{P_t^1, P_t^0\}$, matched parties agree on trade of the good if and only if the following *voluntary trade conditions* are satisfied (Hart and Moore (1988)):

$$b(a) - P_t^1 \geq -P_t^0 + U_t^b, \tag{15}$$

for buyer and

$$P_t^1 - d(a) \geq P_t^0 + U_t^s \tag{16}$$

for seller respectively.

Inequality (15) says that the buyer's payoff from trade under the ex ante contract (the left hand side) is not less than her payoff from no–trade under the ex ante contract (the right hand side). Inequality (16) has a similar meaning for the seller.

If these conditions are satisfied, then the matched parties agree on trade under ex ante contract $\{P_t^1, P_t^0\}$. Thus they can avoid renegotiation in such case. If these inequalities are not satisfied, no–trade outcome becomes status quo and hence they always renegotiate ex ante contract when ex post surplus from trade is positive. However, in the presence of seller's limited liability, renegotiation may fail even when ex post surplus $v(a) - U_t^b - U_t^s$ is positive. The seller makes a renegotiation payment offer $R \geq 0$ to ensure that the buyer accepts it, i.e., $b(a) - R \geq P_t^0 + U_t^b$. If $b(a) - P_t^0 - U_t^b < 0$ holds no such $R \geq 0$ exists to satisfy the seller's limited liability constraint. Thus in that case trade will not occur and the buyer and seller will obtain the payoffs $-P_t^0 + U_t^b$ and $P_t^0 + U_t^s$ respectively. On the other hand, when the buyer makes a renegotiation offer $R \geq 0$, the limited liability constraint of seller is never binding because the seller's acceptance condition requires $R - d(a) \geq P_t^0 + U_t^s$ which yields $R \geq d(a) + P_t^0 + U_t^s \geq 0$.

**Remark**. The limited liability imposed on sellers plays the important role here. Without the limited liability constraint, in any equilibrium the first best outcome can be attained under some mild conditions once we introduce ex ante contracts (See Appendix).

Now the seller's payoff resulting from renegotiation (net of investment disutility $c(a)$) can be written by

$$u_t^s(a) = \begin{cases} \beta \max\{v(a) - U_t^b - U_t^s, 0\} + U_t^s + P_t^0 - c(a) & \text{if } b(a) \geq P_t^0 + U_t^b, \\ \\ U_t^s + P_t^0 - c(a) & \text{otherwise} \end{cases} \tag{17}$$

We define the *cost minimizing investment*, denoted $a^m$, as follows:

$$a^m \equiv \arg\min_{a \in A} d(a) + c(a), \tag{18}$$

which minimizes the total disutility of seller, delivery and investment disutilities $d(a)$ and $c(a)$. We assume that $a^m$ is unique and satisfies the following:

**Assumption 3**. $S(a^m) > v(a^s) - c(a^s)/\beta$.

Assumption 3 says that the seller's bargaining power $\beta$ is not so large relative to the ex ante surplus $S(a^m)$ attained by the cost minimizing investment $a^m$.

We also define the ex ante contract, denoted $C^* = \{P^1, P^0\}$, which implements the cost minimizing investment of seller $a^m$ at the minimum payment of buyer:

$$P^1 = d(a^m) + c(a^m), \quad P^0 = 0.$$

We will call this $C^*$ the cost minimizing contract.

Then we show the following result.

**Proposition 7**. *Suppose that matched parties can write ex ante contract contingent on trade outcome after matching but before sellers' investments and that Assumption 1–3 hold. Then, for small search friction (large $\delta$), there exists a stationary equilibrium in which every matched buyer offers the cost minimizing contract $C^*$ and every matched seller chooses the cost minimizing investment $a^m$ with certainty in every period.*

**Proof**. See Appendix.

The intuition behind Proposition 7 is as follows: Suppose that the reservation values of buyer and seller are stationary and given by

$$\overline{U}^b = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)}, \ \ \overline{U}^s = 0.$$

Suppose also that every matched buyer offers the cost minimizing contract $C^*$ to her matched seller. Then, if the seller accepts this contract and chooses the cost minimizing investment $a^m$, the buyer will obtain the payoff $b(a^m) - P^1 = S(a^m)$ while the seller will obtain the payoff equal to $P^1 - d(a^m) - c(a^m) = 0$. This can be a stationary equilibrium. Thus we have to show two things for large $\delta \in (0,1)$: First, every matched seller has the incentive to choose $a^m$, given the contract $C^*$ and the above reservation values $\overline{U}^b$ and $\overline{U}^s$. Second, every matched buyer has no incentives to offer other contracts than $C^*$, given these reservation values.

First, given the contract $C^*$, if some matched seller deviates to choose $a \neq a^m$, then he can be better off only when renegotiation of $C^*$ will happen (because otherwise he will end up obtaining $P^1 - d(a) - c(a)$ but this is less than the payoff $P^1 - d(a^m) - c(a^m) = 0$ by choosing $a = a^m$). However, under renegotiation the seller's payoff is at most $\beta \max\{v(a) - \overline{U}^b - \overline{U}^s, 0\} + \overline{U}^s - c(a)$ which is less than $\overline{U}^s = 0$ by choosing $a = 0$ because small friction (large $\delta$) makes the reservation values large so that $\overline{U}^b + \overline{U}^s \to S(a^m) > v(a^s) - c(a^s)/\beta$ due to Assumption 3. Thus when the search friction is small enough every matched seller chooses $a^m$ with certainty in order to avoid renegotiation of the contract $C^*$.

Second, can some buyer improve her payoff by offering other contracts than $C^*$? Answer is No. This is because every matched buyer can implement only the cost minimizing investment $a^m$ by offering ex ante contract: If renegotiation occurs under ex ante contract $\{P^1, P^0\}$, this does not result in higher payoff of the buyer than $S(a^m)$. The reason for this is that when renegotiation happens the buyer will obtain the payoff $(1-\beta)\max\{v(a) - \overline{U}^b - \overline{U}^s, 0\} + \overline{U}^b$ which is however less than $\overline{U}^b$ because when $\delta$ is large enough the seller will choose $a = 0$ under renegotiation as we have seen above. Thus, if the buyer can be better off by ex ante contract, it must be the case that the voluntary trade conditions hold and hence renegotiation is avoided. For example suppose that the seller chooses $a \neq a^m$. Then the seller's payoff becomes $P^1 - d(a) - c(a)$ when renegotiation is avoided. We can then show that the voluntary trade conditions must be satisfied with *strict* inequalities due to the seller's limited liability. If the voluntary trade condition for the buyer $b(a) - P^1 \geq -P^0 + U^b$ holds as equality, her payoff is $-P^0 + U^b$ which is not higher than her reservation value $U^b$ due to $P^0 \geq 0$. If the voluntary trade condition for the seller $P^1 - d(a) \geq P^0 + U^s$ holds as equality, then his payoff is $P^0 + U^s - c(a)$ which is less than the payoff attained by slightly reducing the investment to $a'$ ($a' < a$) and inducing renegotiation, $\beta \max\{v(a') - U^b - U^s, 0\} + U^s + P^0 - c(a')$ for $a' < a$. Thus the voluntary trade conditions must hold with strict inequalities both for the buyer and the seller. However, then the seller can slightly change his investment from $a$ to some $a''$ towards $a^m$ and obtain higher payoff $P^1 - d(a'') - c(a'')$ as long as $a \neq a^m$. This argument shows that the buyer can implement the cost minimizing investment $a^m$ and hence obtain at most $S(a^m)$ by offering ex ante contract.

