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Osaka University, Toyonaka, Osaka 560-0043, JAPAN

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Shirou Kuwahara[†] Katsunori Yamada[‡]

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Abstract

This paper presents an endogenous growth model with population growth and an inter-generational spillover of human capital: we consider the "local home environment externality conceptualized by Galor and Tsiddon (1997a). The model will generate a negative relationship between the population growth rate and the per capita GDP growth rate, which is also present in the data. Furthermore, multiple equilibrium paths will result. As far as we know, this is the first paper that derives a multiplicity of steady growth paths in a model with two sources of growth and the Jones technology. The paper also casts a paradox that the GDP growth rate may be higher in the society without the externality than the one in the economy with externality.

JEL classification: O11; O31; O41

Keywords: population growth; multiple equilibria; R&D; Jones technology; the local home environment externality

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†Corresponding author. Assistant Professor, Graduate School of Systems and Information Engineering, University of Tsukuba. Adress: 1-1-1, Ten-noudai, Tsukuba, 305-8571, Japan. E-mail: kuwahara@sk.tsukuba.ac.jp

‡Research Fellow for the Japan Society for the Promotion of Science and Graduate School of Economics, Osaka University. Adress: 1-7 Machikaneyama, Toyonaka, 560-0043, Japan. E-mail: kyamada@econ.osaka-u.ac.jp

1 Introduction

Under the presumption that population grows in the real world, this paper presents an endogenous growth model which is useful to analyze the relationship between the population growth rate and the per capita GDP growth rate. Following the recent developments in the endogenous growth literature, we consider two engines of growth: Romer-type R&D activities and human capital accumulation. Also, following the literature, we adopt the Jones technology to eliminate the "scale effects" from the model. We will show that the model will generate a negative relationship between the population growth rate and the per capita GDP growth rate, which is also present in the data.[1] Furthermore, multiple steady growth paths will result even though we adopt the Jones technology. To obtain the result, the "local home environment externality" conceptualized by Galor and Tsiddon (1997a) plays a crucial role. Interestingly, our stability analyses with a numerical method casts a paradox that the GDP growth rate may be higher in the society without the externality than the one in the economy with externality.

The literature of R&D-based growth model can be divided into three stages.[2] The first stage models include Romer (1990), Grossman and Helpman (1991) and Aghion and Howitt (1992). They consider endogenous technological changes and depict how the economy can show long-run growth. However, due to their assumption of linearity of the innovation functions in R&D sectors, a problem of "scale effects" arises: population growth accelerates the GDP growth rate so that we cannot include the population growth into the model.

Second stage models try to overcome these scale effects by considering the Jones technology (Jones 1995a; 1995b). The Jones technology is a non-linear function of technology-creation-process with labor inputs. This specification is consistent with empirical studies on the relationship between R&D inputs and creation of new technologies. While the Jones technology provides a merit that the scale effects can be eliminated and economists can introduce population growth into the model, there is a demerit that if there is no population growth, the economy does not show long-run growth. That is, the GDP growth rate in these second stage models is positively correlated with the population growth rate. Unfortunately, as it is well-known, this property is not supported by empirical findings (Kelley; 1988, Kelley and Schmidt; 1995 and Ahituv; 2001,

[1] Dalgaard and Kreiner (2001) and Strulik (2005) construct models in which the population growth rate and the per capita GDP growth rate can be positively correlated.

[2] See Arnold (1998).

among others): empirical studies report a negative (or no) correlation between the population growth rate and the per capita GDP growth rate. Moreover, Easterly (1994) and Quah (1996, 1997) suggest that the world economy exhibits complicated patterns of the per capita GDP growth. With this argument, we can easily deduce that the per capita GDP growth rate is far from being tied to the population growth rate.[3]

Finally, third stage models consider two engines of endogenous growth to capture a range of general equilibrium effects on GDP growth. Howitt (1999) and Segerstrom (2000) consider two types of R&D activities: vertical and horizontal R&D technologies. Jones (2001) allows an endogenous choice of fertility and population growth, thus extending the framework of the second stage models. Arnold (1998), Funke and Strulik (2000) and Strulik (2005) consider endogenous human capital accumulation depicted by the well-known Uzawa-Lucas technology in the framework of Romer-type R&D-based growth model.

Following the spirit of Arnold (1998), Funke and Strulik (2000) and Strulik (2005), we develop a model with R&D activities and human capital accumulation. Because we adopt the Jones technology in R&D sector, our model is free from the scale effects and we can analyze the relationship between the population growth and the per capita GDP growth as Dalgaard and Kreiner (2001) and Strulik (2005).[4] One major difference between these previous studies and the present paper is that aggregate human capital can be augmented in two ways, namely the Uzawa-Lucas type human capital investment and the exogenous population growth. The latter mechanism is rather "automatic" and it occurs because we consider spillover effects of human capital among generations: newly born agents are endowed with positive amount of human capital when they enter into the production process. This effect is conceptualized by Galor and Tsiddon (1997a) as the "local home environment externality".[5] Also, we can interpret the effect from the view point of Azariadis and Drazen (1990) as the "inter-generational spillover of human capital".[6]

An important finding under our framework is that the per capita GDP growth rate reflects whole range of general equilibrium effects rather than being tied to the population growth rate. Hence, our model has consequences on sug-

[3] Galor and Weil (2000) construct a model in which historical patterns of economic growth are explained with a non-monotonic relationship between the population growth rate and the technological development. See also Galor (2005).

[4] Funke and Strulik (2000) adopt a linear function in the R&D sector while Arnold (1998) do not consider the population growth.

[5] See also Galor and Tsiddon (1997b).

[6] In Strulik (2005), new agents are endowed with no human capital.

gestions of Easterly (1994) and Quah (1996, 1997). In addition, the correlation between the population growth rate and the per capita GDP growth rate can be negative under certain parametric restrictions. This outcome accommodates to empirical findings of Kelley (1988), Kelley and Schmidt (1995) and Ahituv (2001).

The novelty in the present paper is that multiple steady growth paths may generate even when we adopt the Jones technology.[7] In one equilibrium path, aggregate human capital is augmented solely through exogenous population growth (corner solution case). In the other equilibrium path, there are positive amounts of human capital invested in the Uzawa-Lucas technology (internal solution case). Under plausible parameter sets, we find with a numerical method that both paths can be supported as optimal growth paths. As far as we know, this is the first paper that derives multiple equilibria in a model with two engines of growth and with the Jones technology. We will argue that our model will have potential to give a theoretical reason to the polarization phenomenon discussed by Krugman (1991), Lucas (1993) and Howitt (1994), among others. Then, we hold that the model could integrate two major empirical topics in economic growth literature: the negative correlation between the per capita GDP growth rate and the population growth rate and the polarization phenomenon. Interestingly, our stability analyses also casts a paradox that the GDP growth rate may be higher in the society without the externality than the one in the economy with externality.

The paper is organized as follows. In Section 2, we set up the model. Steady growth paths are analyzed in Section 3. Stability analyses are conducted in Section 4. Section 5 concludes.