One important implication of Proposition 7 is that the equilibrium when ex ante contracts are possible may be less efficient than the equilibrium when no ex ante contracts are possible. Next we will explore this issue.

First note that Proposition 5 still remains true even when we introduce the limited liability constraint of sellers: Suppose that no ex ante contracts are possible. Then the seller's limited liability constraint results in failure of renegotiation only if $b(a) < U_t^b$, even when the ex post surplus $v(a) - U_t^b - U_t^s$ is positive. However, since in the constrained efficient equilibrium we have $b(a^s) > v(a^s) > U^b + U^s = \frac{\delta \alpha x^* S(a^s)}{1 - \delta(1 - \alpha x^*)}$ and hence $b(a^s) > U_t^b$, the limited liability constraint is never binding in the constrained efficient equilibrium identified in Proposition 5.

Now we investigate the effects of ex ante contracts on market efficiency. We compare the social welfare attained in the equilibrium when ex ante contracts are possible with that attained in the equilibrium when they are not. In particular we consider the equilibrium shown in Proposition 7, which we will call *contract equilibrium*, and the equilibrium shown in Lemma 3, which we will call *no–contract equilibrium.* Recall here that the no–contract equilibrium is constrained efficient among all possible equilibria when no ex ante contracts are written at all (Proposition 5). One might think that our comparison between these equilibria is arbitrary because we will rule out other equilibria. However, since our purpose is to demonstrate the possibility that inefficiency of the markets with small friction becomes more severe when ex ante contracts are introduced than when they are not, it is sufficient to show that there exists at least one such equilibrium with ex ante contracts which is less efficient than some equilibrium without ex ante contracts.

By Proposition 7, the social welfare attained in contract equilibrium is given by

$$W^c(\alpha, \delta) \equiv \frac{\alpha S(a^m)}{1 - \delta(1 - \alpha)}, \tag{19}$$

when the discount factor $\delta$ is large because all matched sellers choose the cost minimizing investment $a^m$ with certainty in all periods.

On the other hand, the social welfare in no–contract equilibrium was defined as $W(\delta|\hat{e})$ (see (12)). To make the dependence of $W(\delta|\hat{e})$ on both $\delta$ and $\alpha$ explicit, we can rewrite this as:

$$W^n(\alpha, \delta) \equiv \alpha S(a^s) + (1 - \alpha)\frac{\delta\alpha x^* S(a^s)}{1 - \delta(1 - \alpha x^*)} \tag{20}$$

where $x^* \in [0, 1]$ satisfies $\frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)} = v(a^s) - c(a^s)/\beta$ for $\delta \in (\delta^*, 1)$ and $x^* = 1$ for all $\delta \in (0, \delta^*)$.

We make the following assumption:

**Assumption 4**. $a^s > a^m$.

Then the direct comparison between $W^c(\alpha, \delta)$ and $W^n(\alpha, \delta)$ yields the following result:

**Proposition 8**. *Suppose that Assumption 1–4 hold. Suppose also that the search friction measured by both $\alpha$ and $\delta$ is small, i.e., $\alpha$ and $\delta$ are close to one. Then no–contract equilibrium can attain higher social welfare than contract equilibrium, i.e., $W^n(\alpha, \delta) > W^c(\alpha, \delta)$.*

**Proof**. When $\delta$ tends to be close to one, we have $W^c(\alpha, \delta) \to S(a^m)$ while

$$W^n(\alpha, \delta) \to \alpha S(a^s) + (1 - \alpha)[v(a^s) - c(a^s)/\beta]$$

which is close to $S(a^s)$ when $\alpha \to 1$.

By Assumption 2(v) $S$ is increasing over $[0, a^s]$ and hence we have $S(a^s) > S(a^m)$ due to Assumption 4. Q.E.D.

For Proposition 8 to be meaningful, Assumption 4 must be consistent with Assumption 3. We can give an example to ensure this as follows:

**Example**. Assume that $A = [0, 1]$, $b(a) = 1 + (1 - \rho)a$, $d(a) = 1 - \rho a$ and $c(a) = (1/2)a^2$ where $\rho \in (0, \beta)$. Thus $v(a) = a$. In this quadratic example we can show that $a^s = \beta$, $a^m = \rho$, $S(a^m) = \rho(1 - (1/2)\rho)$, $v(a^s) - c(a^s)/\beta = (1/2)\beta$ and $S(a^s) = \beta(1 - (1/2)\beta)$. Thus we have $a^s = \beta > a^m = \rho$ which satisfies Assumption 4. Also, when $1/2 > \beta$, we have $S(a^m) > v(a^s) - c(a^s)/\beta$ for all $\rho \in (0, \beta)$ which satisfies Assumption 3.

Proposition 8 shows that introduction of ex ante contracts does not mitigate the holdup inefficiency caused by small market friction but rather exaggerates such inefficiency. In fact enlarging the set of available contracts may make the market efficiency worse than the case of limited contracts. Thus this might imply that it is not optimal policy to make markets competitive in developing countries with limited contract enforcement.

### 5.3 Two–Sided Investment

In this subsection we will extend the basic model to allow both matched buyers and sellers to make specific investments. One might think that, if matched buyers also make non–contractible investment as well as sellers do, it becomes more difficult to improve efficiency. However, this argument is not true in the dynamic search market model we are considering here. Surprisingly, we show that in completely symmetric case the two–sided investment may help recover efficiency when the inefficiency identified in Proposition 2 arises in the case of one–sided investment.

To see this, consider the following example:

**Example 1**. Consider the symmetric and two–sided investment case: Each matched buyer chooses match specific investment $a_B \in \{0, 1\}$ as well as each matched seller does so $a_S \in \{0, 1\}$. Party $i$ $(i = B, S)$ incurs the investment cost $ca_i$ where $c > 0$. Also each party has the same ex post bargaining power $\beta = 1/2$. Let $v(a_B, a_S)$ be the trade value of the good and assume that $v(\cdot, \cdot)$ is increasing and

$$S^* \equiv v(1, 1) - 2c > v(1, 0) = v(0, 1).$$

This also implies that it is the first best optimal to implement high investments $a_B = a_S = 1$ from both parties because we have $v(1, 1) - 2c > v(0, 1) - c$ and $v(1, 1) - 2c > v(0, 0)$. Also assume $S^* > 0$.

Then we can show that there exists an equilibrium which implements the first best efficiency when the search friction tends to be small ($\delta \to 1$), even though this is not the case when only one party of matched pair invests.