2 The Model

Our model adopts a Romer-type (1990) R&D production structure, Benhabib-Perli-Xie-type (1994) intermediate goods and final goods production structures and Uzawa (1965) and Lucas (1988) type human capital accumulation process. There are three production sectors: final goods sector, intermediate goods sector, and R&D sector; and four factors: raw labor, human capital, physical capital, and knowledge measured by the variety of intermediate goods.

We consider a continuous time model and we will omit the time script throughout the paper if there is no fear of confusion.

[7] See section 3 for the discussion about the Jones technology and multiple equilibrium paths.

2.1 Final Goods Production Sector

The final goods sector is competitive. The production function is homogenous of degree one and is given by

$$Y = L^{\alpha} H_Y^{\beta} \left(\int_0^A x(j)^{\frac{\gamma}{\zeta}} dj \right)^{\zeta}, \qquad \alpha, \beta, \gamma \in (0,1),\ \zeta \geq 1,\ \text{and} \quad \alpha + \beta + \gamma = 1,$$

where Y, H_Y, L denote the amount of final goods production, human capital employed in the final goods sector, and the raw labor force, respectively. $x(j)$ denotes the amount of intermediate goods supplied by an intermediate goods firm with index j while the variety of the cluster in the intermediate goods sector is measured by A.

The first order conditions (FOCs) in this sector are given as $\frac{\partial Y}{\partial L} = w_L$, $\frac{\partial Y}{\partial H_Y} = w_Y$, and $\frac{\partial Y}{\partial x(j)} = p(j)$, where w_L is the competitive wage paid to the raw labor force, w_Y is the wage paid to human capital devoted to the final goods sector, and $p(j)$ is the price of intermediate good supplied by an intermediate good firm with index j.

2.2 Intermediate Goods Production Sector

Intermediate goods are used to produce the final goods. They are assumed to be supplied monopolistically. One unit of the intermediate good is produced by η units of physical capital, into which the final goods can be translated by one-to-one manner. Hence, the profit of an intermediate firm with index j is given by $\pi(j) = p(j)x(j) - r\eta x(i)$, where r denotes the rental price of capital. By solving the optimization problem of the intermediate goods firm with the optimal conditions in the final goods sector, the following conditions result:

$$x(j) = \left[\frac{\gamma^2}{r\eta\zeta} \right]^{\frac{1}{1-\frac{\gamma}{\zeta}}} \left[L^{\alpha} H_Y^{\beta} \left(\int_0^A x(j)^{\frac{\gamma}{\zeta}} dj \right)^{\zeta - 1} \right]^{\frac{1}{1-\frac{\gamma}{\zeta}}}, \quad \text{and} \quad p(j) = \frac{r\eta\zeta}{\gamma}.$$

In this paper we assume the symmetry in the intermediate goods sector. This assumption results in the equality of the size of intermediate goods firms. Hence, we have $K = \int_0^A \eta x(j) dj = \eta A x$, where K denotes the supply of physical capital. From this condition and the structure of the final goods production sector, we obtain the optimal production of the final goods and the market prices as

$$Y = \eta^{-\gamma} L^{\alpha} H_Y^{\beta} A^{\zeta - \gamma} K^{\gamma}, \tag{1}$$

and

$$r = \frac{\gamma^2}{\zeta} \frac{Y}{K}, \quad w_L = \alpha \frac{Y}{L}, \quad w_Y = \beta \frac{Y}{H_Y} \quad \text{and} \quad \pi = \frac{\gamma(\zeta - \gamma)}{\zeta} \frac{Y}{A}. \tag{2}$$

2.3 R&D Activities

Following the literature, the R&D process is set up as a variety-creating process in the intermediate goods sector. We introduce the Jones technology in the process so that the evolution of a new variety ($\dot{A}$) is given by

$$\dot{A} = BA^{\chi} H_A^{\phi}, \qquad B > 0, \quad \phi \in (0,1), \quad \chi \in [0,1), \tag{3}$$

where H_A denotes the amount of human capital supplied to the R&D activities. The creation of a new variety exhibits the Inada property: $\lim_{H_A \to 0} \frac{\partial \dot{A}}{\partial H_A} = \infty$ and $\lim_{H_A \to \infty} \frac{\partial \dot{A}}{\partial H_A} = 0$. By this structure, the model has no scale effects and we can introduce the population growth into the model.

The term of patent for a newly created variety is assumed to be permanent. Hence, the value of R&D (denoted as v) can be designated as the present value of perpetual monopoly profits: $v(t) \equiv \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds} \pi(\tau) d\tau$. By differentiating this equation with respect to time (t) and by applying the Leibniz's rule, we obtain the well-known no-arbitrage condition which is given as

$$rv = \pi + \dot{v}. \tag{4}$$

Finally, free entry into the R&D sector is secured so that the cost and the benefit of the R&D activity be equal, which provides the following condition

$$v\dot{A} = w_A H_A,$$

where w_A is the wage paid to human capital devoted to the R&D sector. In equilibrium, $w_A = w_Y = w_H$ must hold. Here, w_H can be interpreted as the market price of human capital. Substituting the above equation into (3) gives

$$v = w_H \frac{H_A^{1-\phi}}{BA^{\chi}}. \tag{5}$$

2.4 Population Growth and Human Capital Accumulation

In our model, representative agent is endowed with one unit of raw labor force and a certain amount of human capital. As stated above, raw labor force is in-elastically supplied to the final goods production sector. Furthermore, the population of the economy grows at an exogenously given rate n. The population growth enhances the supply of row labor force (L) constantly.

In this economy, the evolution of *aggregate* human capital depends on two factors. The first element is human capital investment through Uzawa-Lucas technology. And the second element is given by the assumptions of population growth and of spillover effects in human capital accumulation process. Assume that we do not consider the second element. Then, the evolution of aggregate human capital will be obtained as

$$\dot{H} = bH_H, \quad b > 0$$

where H and H_H are the amount of human capital and human capital supplied to the Uzawa-Lucas human capital creation process, respectively. b measures the efficiency in Uzawa-Lucas technology.

In this paper, we introduce spillover effects of human capital among generations and the newly born agents are endowed with positive amount of human capital when they enter into the production process. This effect is conceptualized by Galor and Tsiddon (1997a) as the "local home environment externality" in their discrete time model.[8] Although parents' transfer of education to children, that is, the spillover of human capital, is determined endogenously in Galor and Tsiddon (1997a), we consider an exogenous determination on how much children learn at home from their parents. That is, the intensity of the spillover effects is determined by a parameter. By taking the effects into consideration, the evolution of aggregate human capital can be written as

$$\dot{H} = bH_H + (1-\delta)nH, \quad \delta \in [0,1] \tag{6}$$

where δ represents the intensity of "local home environment externality". H is the current amount of human capital in the economy and nH captures the instantaneous amount of human capital augmented because of population growth when new agents have as much human capital as old agents. See here that when $\delta \in [0,1)$ (spillover case), aggregate human capital increases even without human capital investment ($H_H = 0$). On the other hand, when $\delta = 1$ (no spillover case), (6) reduces to $\dot{H} = bH_H$ and one needs human capital investment ($H_H > 0$) to increase aggregate human capital.[9]

2.5 Optimization Problem of Agents

The utility of a representative agent is given by

$$\int_0^\infty \frac{c(t)^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt \quad (\sigma > 0),$$

[8] An alternative way to interpret the effect is to consider the inter-generational linkage of human skills in Azariadis and Drazen (1990).

[9] Strulik (2005) consider the no spillover case and there are no multiple steady growth paths in his model.

where c, ρ and σ denote the consumption by a representative agent in the dynasty,[10] the subjective discount rate, and the inverse of the elasticity of intertemporal substitution, respectively. Throughout the paper we will impose the following condition about b and ρ to ensure positive investments on human capital if there are no spillover effects in human capital accumulation process

condition 1: $b - \rho > 0$.

The budget constraint of financial assets in per capita terms is given by

$$\dot{k} = rk + w_H(h_A + h_Y) + w_L - c - nk, \tag{7}$$

where k $(= K/L)$ is the per capita financial assets. h_A $(= H_A/L)$ and h_Y $(= H_Y/L)$ represent the per capita human capital devoted to the R&D activities and the per capita human capital devoted to produce the final goods, respectively.

By considering the population growth and the exogenous local home environment externality, we can obtain the per capita human capital evolution when we divide through (6) by L as

$$\dot{h} = bh_H - \delta nh, \tag{8}$$

where h $(= H/L)$ and h_H $(= H_H/L)$ are the per capita human capital and the per capita human capital supplied to the human capital creation process, respectively.[11]

From the objective function and two constraints, the optimal condition about consumption is the usual Keynes-Ramsey rule given by

$$\sigma \frac{\dot{c}}{c} = r - n - \rho. \tag{9}$$

In addition, the optimal condition for the human capital wage reads

$$\frac{\dot{w_H}}{w_H} \leq r - b - (1-\delta)n, \quad \text{with equality whenever} \quad h_H > 0. \tag{10}$$

[10] Hence-force, per capita variables are denoted by lower-case letters.

[11] We can interpret (8) as the following way. If $\delta = 0$ (the spillover is perfect), (8) reduces to $\dot{h} = bh_H$, which means that the per capita human capital is not diluted even with the presence of growing population. This is a simple reflection of the fact that when $\delta = 0$, the newly born agents have the same amount of human capital as their ancestor have. In reality, the spillover effect will not be perfect so that in per capita term the human capital decreases due to the population growth. This effect is captured by $-\delta nh$ in (8). Notice that in the evolution of financial wealth the diluting effect of population growth is perfect so that in (7) we have the term of $-nk$ instead of $-\delta nh$.

This condition indicates that the growth rate in human capital wage must be sufficiently high compared to the interest rate in order to ensure positive investment in the human capital creation process.

Finally, the transversality conditions are given as follows:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\mu_t h_t = 0, \tag{11}$$

where λ and μ are the shadow prices of per capita financial assets (k) and per capita human capital (h), respectively.

The equilibrium dynamics of the economy can be depicted by (2), (4), (5), (7) – (11). Finally, the market clearing condition of human capital imposes $H = H_Y + H_A + H_H$. For later reference, we define $u_A \equiv H_A/H$, $u_H \equiv H_H/H$ and $u_Y \equiv H_Y/H$ so that $u_A + u_Y + u_H = 1$.

3 Analyses of the Steady Growth Paths

In this section we confine our attention to the case of the steady growth path (SGP), and derive some features of the model. In the present model, we have two SGPs; one path for the internal solution case (with $H_Y > 0$, $H_A > 0$, $H_H > 0$) and the other path for the corner solution case (with $H_Y > 0$, $H_A > 0$, $H_H = 0$).

To understand the importance to consider the local home environment externality and population growth at the same time, with the presence of the Jones technology, assume that there is no local home environment externality. In this case, if $H_H = 0$, there is no growth of (aggregate) human capital. This situation implies that the variety in intermediate goods should be constant over time ($\dot{A}/A = 0$) along this SGP.[12] From the structure of the Jones technology and $A > 0$, this means $H_A = 0$. However, $H_A = 0$ in front of the Jones technology is not optimal because the marginal benefit of infinitesimal increase of H_A from zero is infinite with the Jones technology ($\lim_{H_A\to 0} \frac{\partial \dot{A}}{\partial H_A} = \infty$). Then, when we impose the Jones technology the internal solution case will result as the only optimal path without the local home environment externality.

In our model, local home environment externality is taken into consideration. Hence, even if $H_H = 0$ there can be positive growth in aggregate human capital. Hence, even with the Jones technology, corner solution case could be optimal under certain parameter restrictions. Moreover, it is easy to see that both SGPs can be realized under the same parameter set: we will have multiple SGPs in our model.

[12] To see this, refer to (12) below.

In the following sections, we denote the growth rate of the variable x by g_x in the internal solution case and by $\bar{g_x}$ in the corner solution case.

3.1 The Internal Solution Case

First assume the internal solution case. In this case, we have $H_H > 0$ so that (10) is satisfied with equality.

From (8), we can see that $g_H = g_{H_H}$ on the SGP. Therefore, $g_{H_A} = g_{H_Y} = g_{H_H} = g_H$ are easily derived.[13] From (3) and $g_{H_A} = g_H$, the following condition is necessary:

$$g_A = \frac{\phi}{1-\chi} g_H. \tag{12}$$

(12) relates the growth rate of variety (g_A) to the growth rate of aggregate human capital. Notice that the Jones technology affects g_A not through the *level* parameter (B) but through the efficiency parameters (χ and ϕ).

From the resource constraint of the final goods, $\dot{K} = Y - C$, we have $g_C = g_K = g_Y$, where $C \equiv cL$ is the total consumption. From this and the production function of the final goods, g_Y is obtained as

$$g_Y = \frac{1}{1-\gamma} \left\{ \alpha n + \beta g_H + (\zeta - \gamma) g_A \right\}. \tag{13}$$

By combining (12) and (13), we obtain the following relationship between g_Y and g_H

$$g_Y = \frac{1}{1-\gamma} \{ \alpha n + \Psi g_H \}, \tag{14}$$

where $\Psi \equiv \beta + \frac{\zeta - \gamma}{1-\chi} \phi (> 0)$.

Next, with (10) and $w_H = \beta \frac{Y}{H_Y}$, we have

$$g_Y - g_H = r - b - (1-\delta) n. \tag{15}$$

Moreover, on the SGP, (9) is written into

$$g_Y - n = \frac{1}{\sigma} (r - n - \rho). \tag{16}$$

From (15) and (16), we obtain another relationship between g_Y and g_H as

$$g_H = (1-\sigma)(g_Y - n) + (1-\delta) n + b - \rho. \tag{17}$$

As it can be seen from (14) and (17), the GDP growth rate (g_Y) and the aggregate human capital growth rate (g_H) are inter-dependently determined,

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[13] The proof of uniqueness of allocation of H among H_A, H_Y and H_H on the SGP is straightforward and is available from the authors upon request.

which is not the case in Arnold (1998). This is because we consider a general CRRA felicity function rather than a log-separable one.