To see this result, first consider the case of one–sided investment. Then suppose that $\delta$ is so high that there exist no equilibria in which every matched investing party chooses high investment with probability one as shown by Proposition 2.

Now we consider the two–sided investment case that both parties of each match choose investments, $a_i \in \{0,1\}$, $i = B, S$. Then we will show that there exists a stationary equilibrium in which every matched party chooses high investment $a_B = a_S = 1$ with probability one.

To see this, suppose that the reservation values $U_B^*$ and $U_S^*$ are given as follows:

$$U_B^* + U_S^* = \frac{\delta\alpha S^*}{1-\delta(1-\alpha)}.$$

Then consider the investment incentive of one party of a matched pair, say $i$, given the other party choosing high investment $a_j = 1$. If $i$ chooses $a_i = 1$ as well, then he obtains the payoff:

$$\frac{1}{2}\{v(1,1) - U_B^* - U_S^*\} + U_i^* - c.$$

Here the ex post surplus $v(1,1) - U_B^* - U_S^*$ is always positive for all $\delta \in (0,1)$.

On the other hand, if $i$ chooses $a_i = 0$, then he can obtain the following payoff:

$$\frac{1}{2}\max\{v(0,1) - U_B^* - U_S^*, 0\} + U_i^*.$$

Then, since $v(1,1) - 2c > v(0,1)$ and definition of $U_B^* + U_S^*$, the term $v(0,1) - U_B^* - U_S^*$ converges to $v(0,1) - [v(1,1) - 2c] < 0$ as $\delta \to 1$. Thus ex post surplus is negative, which shows that $i$'s payoff becomes $U_i^*$ by choosing $a_i = 0$. In particular, there exists some $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$ we have $v(0,1) - U_B^* - U_S^* < 0$. Thus party $i$ chooses $a_i = 1$ for all $\delta \in (\overline{\delta}, 1)$, if the following inequality is satisfied

$$\frac{1}{2}\{v(1,1) - U_B^* - U_S^*\} + U_i^* - c \geq U_i^*, \quad i = B, S.$$

In fact we derive

$$\begin{aligned}
&\frac{1}{2}\{v(1,1) - U_B^* - U_S^*\} - c \\
&= \frac{1}{2}\left\{v(1,1) - \frac{\delta\alpha[v(1,1) - 2c]}{1-\delta(1-\alpha)}\right\} - c \\
&> \frac{1}{2}\{v(1,1) - [v(1,1) - 2c]\} - c \\
&= \frac{1}{2}2c - c \\
&= 0.
\end{aligned}$$

Thus there exists a stationary equilibrium in which all matched parties choose the first best investment $a_B = a_S = 1$, given their reservation values

$U_B^*$ and $U_S^*$ where $U_B^* + U_S^* = \frac{\delta\alpha S^*}{1-\delta(1-\alpha)}$.

However, this result depends on the assumption that both buyers and sellers have the same investment cost as well as the same bargaining power. In fact we can show that, when buyers and sellers are heterogeneous with respect to their investment costs and bargaining powers, the inefficiency result shown in Proposition 2 still appears. More precisely, we show that there generically exist no equilibria in which both matched buyer and seller choose high investments ($a_B = a_s = 1$) with probability one.

To see this, consider the following example:

**Example 2**. We modify Example 1 as follows: The buyer's cost of choosing high investment $c_B$ differs from the seller's cost $c_S$ ($c_B \neq c_S$).

Then the following incentive compatibility constraints must be satisfied if there exists a stationary equilibrium in which every matched pair chooses high investment $a_B = a_S = 1$ (first best investment) with probability one:

$$\beta\{v(1,1) - U_B^* - U_S^*\} + U_S^* - c_S \geq U_S^*$$

and

$$(1-\beta)\{v(1,1) - U_B^* - U_S^*\} + U_B^* - c_B \geq U_B^*$$

where $U_i^*$ denotes the equilibrium reservation value of party $i = B, S$. The reason why the above inequalities must hold is that each matched party can choose low investment ($a_i = 0$) and ensure at least the reservation value $U_i^*$ by himself.

Also the equilibrium reservation values $U_B^*$ and $U_S^*$ must satisfy

$$U_B^* + U_S^* = \frac{\delta\alpha[v(1,1) - c_B - c_S]}{1-\delta(1-\alpha)}.$$

Let $\delta \to 1$. Then the above inequalities can be written by

$$\beta(c_B + c_S) \geq c_S$$

and

$$(1-\beta)(c_B + c_S) \geq c_B.$$

Since these inequalities must hold as equalities, we must have

$$\frac{1-\beta}{\beta} = \frac{c_B}{c_S}.$$

However, a small perturbation of the parameter values violates this equality, which shows that there generically exist no equilibria in which both matched buyer and seller choose high investments (the first best investment) with probability one.

Example 2 shows that the inefficiency result shown by Proposition 2 appears again even when both parties of matched pair make specific investments, if a slight perturbation of the parameters breaks exact symmetry between buyers and sellers. In this sense our inefficiency result shown in Proposition 2 is generically robust even when both parties of matched pair make specific investments.

# 6 Conclusion

In this paper we have investigated the holdup problem in the search market in which many buyers and sellers search for trading partners and specific investments are made before trade but after match. Then we have shown that the holdup problem imposes more serious inefficiency when search friction becomes smaller: When search friction tends to be small, equilibrium investment is dropped down to zero and trade is delayed with positive probability for infinitely many times. Thus equilibrium delay of trade must occur with non–trivial probability even in the complete information setting.

# 7 Appendix

## 7.1 Proof of Lemma 1

. Every matched seller's equilibrium payoff $u_t^s$ in period $t$ is given by

$$u_t^s \equiv \int_A \{\beta \max\{v(a) - U_t^s - U_t^b, 0\} + U_t^s - c(a)\} df_t.$$

Let $u_S(a)$ be the function of $a$:

$$u_S(a) \equiv \beta \max\{v(a) - U_t^s - U_t^b, 0\} + U_t^s - c(a).$$

Then $u_S(a)$ can be written by

$$u_S(a) = \begin{cases} \beta\{v(a) - U_{t+1}^s - U_{t+1}^b\} + U_{t+1}^s - c(a) & \text{for } a \text{ such that } v(a) > U_t^s + U_t^b, \\ U_t^s - c(a) & \text{otherwise.} \end{cases}$$

Now suppose that $f_t(A \setminus \{0, a^s\}) > 0$. In particular take any $a'' \in A \setminus \{0, a^s\}$ and suppose that $u_S(a'') \geq \max\{u_S(0), u_S(a^s)\}$. Then, since any seller can choose $a = 0$ and guarantee at least the payoff $U_t^s$, we must have $u_S(a'') \geq U_t^s$, which can be written by

$$\beta \max\{v(a'') - U_t^b - U_t^s, 0\} \geq c(a'').$$

Then, since $a'' > 0$ and $c(a'') > 0$, $v(a'') > U_t^b + U_t^s$ must hold and hence the above inequality yields $v(a'') - c(a'')/\beta \geq U_t^b + U_t^s$, which in turn implies

$$v(a^s) - c(a^s)/\beta > v(a'') - c(a'')/\beta \geq U_t^b + U_t^s$$

by definition of $a^s$ and Assumption 1. Thus $v(a^s) > U_t^b + U_t^b$ is satisfied and then some matched seller will deviate to choose $a^s$ with certainty and obtain a higher payoff than $u_S(a'')$. This is a contradiction. Thus $f_t(A\backslash\{0, a^s\}) = 0$ must hold in any period $t$. Q.E.D.