From (14) and (17), g_Y is obtained as

$$g_Y = \frac{\alpha + (\sigma - \delta)\Psi}{\Upsilon} n + \frac{\Psi}{\Upsilon}(b - \rho), \tag{18}$$

where $\Upsilon \equiv \alpha + \beta - (1 - \sigma)\Psi$. We can re-write (18) into a per capita form as

$$g_y = \frac{(\underline{\delta} - \delta)\Psi}{\Upsilon} n + \frac{\Psi}{\Upsilon}(b - \rho), \tag{19}$$

where g_y is the per capita GDP growth rate and $\underline{\delta} = \frac{\Psi - \beta}{\Psi} \in (0, 1)$. We can see from (19) that g_y is a linear combination of the factor of the population growth rate n and the factor of contribution of Uzawa-Lucas type human capital accumulation $b - \rho$. Hence, the signs of Υ and $(\underline{\delta} - \delta)$ determine the relation ship between n and g_y. Notice here that if $\Upsilon < 0$, then an unfamiliar implication results: in that case, $(b - \rho)$ affects g_y negatively. In this paper we will exclude this case by imposing the following condition

condition 2: $\Upsilon > 0$

With the condition 2, we can have a negative correlation between the population growth rate and the per capita GDP growth rate when $\underline{\delta} - \delta < 0$ is satisfied. The relationship is supported by empirical findings such as Kelley (1988), Kelley and Schmidt (1995) and Ahituv (2001).

Next we should investigate conditions to ensure the internal solution obtained above. First, from the internal solution assumption ($H_H > 0$) and from (8), $\dot{h}/h = bh_H/h - \delta n > -\delta n$. With a little algebra, this condition can be re-written into

$$b > \rho - \frac{1 - \sigma}{\alpha + \beta} \left(\underline{\delta} - \delta\right) \Psi n. \tag{20}$$

Secondly, with the transversality conditions (11), a little algebra gives the other constraint as

$$b \left\{ \begin{array}{c} > \\ < \end{array} \right\} \frac{\alpha + \beta}{1 - \sigma} \Psi \rho - (\underline{\delta} - \delta) n, \quad \text{for} \left\{ \begin{array}{c} \sigma < 1 \\ \sigma > 1 \end{array} \right. . \tag{21}$$

When $\sigma = 1$, the transversality conditions are automatically satisfied. We obtain that the SGP of the internal solution can generate under the parameter space satisfying both (20) and (21).

3.2 The Corner Solution Case

This section investigates the corner solution case ($H_H = 0$). By remembering that (10) is not satisfied with equality and by replicating the procedure in the previous section with the condition $H_H = 0$, we can obtain the GDP growth rate ($\bar{g}_Y$) and the per capita GDP growth rate ($\bar{g}_y$) in the corner solution case as

$$\bar{g}_Y = \frac{\Psi}{\alpha + \beta}(\underline{\delta} - \delta)n + n, \tag{22}$$

and

$$\bar{g}_y = \frac{\Psi}{\alpha + \beta}(\underline{\delta} - \delta)n. \tag{23}$$

(23) shows that the per capita GDP growth rate exhibits a *semi-endogenous growth* property: the growth rate is pinned down to the population growth rate as in the second stage models.

In turn, we should investigate parametric conditions to generate the corner solution case. This is done by examining (10) when it holds with inequality. Because $\bar{g}_{w_H} = \bar{g}_Y - \bar{g}_H$ on the SGP, the condition given by (10) can be written into

$$\bar{g}_{w_H} = \bar{g}_Y - \bar{g}_H < r - b - (1 - \delta)n. \tag{24}$$

With (9) and the condition that $\bar{g}_Y = \bar{g}_C$ on the SGP, the following condition is obtained

$$\sigma(\bar{g}_Y - n) = r - \rho - n. \tag{25}$$

In the corner solution case, $\bar{g}_H = (1 - \delta)n$. Also we can eliminate r and $\bar{g}_Y$ from (25) with (24) and with (22). Hence, we obtain the parametric condition for the corner solution case as

$$b > \rho - \frac{1 - \sigma}{\alpha + \beta}(\underline{\delta} - \delta)\Psi n. \tag{26}$$

Notice that the partition given by (26) is equivalent to the one given by (20) in the internal solution case.

Finally, a negative correlation between n and $\bar{g}_y$ can be obtained when $(\underline{\delta} - \delta) < 0$ with the condition 2, as in the internal solution case.

4 Stability of the SGPs

In this section, we will investigate the local stability of two SGPs with a numerical method. We calibrate the model in accordance with strains of previous literature and examine the stability of the dynamical system on the SGPs.

4.1 The internal solution case

In the internal solution case, the dynamical system is given by the following seven equations of seven variables consisting of $\{L, K, H, A, u_Y, u_A, v\}$

$$
\begin{aligned}
\dot{L} &= nL, && (27)\\
\dot{K} &= Y(L,H,K,A,u_Y) - C, && (28)\\
\sigma\left(\frac{\dot{C}}{C} - n\right) &= r(L,H,K,A,u_Y) - \rho - n, && (29)\\
\frac{\dot{w}_H}{w_H} &= r(L,H,K,A,u_Y) - b - (1-\delta)n, && (30)\\
\dot{v} + \pi &= r(L,H,K,A,u_Y)v, && (31)\\
\dot{H} &= b(1-u_A-u_Y)H + (1-\delta)nH, && (32)
\end{aligned}
$$

and

$$\dot{A} = BA^{\chi}(u_A H)^{\phi}. \tag{33}$$

Notice here that (i) u_H does not appear in the system because u_H is determined by the condition $u_A + u_Y + u_H = 1$ when u_Y and u_A are given, (ii) Y is given by $Y = \eta^{-\gamma}L^{\alpha}(u_Y H)^{\beta}A^{\zeta-\gamma}K^{\gamma}$ from (1), (iii) r is given by $r = \frac{\gamma^2}{\zeta}\frac{Y}{K}$ from (2), (iv) π is given by $\pi = \frac{\gamma(\zeta-\gamma)}{\zeta}\frac{Y}{A}$ from (2) and (v) w_H is given by $w_H = \beta\frac{Y}{u_Y H}$ from (2), or equivalently, w_H is given by the condition $v = w_H\frac{(u_A H)^{1-\phi}}{BA^{\chi}}$ from (5).

In order to describe the system with variables which are stationary on the steady growth path, we define the following variables[14]

$$\tilde{c} \equiv \frac{C}{L^{\frac{\alpha}{1-\gamma}}H^{\frac{\beta}{1-\gamma}}A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \tilde{k} \equiv \frac{K}{L^{\frac{\alpha}{1-\gamma}}H^{\frac{\beta}{1-\gamma}}A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \text{and} \quad \xi \equiv \frac{H}{A^{\frac{1-\chi}{\phi}}}. \tag{34}$$

By using these variables, the dynamical system of the internal solution case can be reduced to the system of five equations consisting of five variables: $\{\xi, \tilde{c}, \tilde{k}, u_A, u_Y\}$. By construction, these five variables are constant on the steady growth path.

The intensive form dynamical system can be obtained as follows[15]

$$\dot{\tilde{k}} = \left[\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}} - \frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{k}, \tag{35}$$

$$
\begin{aligned}
\dot{\tilde{c}} \;=\; & \left[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}) - \rho - n\} + n \right.\\
& \left. -\frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{c}, \quad (36)
\end{aligned}
$$

[14] See (12) for the structure of ξ, and (13) for the structure of $\tilde{c}$ and $\tilde{k}$.

[15] See appendix a for a detailed description on how we have the result.

$$\dot{\xi} = \{b(1-u_A-u_Y)+(1-\delta)n - \frac{1-\chi}{\phi}Bu_A^{\phi}\xi^{\phi}\}\xi, \tag{37}$$

$$\dot{u_A} = \left[-b(1-u_A-u_Y)+b+\phi(1-\delta)n+\left\{\chi - \frac{\gamma(\zeta-\gamma)}{\zeta\beta}\frac{u_Y}{u_A}\right\}Bu_A^{\phi}\xi^{\phi}\right]\frac{u_A}{1-\phi}, \tag{38}$$

and

$$\begin{aligned} \dot{u_Y} &= \left\{-b(1-u_A-u_Y)+\frac{\{\alpha+\beta(1-\delta)\}n}{1-\beta}\right. \\ &\quad \left.+\frac{(\zeta-\gamma)B}{1-\beta}u_A^{\phi}\xi^{\phi}+\frac{\gamma}{1-\beta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}-\frac{\tilde{c}}{\tilde{k}})\right\}u_Y. \end{aligned} \tag{39}$$

Next, in order to analyze the local stability of the dynamical system in the internal solution case, we will derive the steady-growth-path-values of $\{u_Y^{sgp}, u_A^{sgp}, \xi^{sgp}, \tilde{c}^{sgp}, \tilde{k}^{sgp}\}$. The SGP values are obtained as

$$u_A^{sgp} = (1-u_H^{sgp})\left\{1+\frac{(r-g_Y+g_A)\beta\zeta}{\gamma(\zeta-\gamma)g_A}\right\}^{-1}, \tag{40}$$

$$u_Y^{sgp} = 1-u_A^{sgp}-u_H^{sgp}, \tag{41}$$

$$\xi^{sgp} = \left(\frac{g_A}{B}\right)^{\frac{1}{\phi}}\frac{1}{u_A^{sgp}}, \tag{42}$$

$$\tilde{k}^{sgp} = \left(\frac{r\zeta\eta^{\gamma}}{\gamma^2(u_Y^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}, \tag{43}$$

and

$$\tilde{c}^{sgp} = (\frac{r\zeta}{\gamma^2}-g_Y)\tilde{k}^{sgp}, \tag{44}$$

where r denotes the interest rate in the inner solution case and $u_H^{sgp} = \frac{g_H-(1-\delta)n}{b}$.

4.2 The Corner Solution Case

In the corner solution case, the dynamical system is given by the following six equations of six variables consisting of $\{L, K, H, A, u_Y, v\}$

$$\begin{aligned} \dot{L} &= nL, & (45)\\ \dot{K} &= Y(L,H,K,A,u_Y)-C, & (46)\\ \sigma\left(\frac{\dot{C}}{C}-n\right) &= r(L,H,K,A,u_Y)-\rho-n, & (47)\\ \dot{v}+\pi &= r(L,H,K,A,u_Y)v, & (48)\\ \dot{H} &= (1-\delta)nH, & (49) \end{aligned}$$

and

$$\dot{A} = BA^{\chi}\{(1-u_Y)H\}^{\phi}. \tag{50}$$

Notice here that (i) u_H and u_A do not appear in the system because $u_H = 0$ and u_A is determined by the condition $u_A + u_Y = 1$ when u_Y is given,[16] (ii) Y is given by $Y = \eta^{-\gamma} L^{\alpha}(u_Y H)^{\beta} A^{\zeta-\gamma} K^{\gamma}$ from (1), (iii) r is given by $r = \frac{\gamma^2}{\zeta}\frac{Y}{K}$ from (2), (iv) π is given by $\pi = \frac{\gamma(\zeta-\gamma)}{\zeta}\frac{Y}{A}$ from (2) and (v) v is determined by $v = w_H \frac{\{(1-u_Y)H\}^{1-\phi}}{BA^{\chi}}$ from (5) whereas w_H is given by $w_H = \beta\frac{Y}{u_Y H}$ from (2).

Likely to the internal solution case, we define the following variables in order to obtain the intensive form dynamical system in the corner solution case

$$\tilde{c}_c \equiv \frac{C}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \tilde{k}_c \equiv \frac{K}{L^{\frac{\alpha}{1-\gamma}} H^{\frac{\beta}{1-\gamma}} A^{\frac{\zeta-\gamma}{1-\gamma}}}, \quad \text{and} \quad \xi_c \equiv \frac{H}{A^{\frac{1-\chi}{\phi}}}, \tag{51}$$

By using these variables, the dynamical system of the corner solution case can be reduced to the system of four equations consisting of four variables: $\{\xi_c, \tilde{c}_c, \tilde{k}_c, u_{Y,c}\}$ where the subscript c denotes "corner solution case". By construction, these four variables are constant on the steady growth path.

The intensive form dynamical system in the corner solution case can be obtained as follows[17]

$$\begin{aligned}\dot{\tilde{c}}_c \; &= \; \left[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma} u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1}) - \rho - n\} + n \right.\\ &\left. -\frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\tilde{c}_c,\end{aligned} \tag{52}$$

$$\dot{\tilde{k}}_c = \left[\eta^{-\gamma} u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} - \frac{\tilde{c}_c}{\tilde{k}_c} - \frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\tilde{k}_c, \tag{53}$$

$$\dot{\xi}_c = \{(1-\delta)n - \frac{1-\chi}{\phi}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\}\xi_c, \tag{54}$$

and

$$\begin{aligned}\dot{u}_{Y,c} \; &= \left[\{(\phi-\beta)(1-\delta)-\alpha\}n + \frac{(\gamma-\zeta)\gamma}{\zeta}\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} + \gamma\frac{\tilde{c}_c}{\tilde{k}_c}\right. \\ &\left. +(\chi-\zeta+\gamma-\frac{\gamma(\zeta-\gamma)}{\beta\zeta}\frac{u_{Y,c}}{1-u_{Y,c}})B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\frac{u_{Y,c}(1-u_{Y,c})}{\beta-1+(\phi-\beta)u_{Y,c}}.\end{aligned} \tag{55}$$

Next, in order to analyze the local stability of the dynamical system in the corner solution case, we will derive the steady-growth-path-values of $\{u_{Y,c}^{sgp}, \xi_c^{sgp}, \tilde{c}_c^{sgp}, \tilde{k}_c^{sgp}\}$. They are determined as

$$u_{Y,c}^{sgp} = \frac{\Omega}{1+\Omega}, \quad \text{where} \quad \Omega = \beta\zeta(\bar{r} - \bar{g}_Y + \bar{g}_A)/(\gamma(\zeta-\gamma)\bar{g}_A) \tag{56}$$

$$\xi_c^{sgp} = \left(\frac{\bar{g}_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{(1-u_{Y,c}^{sgp})}, \tag{57}$$

[16] It is easy to see that since u_A drops off the dynamical system, the dimension of the system reduces in the corner solution case to six from seven in the internal solution case.