### 7.2 Proof of Lemma 2

First suppose that $a_t = a^s$ was realized according to the equilibrium strategy $x_t$ in period $t$. Then, since $a^s$ was realized, $x_t > 0$ must be satisfied. Since any seller can choose $a = 0$ and guarantee at least the payoff $U_t^s$, we must have

$$\beta \max\{v(a^s) - U_t^b - U_t^s, 0\} \geq c(a^s),$$

which in turn implies $v(a^s) > U_t^b + U_t^s$ due to $c(a^s) > 0$. Since ex post surplus $v(a^s) - U_t^b - U_t^s$ is positive, trade must occur after $a_t = a^s$ was realized.

Second suppose that $a_t = 0$ was realized. This implies $x_t < 1$. Then, if $v(0) \geq U_t^b + U_t^s$ holds, we have $v(a^s) > U_t^b + U_t^s$ and hence the seller is better off by choosing $a^s$ with certainty due to Assumption 1. Thus it must be the case that

$$v(0) < U_t^b + U_t^s$$

which shows that ex post surplus $v(0) - U_t^b - U_t^s$ is negative and thus trade never occurs after $a_t = 0$ was realized. Q.E.D.

### 7.3 Proof of Proposition 1

By Lemma 1, we can identify the seller's equilibrium strategy $f_t$ by the probability $x_t \in [0, 1]$ putting on $a = a^s$ and the probability $1 - x_t$ on $a = 0$ respectively. Thus we will write the equilibrium strategy as $x_t$ instead of $f_t$.

Take any equilibrium path $\{x_t, U_t^b, U_t^s\}_{t=0}^{\infty}$. Then by Lemma 1 and 2, the sum of the reservation values of a buyer and a seller can be written as follows:

$$U_t^b + U_t^s = \delta\{\alpha x_{t+1} S(a^s) + (1 - \alpha x_{t+1})(U_{t+1}^b + U_{t+1}^s)\} \qquad \text{(A1)}$$

because every matched seller chooses the static equilibrium investment $a^s$ with probability $x_t \in [0, 1]$ in period $t$, only in which case trade occurs and the trade value $S(a^s)$ is realized.

First we show that $U_t^b + U_t^s < S(a^s)$ for all $t$. Suppose that $U_T^b + U_T^s \geq S(a^s)$ for some period $T$. Then by (A1) we have

$$\begin{aligned} S(a^s) &\leq U_T^b + U_T^s \\ &= \delta\{\alpha x_{T+1} S(a^s) + (1 - \alpha x_{T+1})(U_{T+1}^b + U_{T+1}^s)\} \\ &\leq \delta\alpha x_{T+1} S(a^s) + (1 - \delta\alpha x_{T+1})(U_{T+1}^b + U_{T+1}^s) \end{aligned}$$

which shows $S(a^s) \leq U^b_{T+1} + U^s_{T+1}$ because $1 > \delta\alpha x_{T+1}$. Repeating this argument, we have $U^b_k + U^s_k \geq S(a^s)$ for all $k \geq T$. This then implies that $U^b_k + U^s_k \geq S(a^s) > v(a^s) - c(a^s)/\beta$ and hence $x_k = 0$ for all $k \geq T$. Then, by Lemma 2 trade never occurs in all periods $k \geq T$ and thus every player obtains his or her outside payoff, zero, in each period from period $T$ forever, irrespective of matching a partner. However, then $U^i_k = 0$ for all $k \geq T$ and $i = b, s$, which contradicts $U^b_k + U^s_k \geq S(a^s) > 0$ for all $k \geq T$. Thus we must have $S(a^s) > U^b_t + U^s_t$ for all $t$.

$U^i_t \geq 0$ must also hold for all $t$ and $i = b, s$ because any player can always obtain his or her outside payoff, zero, in any period by rejecting trade.

Then we derive

$$\begin{aligned} U^b_t + U^s_t &= \delta\{\alpha x_{t+1} S(a^s) + (1 - \alpha x_{t+1})(U^b_{t+1} + U^s_{t+1})\} \\ &\leq \delta\{\alpha S(a^s) + (1-\alpha)(U^b_{t+1} + U^s_{t+1})\} \\ &= \delta\left\{\alpha S(a^s)\frac{1 - \delta^{T-1}(1-\alpha)^{T-1}}{1 - \delta(1-\alpha)} + \delta^T(U^b_T + U^s_T)\right\} \end{aligned}$$

where the inequality follows from the fact that $S(a^s) > U^b_t + U^s_t$ for all $t$ in any equilibrium. Then, by taking large enough $T$ ($T \to \infty$) and noting that $U^b_T + U^s_T$ is bounded from above by $S(a^s)$ and from below by zero, for the above inequality to hold for any $T$, we must have

$$U^b_t + U^s_t \leq \frac{\delta\alpha S(a^s)}{1 - \delta(1-\alpha)}, \quad \forall\, t.$$

Thus, the matched seller's payoff function $u_S(a)$ defined by

$$u_S(a) = \beta \max\{v(a) - U^s_t - U^b_t, 0\} + U^s_t - c(a),$$

can attain its maximum at $a = a^s$ because

$$v(a^s) - c(a^s)/\beta > \frac{\delta\alpha S(a^s)}{1 - \delta(1-\alpha)} \geq U^b_t + U^s_t$$

for all $\delta \in (0, \delta^*)$. This shows that each matched seller chooses $a^s$ with probability one. Q.E.D.

### 7.4 Proof of Proposition 2

Now take an equilibrium path $e = \{x_t, U^b_t, U^s_t\}^\infty_{t=0}$. Then we show that $\sup L(e) = +\infty$.

Suppose contrary to the claim that $\sup L(e) < +\infty$ in some equilibrium $e$. Then there must exist some period $T$ such that we have $x_t = 1$ for any $t \geq T$.

Consider period $T + 1$. Then we first show that

$$U^b_{T+1} + U^s_{T+1} = \frac{\delta\alpha S(a^s)}{1 - \delta(1-\alpha)}.$$

To see this, first suppose that $U^b_{T+1} + U^s_{T+1} > \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}$. Then we can show that $U^b_{T+2} + U^s_{T+2} > U^b_{T+1} + U^s_{T+1}$ because, if not, since $x_{T+2} = 1$ we have

$$\begin{aligned} U^b_{T+1} + U^s_{T+1} &= \delta\{\alpha S(a^s) + (1-\alpha)(U^b_{T+2} + U^s_{T+2})\} \\ &\leq \delta\{\alpha S(a^s) + (1-\alpha)(U^b_{T+1} + U^s_{T+1})\} \end{aligned}$$

which shows $U^b_{T+1}+U^s_{T+1} \leq \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}$, a contradiction. Thus $U^b_{T+2}+U^s_{T+2} > U^b_{T+1}+U^s_{T+1}$. Repeating this argument, we must have $U^b_k+U^s_k > U^b_{k-1}+U^s_{k-1}$ for all $k \geq T+2$. However, then there must exist some period $K$ ($K \geq T+1$) such that $U^b_K + U^s_K > v(a^s) - c(a^s)/\beta$, which implies $x_K = 0$ but this contradicts $x_t = 1$ for all $t > T$. Thus we must have $U^b_{T+1}+U^s_{T+1} \leq \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}$. Next suppose that $U^b_{T+1} + U^s_{T+1} < \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}$. Then, by a similar argument, we can show that $U^b_k + U^s_k > U^b_{k+1} + U^s_{k+1}$ for all $k \geq T$. However, then $U^b_M + U^s_M < 0$ for some period $M > T$, which contradicts to $U^i_t \geq 0$ for all $t$ and $i = b, s$. Thus we must have

$$U^b_{T+1} + U^s_{T+1} = \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}.$$

Then, given this value, each matched seller's equilibrium payoff in period $T+1$ is written by

$$u^s_{T+1} \equiv \beta\{v(a^s) - U^b_{T+1} - U^s_{T+1}\} + U^s_{T+1} - c(a^s)$$

because $x_{T+1} = 1$. However, by substituting $U^b_{T+1} + U^s_{T+1} = \frac{\delta\alpha S(a^s)}{1-\delta(1-\alpha)}$ into this expression, we can verify that for any $\delta \in (\delta^*, 1)$ $u^s_{T+1}$ must be less than $U^s_{T+1}$ which can be attained at least by choosing $a_{T+1} = 0$. This is a contradiction. Thus we must have $\sup L(e) = +\infty$. Q.E.D.

### 7.5 Proof of Proposition 3

For a positive integer $K \geq 1$, define the sequence $\{U^N_t\}^K_{k=1}$ recursively as follows:

$$U^N_k = \delta U^N_{k+1}, \quad k = 1, 2, ..., K-1,$$

and

$$U^N_K = \delta\{\alpha S(a^s) + (1-\alpha)U^I\}$$

where

$$U^I = \delta U^N_1.$$

Solving these values, we obtain

$$U^N_k = \frac{\delta^{K-k+1}\alpha S(a^s)}{1-\delta^{K+1}(1-\alpha)}, \quad k = 1, 2, ..., K,$$

and

$$U^I = \frac{\delta^{K+1}\alpha S(a^s)}{1-\delta^{K+1}(1-\alpha)}.$$

Since $U_1^N > U^I$ for all $\delta \in (0,1)$ and both $U_1^N$ and $U^I$ are increasing in $\delta$, we can find some $\underline{\delta}_K$ and $\overline{\delta}_K$ such that for all $\delta \in (\underline{\delta}_K, \overline{\delta}_K)$,

$$U_1^N = \frac{\delta^K \alpha S(a^s)}{1-\delta^{K+1}(1-\alpha)} > v(a^s) - c(a^s)/\beta > U^I = \frac{\delta^{K+1}\alpha S(a^s)}{1-\delta^{K+1}(1-\alpha)}.$$

Then, since $U_k^N > U_{k-1}^N$ for all $k = 2, 3, ..., K$, we verify that $U_k^N > v(a^s) - c(a^s)/\beta$ for all $k = 1, 2, ..., K$.

Now consider the following strategies of matched sellers: In period $t = \tau K + \tau - 1$ ($\tau = 1, 2, ...$) all matched sellers choose the static equilibrium investment $a^s$ with certainty. In all other periods all matched sellers choose the least costly investment, zero, with certainty.

We also define the reservation values $\{U_t^b, U_t^s\}_{t=0}^{\infty}$ as follows:

$$U_t^b + U_t^s = \begin{cases} U^I & \text{for } t = \tau K + \tau - 1 \ (\tau = 1, 2, ...), \\ \\ U_k^N & \text{for } t = k - 1 \ (\text{mod } K+1),\ k = 1, 2, ..., K. \end{cases}$$

Then, the above strategies become optimal, given these reservation values because $\beta\{v(a^s) - U_t^b - U_t^s\} \geq c(a^s)$ for all $t = \tau K + \tau - 1$ ($\tau = 1, 2, ...$), which implies choosing $a_t = a^s$ is optimal, while $\beta\{v(a^s) - U_t^b - U_t^s\} < c(a^s)$ for all other periods, which imply choosing $a_t = 0$ is optimal.

Finally, the above sellers' strategies are consistent with the construction of the reservation values. Q.E.D.

### 7.6 Proof of Proposition 5

By Proposition 1 we know that equilibrium is unique and same as the static one when $\delta \in (0, \delta^*)$: In every period matched seller chooses the static equilibrium investment $a^s$ with probability one. Thus this equilibrium is trivially constrained efficient because no other equilibria exist.

Next we will turn to the case that $\delta \in (\delta^*, 1)$. In this case multiple equilibria arise for some discount factors as we have shown in Proposition 3 and 4.

Consider the equilibrium $\hat{e}$ shown in Lemma 3 and note that the social welfare under $\hat{e}$ can be written by

$$\begin{aligned} W(\delta|\hat{e}) &= \alpha\hat{x}_0 S(a^s) + (1-\alpha\hat{x}_0) + (1-\alpha)(\hat{U}_t^b + \hat{U}_t^s)\} \\ &= \alpha S(a^s) + (1-\alpha)[v(a^s) - c(a^s)/\beta] \end{aligned}$$

for $\delta \in (\delta^*, 1)$ because

$$v(a^s) - c(a^s)/\beta = \frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)}.$$

and

$$\hat{U}_t^b + \hat{U}_t^s = \frac{\delta\alpha x^* S(a^s)}{1-\delta(1-\alpha x^*)} \quad \forall\, t.$$

Now take any other equilibrium, $\tilde{e} = \{\tilde{x}_t, \tilde{U}_t^b, \tilde{U}_t^s\}_{t=0}^{\infty}$, than $\hat{e}$. Then we have the social welfare under $\tilde{e}$ as

$$W(\delta|\tilde{e}) = \alpha\tilde{x}_0 S(a^s) + (1-\alpha\tilde{x}_0)(\tilde{U}_0^b + \tilde{U}_0^s)$$

where

$$\tilde{U}_t^b + \tilde{U}_t^s = \delta\{\alpha\tilde{x}_{t+1}S(a^s) + (1-\alpha\tilde{x}_{t+1})(\tilde{U}_{t+1}^b + \tilde{U}_{t+1}^s)\},\ \ t = 1, 2, ...$$

We will consider the following separate cases:

CASE 1: $v(a^s) - c(a^s)/\beta \geq \tilde{U}_0^b + \tilde{U}_0^s$.