[17] See appendix b for a detailed derivation.

$$\tilde{k}_c^{sgp} = \left(\frac{\bar{r}\zeta\eta^\gamma}{\gamma^2 (u_{Y,c}^{sgp})^\beta} \right)^{\frac{-1}{1-\gamma}}, \tag{58}$$

and

$$\tilde{c}_c^{sgp} = (\frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y)\tilde{k}_c^{sgp}, \tag{59}$$

where $\bar{r}$ is the interest rate in the corner solution case.

4.3 Calibration and Discussion

We calibrate our model as follows. First, following Benhabib et al. (1994), we can designate the labor share (S_L), the share of human capital (S_H) and the capital share (S_K) in the final goods sector by $S_L = \alpha/(\alpha + \beta + (\gamma^2/\zeta))$, $S_H = \beta/(\alpha+\beta+(\gamma^2/\zeta))$ and $S_K = (\gamma^2/\zeta)/(\alpha+\beta+(\gamma^2/\zeta))$, respectively. We set $S_L = 0.25$, $S_H = 0.5$ and $S_K = 0.25$ following Benhabib et al. (1994). $\zeta = 2$ is cited from them as well. These conditions require that $\alpha = 0.15$, $\beta = 0.30$, and $\gamma = 0.55$. We set the subjective discount rate ρ to be 0.04, and the population growth rate n to be 0.03. The latter figure is drawn from Kortum (1997). We set $\eta = 0.34$, which gives some 25% markup in the intermediate goods sector, as is the case of Benhabib et al. (1994). b is set to be 0.05 from Lucas (1988), Benhabib and Perli (1994) and Funke and Strulik (2000). We set $\sigma = 0.9$. ϕ and χ must be determined such that the Jones technology exhibits diminishing returns to scale and we choose $\phi = 0.3$ and $\chi = 0.5$. The level parameter in the R&D process is set to be $B = 1$.

Finally, δ should be determined. This is the parameter to determine the intensity of spillover effects in human capital accumulation process. We examine with $\delta = 0.8$ and $\delta = 0.9$. We will investigate the case $\delta = 1$ (no spillover case) in order to compare the result with those in the economy with the local home environment externality.

Table 1 shows some key economic variables which result from the parameter sets. Notice also that the condition 2 is satisfied with these parameters. The sign of $\underline{\delta} - \delta$ is negative so that the population growth affects the per capita GDP growth rate negatively, which is also present in the data. Most importantly, conditions (20), (21) and (26) are satisfied so that we have multiple SGPs under the parameter sets.

Now we examine the local stability of two SGPs.[18] In the internal solution case, the system consists of $\{\xi, \tilde{c}, \tilde{k}, u_A, u_Y\}$ and there are three jump variables

[18] The matlab codes to calculate the roots of the system are available from the authors upon requests.

and two state variables. On the other hand, in the corner solution case, the system consists of $\{\xi_c, \tilde{c}_c, \tilde{k}_c, u_{Y,c}\}$ and there are two jump variables and two state variables. Because we adopt the production structure of Benhabib et al. (1994), it seems that local indeterminacy will result.[19] However, our investigations will show that the SGP of the corner solution is robustly saddle stable, and that the SGP of the internal solution case will be either saddle stable or unstable in the sense that there are too many explosive roots. At this point, we suspect that this contrast between Benhabib et al. (1994) and the present analysis is attributable to assumptions in R&D process; there are scale effects in Benhabib et al. (1994) while we adopt the Jones technology thereby eliminating the increasing returns effects in the R&D process, which would be a chief source of indeterminacy in Benhabib et al. (1994).

Table 2 shows the eigen values of the matrix which we obtain by linearizing the dynamical systems around the SGPs when $\delta = 0.9$ and $\delta = 0.8$. As table 2 suggests, when $\delta = 0.9$ both of the SGPs are locally saddle stable. With the results it can be said that both SGPs can be realized as optimal paths. In the case where we have multiple equilibria, the equilibrium path to be realized depends on the self-fulfilling expectations of households; and hence, our result provides a reason to explain the polarization of economies discussed by Krugman (1991), Lucas (1993) and Howitt (1994). We argue that multiplicity in human capital investment decisions could cause the phenomenon.

If $\delta = 0.8$, the situation changes (see table 2). In this case, the SGP of the internal solution case cannot be realized as an optimal path because the number of unstable roots is bigger than the number of the jump variables in the system. The SGP of the corner solution case is still locally saddle stable. In this case, though the economy potentially has two SGPs, only the corner solution path will realize. In this case, our model would have consequences on the discussion by Lucas (1993): we may need a miracle to attain a higher growth rate with positive investments on human capital.[20]

It is interesting to compare above results with the case where we have no spillover effects ($\delta = 1$). In the case of $\delta = 1$, we do not have multiple equilibria and only the internal solution case will result. The key economic variables are in table 3. As tables 1 and 3 suggest, the GDP growth rate in the economy

[19] In the present model, there will be no mechanism to bring indeterminacy from the Lucas-type human capital production process because in the present model there is no externality of human capital in the final goods production sector. See also Behnabib and Perli (1994).

[20] The probability that the internal solution case realizes is zero in this case in the Lebesgue sense.

when $\delta = 1$ is higher than that of the corner solution case when $\delta = 0.8$. Moreover, the SGP when $\delta = 1$ is locally saddle stable (see table 4). This result casts a paradox: it may seem that if there is positive externality in human capital accumulation process, the local home environment externality, then the growth rate will be higher than the one we have when there is no local home environment externality. However, our results hold that it is not always the case. The economy may be trapped in a slowly growing path without human capital investments due to the externality.

Actually, the situation that the growth rate in the society may be low when the local home environment externality matters is illustrated by Galor and Weil (2000). Quoting Schultz (1964), they argue that

> $\cdots$ when productive technology has been constant [and low] for a long period of time, farmers will have learned to use their resources efficiently. Children will acquire knowledge of how to deal with this environment directly from observing their parents, and formal schooling will have little economic value $\cdots$. [Oded Galor and David N. Weil, "Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond," p810]

This argument will be, again, a reflection of difficulty to attain "modern economic growth" through the augmentation of human capital.