In this case we have

$$\begin{aligned} W(\delta|\tilde{e}) &= \alpha\tilde{x}_0 S(a^s) + (1-\alpha\tilde{x}_0)(\tilde{U}_0^b + \tilde{U}_0^s) \\ &\leq \alpha\tilde{x}_0 S(a^s) + (1-\alpha\tilde{x}_0)[v(a^s) - c(a^s)/\beta] \\ &\leq \alpha S(a^s) + (1-\alpha)[v(a^s) - c(a^s)/\beta] \\ &= W(\delta|\hat{e}) \end{aligned}$$

where the first inequality follows from our supposition above and the second inequality from the fact that $S(a^s) = v(a^s) - c(a^s) > v(a^s) - c(a^s)/\beta$ respectively.

Thus the equilibrium $\tilde{e}$ never attains higher welfare than $\hat{e}$.

CASE 2: $v(a^s) - c(a^s)/\beta < \tilde{U}_0^b + \tilde{U}_0^s$.

In this case we have $\tilde{x}_0 = 0$. Let $T > 0$ be the period such that $\tilde{x}_t = 0$ for any $t < T$ but $\tilde{x}_T > 0$. [11]

Then, by $\tilde{x}_T > 0$, we must have

$$v(a^s) - c(a^s)/\beta \geq \tilde{U}_T^b + \tilde{U}_T^s.$$

Under the equilibrium path $\tilde{e}$, we then obtain

$$\begin{aligned} W(\delta|\tilde{e}) &= \delta^T\{\alpha\tilde{x}_T S(a^s) + (1-\alpha\tilde{x}_T)(\tilde{U}_{T+1}^b + \tilde{U}_{T+1}^s)\} \\ &\leq \delta^T\{\alpha\tilde{x}_T S(a^s) + (1-\alpha\tilde{x}_T)[v(a^s) - c(a^s)/\beta] \\ &\leq \delta\{\alpha\tilde{x}_T S(a^s) + (1-\alpha\tilde{x}_T)[v(a^s) - c(a^s)/\beta]\} \\ &\leq \delta\{\alpha S(a^s) + (1-\alpha)[v(a^s) - c(a^s)/\beta]\} \\ &< \alpha S(a^s) + (1-\alpha)[v(a^s) - c(a^s)/\beta] \\ &= W(\delta|\hat{e}) \end{aligned}$$

[11] If $x_t = 0$ for all $t$, then by Lemma 2 no trades are realized in all periods and hence $W(\delta|\tilde{e}) = 0$, which is less than $W(\delta|\hat{e}) > 0$.

where the first inequality follows from $v(a^s)-c(a^s)/\beta \geq \tilde{U}^b_K+\tilde{U}^s_K$, the second inequality from $\delta < 1$, the third inequality from $S(a^s) > v(a^s)-c(a^s)/\beta$ and the last equality from the definition of $\hat{e}$ respectively. Thus there exist no other equilibria which is more efficient than $\hat{e}$ in this case as well. Q.E.D.

### 7.7 Proof of Proposition 6

Suppose not, i.e., $\sup\tilde{L}(e) < +\infty$ in some equilibrium $e$. This means that there exists some finite time $T \geq 0$ such that for all $t \geq T$ we have $x_t(\theta, \psi) = 1$ for all $(\theta, \psi) \in \Theta \times \Psi$. Thus, by using a smilar logic to the proof of Proposition 2, we can show that the sum of the reservation values must be stationary from period $T$ onward and hence be constant at $U_b + U_s$, i.e.,

$$U^b + U^s = \delta\{\alpha E_{\theta,\psi}[S(a^s(\theta,\psi),\theta,\psi)] + (1-\alpha)(U_b + U_S)\}$$

where $S(a,\theta,\psi) \equiv v(a,\theta) - c(a,\psi)$ is the ex ante total surplus.

Solving this, we have

$$U^b + U^s = \frac{\alpha\delta E_{\theta,\psi}[S(a^s(\theta,\psi),\theta,\psi)]}{1-\delta(1-\alpha)}.$$

Take some $(\theta'', \psi'') \in \Theta\times\Psi$ such that $S(a^s(\theta'',\psi''),\theta'',\psi'') \leq E_{\theta,\psi}[S(a^s(\theta,\psi),\theta,\psi)]$. Then we obtain

$$v(a^s(\theta'',\psi''),\theta) - c(a^s(\theta'',\psi''),\psi'')/\beta < S(a^s(\theta'',\psi''),\theta'',\psi'') \leq E_{\theta,\psi}[S(a^s(\theta,\psi),\theta,\psi)]$$

which shows that there exists some $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$ we must have

$$v(a^s(\theta'',\psi'')) - c(a^s(\theta'',\psi''))/\beta < \frac{\delta\alpha E_{\theta,\psi}[S(a^s(\theta,\psi),\theta,\psi)]}{1-\delta(1-\alpha)}.$$

Thus seller of type $\psi''$ will never choose $a^s(\theta'',\psi'')$ when he matches with buyer of type $\theta''$ in any period $t \geq T$, which implies $x_t(\theta'',\psi'') = 1$ for all $t \geq T$. However this is a contradiction. Q.E.D.

### 7.8 Proof of Proposition 7

Consider the following reservation values:

$$U^b_t = \overline{U}^b \equiv \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)}, \;\; U^s_t = \overline{U}^s \equiv 0 \;\; \forall\, t.$$

Then we will show that, given these reservation values, each matched buyer optimally offers the cost minimizing contract $C^*$ and each matched seller chooses the cost minimizing investment $a^m$ with certainty in any period.

**Lemma A1**. *Given $C^*$ and $(\overline{U}^b, \overline{U}^s)$, every matched seller optimally chooses*

$a^m$ *with certainty when* $\delta$ *is large.*

**Proof**. Suppose that matched seller chooses $a^m$. Then the voluntary trade conditions under $C^*$ are satisfied:

$$b(a^m) - P^1 = S(a^m) > P^0 + \overline{U}^b = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)},$$

and

$$P^1 - d(a^m) = c(a^m) > P^0 + \overline{U}^s = 0.$$

Thus, if the seller chooses $a^m$, he obtains the payoff $P^1 - d(a^m) - c(a^m) = 0$.

Now suppose that the seller chooses $a \neq a^m$. Then, if the voluntary trade conditions still hold, he obtains the payoff $P^1 - d(a) - c(a)$ which is less than zero by definition of $a^m$. If the voluntary trade conditions are not satisfied, then renegotiation may happen and hence the seller can obtain at most

$$u^s(a) = \beta \max\{v(a) - \overline{U}^s - \overline{U}^b, 0\} + \overline{U}^s - c(a).$$

Note here that, if $b(a) < \overline{U}^b$, the seller's renegotiation payment offer does not satisfy his limited liability and hence renegotiation fails, in which case the seller obtains $\overline{U}^s - c(a)$. Thus the most profitable case for the seller is that $b(a) > \overline{U}^b$ and thus renegotiation succeeds, in which case his payoff is given as $u^s(a)$ above. However, by Assumption 3, for large $\delta$ we have

$$\overline{U}^b + \overline{U}^s = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)} > v(a^s) - c(a^s)/\beta.$$

Thus $\max_{a\in A} u^s(a) = \overline{U}^s = 0$ when $\delta$ is high enough, which shows that the seller's deviation payoff is not higher than zero. Thus in either case the seller has no incentives to deviate from choosing $a^m$. Q.E.D.

**Lemma A2**. *Every matched buyer optimally offers* $C^*$ *when* $\delta$ *is large, given* $(\overline{U}^b, \overline{U}^s)$.

**Proof**. If a matched buyer offers $C^*$, then she can obtain the payoff $S(a^m)$ by Lemma A1 when $\delta$ is large. Then we will show that the buyer cannot obtain higher payoff by offering different contracts.

To see this, suppose that some matched buyer offers $\hat{C} = \{\hat{P}^1, \hat{P}^0\} \neq C^*$ and that her matched seller's investment is realized as $\hat{a}$ according to his mixed strategy, given this contract. We can suppose that $\hat{a} > 0$ because, if the seller's mixed strategy puts all positive probabilities on $a = 0$, then the buyer's payoff must be at most $S(0)$, which is less than his reservation value $\overline{U}^b$ when $\delta$ is large enough.[12]

[12] For large $\delta$, $U^b$ can be close to $S(a^m)$ which is higher than $S(0)$ because $S(a^m) = b(a^m) - (d(a^m) + c(a^m)) > b(a^m) - (d(0) + c(0)) > S(0) = b(0) - d(0) - c(0)$.

Suppose first that the voluntary trade conditions are satisfied under $\hat{C}$:

$$b(\hat{a}) - \hat{P}^1 \geq -\hat{P}^0 + \overline{U}^b, \tag{A2}$$

$$\hat{P}^1 - d(\hat{a}) \geq \hat{P}^0 + \overline{U}^s. \tag{A3}$$

Thus the seller obtains the payoff $\hat{P}^1 - d(\hat{a}) - c(\hat{a})$.

Note also that the buyer can guarantee the payoff $S(a^m)$ by offering $C^*$ because the seller accepts it and chooses $a^m$ following no renegotiation (due to Lemma A1). Then, since $S(a^m) > \overline{U}^b$, the buyer's deviation payoff must be strictly higher than his reservation value $\overline{U}^b$. From this, the above first voluntary trade condition (A2) must be satisfied with strict inequality because otherwise the buyer's payoff is not higher than the reservation value $U^b$, i.e., $-\hat{P}^0 + \overline{U}^b \leq \overline{U}^b$ due to $\hat{P}^0 \geq 0$.

We will then show that $\hat{a} = a^m$ must hold. To see this, suppose $\hat{a} \neq a^m$ and consider two separate cases:

CASE 1: $\hat{P}^1 - d(\hat{a}) = \hat{P}^0 + \overline{U}^s$.
In this case the seller obtains the payoff $\hat{P}^1 - d(\hat{a}) - c(\hat{a})$. However, if the seller slightly reduces investment from $\hat{a} > 0$ and chooses $a' < \hat{a}$ (this is possible because $A$ is closed interval and $c$ is continuous), then renegotiation may or may not occur. In either case the seller obtains at least the following payoff

$$U^s + \hat{P}^0 - c(a') = \hat{P}^1 - d(\hat{a}) - c(a')$$

which is higher than the supposed payoff $\hat{P}^1 - d(\hat{a}) - c(\hat{a})$ due to $c(\hat{a}) > c(a')$. Thus the seller has the incentive to deviate from $\hat{a}$.

CASE 2: $\hat{P}^1 - d(\hat{a}) > \hat{P}^0 + \overline{U}^s$.
In this case, if the seller can slightly change investment level from $\hat{a}$ towards $a^m$ and choose $a'$, then the voluntary trade conditions can be still satisfied because $b$ and $d$ are continuous and we already know that the voluntary trade condition for the buyer (A2) is strictly satisfied under $a = \hat{a}$. Thus the seller can obtain the following payoff

$$\hat{P}^1 - d(a') - c(a')$$

which is higher than the supposed payoff $\hat{P}^1 - d(\hat{a}) - c(\hat{a})$ (by $a' \to a^m$). Thus the seller has the incentive to deviate from $\hat{a}$.

The above CASE 1 and 2 show that $\hat{a} = a^m$. However, for the purpose of implementing $a = a^m$ given the reservation values $\overline{U}^b$ and $\overline{U}^s$, it is the most profitable for the buyer to offer the contract $C^*$ because then she can extract the full surplus $S(a^m)$ for implementing $a^m$. This implies that each matched buyer has no incentives to offer other contracts than $C^*$, given the reservation values $(\overline{U}^b, \overline{U}^s)$.

Thus the remaining case is only that renegotiation happens under the deviation contract $\hat{C}$. In that case the payoffs of the parties are determined by the ex post bargaining. Then the seller will obtain the following payoff from the supposed investment level $\hat{a}$:

$$u^s(\hat{a}) = \beta \max\{v(\hat{a}) - \overline{U}^b - \overline{U}^s, 0\} + \overline{U}^s + \hat{P}_0 - c(\hat{a}).$$

However, since $\overline{U}^b + \overline{U}^s = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)} > v(a^s) - c(a^s)/\beta$ for large $\delta$ under Assumption 3, we have

$$u^s(\hat{a}) \leq \overline{U}^s + \hat{P}^0 - c(\hat{a}).$$

Thus the seller must have chosen $\hat{a} = 0$ with certainty.

Then the buyer's payoff can be at most

$$(1-\beta)\max\{v(0) - \overline{U}^b - \overline{U}^s, 0\} + \overline{U}^b - \hat{P}^0$$

which is however not larger than $\overline{U}^b$ because we have $\overline{U}^b + \overline{U}^s = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)} > v(a^s) - c(a^s)/\beta > v(0)$. Since the buyer can obtain $S(a^m)$ by offering the ex ante contract $C^*$ and $S(a^m) > \overline{U}^b$, she optimally offers the contract $C^*$. Q.E.D.

Lemma A1 and A2 have established the result that every matched buyer offers the cost minimizing contract $C^*$ and every matched seller chooses $a^m$ with certainty, given $\overline{U}^b = \frac{\delta\alpha S(a^m)}{1-\delta(1-\alpha)}$ and $\overline{U}^s = 0$. Also these reservation values are consistent with the contract offer of buyers and investment choice of sellers as well. Thus this can be a stationary equilibrium. Q.E.D.

## 7.9 Proof of the Remark

We make the mild assumption that $d(a) + c(a)$ is increasing in $a \in (a^m, \overline{a}]$ (this will hold when $d + c$ is convex function). We define the first best investment as $a^*$ which maximizes ex ante surplus $S(a)$. Then, fix any pair of reservation values $(U_t^b, U_t^s)$ in period $t$ and consider the following ex ante contract $\{P_t^1, P_t^0\}$:

$$b(a^*) - U_t^b = P_t^1 - P_t^0 \geq d(a^*) + U_t^s,$$

which is offered by a matched buyer. Note here that $U_t^b + U_t^s < S(a^*)$ for all $t$ and hence such $P_t^1$ and $P_t^0$ exist for these inequalities to be satisfied.