5 Concluding Remarks

This paper constructs an endogenous growth model with R&D and human capital accumulation following the line developed by Arnold (1998), Funke and Strulik (2000) and Strulik (2005). We allow for exogenous population growth and the local home environment externality of human capital. Under these assumptions, we obtain a negative correlation between the population growth rate and the per capita GDP growth rate. Moreover, we show that multiple equilibria might generate. Hence, our model would have consequences on the polarization phenomenon which is discussed by Krugman (1991), Lucas (1993) and Howitt (1994). Then, we argue that the model could integrate two major empirical topics in economic growth literature. The paper also casts a paradox that the GDP growth rate may be higher in the society without the externality than the one in the economy with externality.

As a final remark, it is important to point out that labor supply is exogenous in the model. We believe that a promising extension to our model would be to either endogenize the fertility rate or allow agents to make labor-leisure choices. We leave these tasks for future research.

6 Appendicies

6.1 Appendix a

We can derive the intensive form dynamical system in the internal solution case as the following way. First take the natural logarithm of (34) and differentiate the equations with time. Hence, we will obtain the dynamic evolution of ξ, $\tilde{c}$ and $\tilde{k}$ as

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{C}}{C} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{60}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{61}$$

and

$$\frac{\dot{\xi}}{\xi} = \frac{\dot{H}}{H} - \frac{1-\chi}{\phi}\frac{\dot{A}}{A}. \tag{62}$$

From (32), we obtain that $\frac{\dot{H}}{H} = b(1-u_A-u_Y)+(1-\delta)n$. From (33), we see $\frac{\dot{A}}{A} = Bu_A^{\phi}\xi^{\phi}$ and from (27) we have $\dot{L}/L = n$. From (29) and (2), $\frac{\dot{C}}{C}$ is obtained as

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(\frac{\gamma^2}{\zeta}\frac{Y}{K} - \rho - n) + n. \tag{63}$$

$\frac{Y}{K}$ is obtained from (1) that

$$\frac{Y}{K} = \eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}. \tag{64}$$

From the resource constraint, we have $\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K}$. By this condition, we have

$$\frac{\dot{K}}{K} = \eta^{\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}}. \tag{65}$$

Substituting these conditions into (60) – (62) ,we will have the following three equations

$$\dot{\tilde{k}} = \left[\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1} - \frac{\tilde{c}}{\tilde{k}} - \frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{k}, \tag{66}$$

$$\begin{aligned} \dot{\tilde{c}} \quad = \quad & \left[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma}u_Y^{\beta}\tilde{k}^{\gamma-1}) - \rho - n\} + n \right. \\ & \left. -\frac{\alpha n}{1-\gamma} - \frac{\beta(1-\delta)n + \beta b(1-u_Y-u_A)}{1-\gamma} - \frac{(\zeta-\gamma)}{(1-\gamma)}Bu_A^{\phi}\xi^{\phi}\right]\tilde{c}, \end{aligned} \tag{67}$$

$$\dot{\xi} = \{b(1 - u_A - u_Y) + (1 - \delta)n - \frac{1 - \chi}{\phi} B u_A^{\phi} \xi^{\phi}\}\xi. \tag{68}$$

Next, from (30) and (5) we have

$$\frac{\dot{v}}{v} + \chi \frac{\dot{A}}{A} + (\phi - 1)(\frac{\dot{u}_A}{u_A} + \frac{\dot{H}}{H}) = \frac{\dot{w}_H}{w_H} = r - b - (1 - \delta)n. \tag{69}$$

Substituting the value of π from (2) into (31), we obtain the growth rate of v as follows:

$$\frac{\dot{v}}{v} = r - \frac{\gamma(\zeta - \gamma)}{\beta\zeta} \frac{u_Y}{u_A} B u_A^{\phi} \xi^{\phi}. \tag{70}$$

Substituting $\frac{\dot{H}}{H} = b(1 - u_A - u_Y) + (1 - \delta)n$, $\frac{\dot{A}}{A} = B u_A^{\phi} \xi^{\phi}$ and (70) into (69), we obtain the dynamics of u_A as

$$\dot{u_A} = \left[-b(1 - u_A - u_Y) + b + \phi(1 - \delta)n + \left\{\chi - \frac{\gamma(\zeta - \gamma)}{\zeta\beta} \frac{u_Y}{u_A}\right\} B u_A^{\phi} \xi^{\phi}\right] \frac{u_A}{1 - \phi}. \tag{71}$$

From (2) we have $\frac{\dot{w}_H}{w_H} = \frac{\dot{Y}}{Y} - (\frac{\dot{H}}{H} + \frac{\dot{u}_Y}{u_Y})$. With this condition and (30), the following equation is obtained

$$r - b - (1 - \delta)n = \frac{\dot{w}_H}{w_H} = \frac{\dot{Y}}{Y} - (\frac{\dot{H}}{H} + \frac{\dot{u}_Y}{u_Y}). \tag{72}$$

From the production function, $\frac{\dot{Y}}{Y} = \alpha n + \beta \left(\frac{\dot{H}}{H} + \frac{\dot{u}_Y}{u_Y}\right) + (\zeta - \gamma)\frac{\dot{A}}{A} + \gamma \frac{\dot{K}}{K}$. With these arguments, we have the dynamics of u_Y as follows

$$\begin{aligned} \dot{u_Y} \quad = \quad & \left\{-b(1 - u_A - u_Y) + \frac{\{\alpha + \beta(1 - \delta)\}n}{1 - \beta}\right. \\ & \left. + \frac{(\zeta - \gamma)B}{1 - \beta} u_A^{\phi} \xi^{\phi} + \frac{\gamma}{1 - \beta}(\eta^{-\gamma} u_Y^{\beta} \tilde{k}^{\gamma - 1} - \frac{\tilde{c}}{\tilde{k}})\right\} u_Y. \end{aligned} \tag{73}$$

(66),(67),(68),(71) and (73) depict the dynamical system of the model.