Given this contract, the voluntary trade conditions hold if the seller chooses $a^*$. In that case the seller obtains $P_t^1 - d(a^*) - c(a^*)$. On the other hand, if the seller chooses $a < a^*$, then these conditions will not hold (since $b(a) < b(a^*)$ for $a < a^*$) and hence renegotiation will occur. Then the seller obtains the payoff $\beta \max\{v(a) - U_t^b - U_t^s, 0\} + U_t^s + P_t^0 - c(a)$. However, this is less than the payoff $P_t^1 - d(a^*) - c(a^*)$ obtained by choosing $a^*$. Since the seller has no incentives to choose $a > a^*$,[13] the seller optimally chooses $a^*$. Then the buyer can extract the full surplus $S(a^*)$ by setting $P_t^0 = U_t^b - S(a^*)$. Q.E.D.

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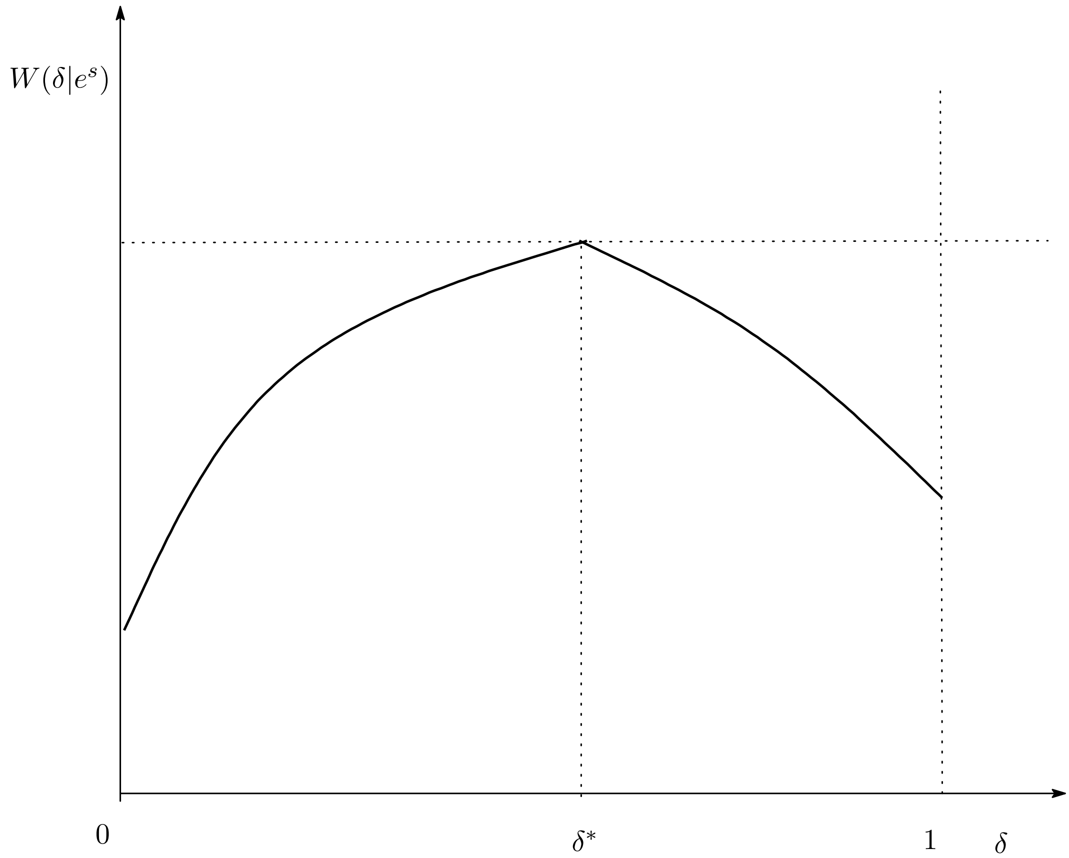


Figure 1:

Social Welfare in Stationary Equilibrium

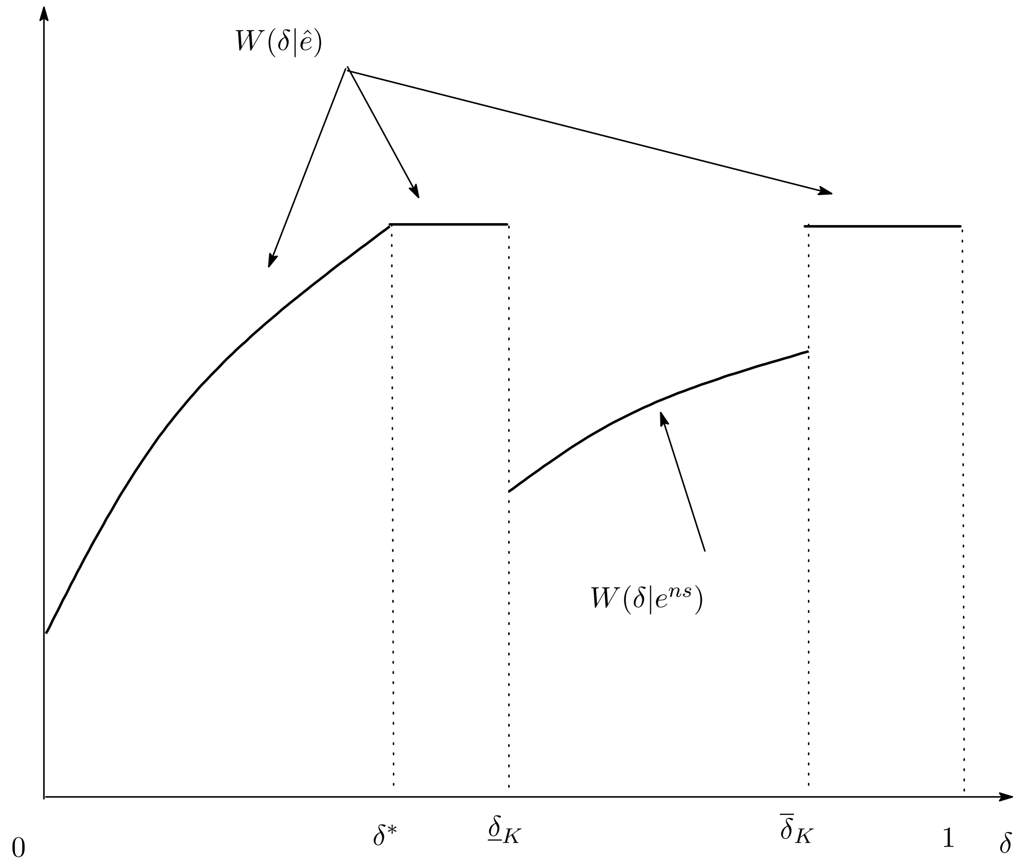


Figure 2:

Social Welfare in Non–Stationary Equilibrium