Next we will derive the SGP values of $\{u_Y^{sgp}, u_A^{sgp}, \xi^{sgp}, \tilde{c}^{sgp}, \tilde{k}^{sgp}\}$. From (15), the interest rate on the SGP in the internal solution case is determined as

$$r = g_Y - g_H + b + (1 - \delta)n. \tag{74}$$

From (31), we can have $r = \frac{\dot{v}}{v} + \frac{\pi}{v}$. Also, from (2), we have that the growth rate of π equals $g_Y - g_A$. With these conditions and the fact that the growth rates of π and v are same, we have

$$\frac{\pi}{v} = r - g_Y + g_A. \tag{75}$$

We can solve above equation with (2), (5) and $g_A = B(u_A^{sgp} \xi^{sgp})^{\phi}$ as

$$\frac{\gamma(\zeta - \gamma) g_A}{\beta\zeta} \frac{u_Y^{sgp}}{u_A^{sgp}} = r - g_Y + g_A. \tag{76}$$

From (32), on the SGP we have

$$g_H = b(1 - u_Y^{sgp} - u_A^{sgp}) + (1-\delta)n = bu_H^{sgp} + (1-\delta)n. \tag{77}$$

Then, we have $u_H^{sgp} = \frac{g_H-(1-\delta)n}{b}$ where g_H is already obtained with (17) and (18). From the resource constraint we obtain

$$u_A^{sgp} = 1 - u_Y^{sgp} - u_H^{sgp}. \tag{78}$$

Hence, from (76) and (78) we obtain u_Y^{sgp} and u_A^{sgp} respectively as

$$u_A^{sgp} = (1 - u_H^{sgp})\left\{1 + \frac{(r - g_Y + g_A)\beta\zeta}{\gamma(\zeta-\gamma)g_A}\right\}^{-1}, \tag{79}$$

and

$$u_Y^{sgp} = 1 - u_A^{sgp} - u_H^{sgp}. \tag{80}$$

ξ^{sgp} follows immediately due to $g_A = B(u_A^{sgp}\xi^{sgp})^{\phi}$ as

$$\xi^{sgp} = \left(\frac{g_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{u_A^{sgp}}. \tag{81}$$

Next, from the resource constraint ($\dot{K} = Y - C$) and (2) we have

$$\frac{C}{K} = \frac{r\zeta}{\gamma^2} - g_Y = \frac{\tilde{c}^{sgp}}{\tilde{k}^{sgp}}. \tag{82}$$

Moreover, from $\frac{Y}{K} = \frac{r\zeta}{\gamma^2}$ and (1) we have

$$\eta^{-\gamma}L^{\alpha}(u_Y H)^{\beta}A^{\zeta-\gamma}K^{\gamma-1} = \frac{r\zeta}{\gamma^2}. \tag{83}$$

From the definition of $\tilde{k}$ and with a little algebra, we have

$$\tilde{k}^{sgp} = \left(\frac{r\zeta\eta^{\gamma}}{\gamma^2(u_Y^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}. \tag{84}$$

Finally, with $\tilde{k}^{sgp}$, $\tilde{c}^{sgp}$ is obtained from (82) as

$$\tilde{c}^{sgp} = (\frac{r\zeta}{\gamma^2} - g_Y)\tilde{k}^{sgp}. \tag{85}$$

6.2 Appendix b

We can derive the intensive form dynamical system in the corner solution case as the following way. First take the natural logarithm of (51) and differentiate the equations with time. Hence, we will obtain respectively the dynamic evolution of $\tilde{c}_c$, $\tilde{k}_c$ and ξ_c as

$$\frac{\dot{\tilde{c}}_c}{\tilde{c}_c} = \frac{\dot{C}}{C} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{86}$$

$$\frac{\dot{\tilde{k}}_c}{\tilde{k}_c} = \frac{\dot{K}}{K} - \frac{\alpha}{1-\gamma}\frac{\dot{L}}{L} - \frac{\beta}{1-\gamma}\frac{\dot{H}}{H} - \frac{\zeta-\beta}{1-\gamma}\frac{\dot{A}}{A}, \tag{87}$$

and

$$\frac{\dot{\xi}_c}{\xi_c} = \frac{\dot{H}}{H} - \frac{1-\chi}{\phi}\frac{\dot{A}}{A}. \tag{88}$$

In the corner solution case, the growth rate of aggregate human capital is determined as $\frac{\dot{H}}{H} = (1-\delta)n$ from (49). Also, the growth rate of the variety is obtained as $\frac{\dot{A}}{A} = B(1-u_{Y,c})^{\phi}\xi_c^{\phi}$ from (50) whereas $\frac{\dot{L}}{L} = n$ from (45).

From the resource constraint, we have $\frac{\dot{K}}{K} = \frac{Y}{K} - \frac{C}{K}$. By this condition, we have $\bar{g}_K = \bar{g}_Y = \bar{g}_C = \bar{g}_Y$ on the SGP. From (47) and (2), $\frac{\dot{C}}{C}$ is obtained as

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(\frac{\gamma^2}{\zeta}\frac{Y}{K} - \rho - n) + n. \tag{89}$$

Finally, $\frac{Y}{K}$ is obtained as

$$\frac{Y}{K} = \eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} \tag{90}$$

With these conditions, we have

$$\begin{aligned} \dot{\tilde{c}}_c \;\; = \;\; & \left[\frac{1}{\sigma}\{\frac{\gamma^2}{\zeta}(\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1}) - \rho - n\} + n \right. \\ & \left. -\frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\tilde{c}_c, \end{aligned} \tag{91}$$

$$\dot{\tilde{k}}_c = \left[\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} - \frac{\tilde{c}_c}{\tilde{k}_c} - \frac{\alpha}{1-\gamma}n - \frac{\beta(1-\delta)}{1-\gamma}n - \frac{(\zeta-\gamma)}{(1-\gamma)}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\tilde{k}_c, \tag{92}$$

and

$$\dot{\xi}_c = \{(1-\delta)n - \frac{1-\chi}{\phi}B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\}\xi_c. \tag{93}$$

With (48), (2) and (5) a little algebra leads to the evolution of $u_{Y,c}$ as

$$\begin{aligned} \dot{u}_{Y,c} \;\; = & \left[\{(\phi-\beta)(1-\delta)-\alpha\}n + \frac{(\gamma-\zeta)\gamma}{\zeta}\eta^{-\gamma}u_{Y,c}^{\beta}\tilde{k}_c^{\gamma-1} + \gamma\frac{\tilde{c}_c}{\tilde{k}_c}\right. \\ & \left. +(\chi-\zeta+\gamma-\frac{\gamma(\zeta-\gamma)}{\beta\zeta}\frac{u_{Y,c}}{1-u_{Y,c}})B(1-u_{Y,c})^{\phi}\xi_c^{\phi}\right]\frac{u_{Y,c}(1-u_{Y,c})}{\beta-1+(\phi-\beta)u_{Y,c}}. \end{aligned} \tag{94}$$

(91) – (94) regulate the dynamics of the system.

Next we will derive the SGP values of $\{u_{Y,c}^{sgp}, \xi_c^{sgp}, \tilde{c}_c^{sgp}, \tilde{k}_c^{sgp}\}$. From $\bar{g}_Y = \bar{g}_C$, (22) and (47), the interest rate on the SGP in the corner solution case is determined as

$$\bar{r} = \frac{\Psi}{\alpha+\beta}(\underline{\delta}-\delta)n\sigma + \rho + n. \tag{95}$$

From (48), we can have $\bar{r} = \frac{\dot{v}}{v} + \frac{\pi}{v}$. Also, from (2), we have that the growth rate of π equals $\bar{g}_Y - \bar{g}_A$. With these conditions and the fact that the growth rates of π and v are same, we have

$$\frac{\pi}{v} = \bar{r} - \bar{g}_Y + \bar{g}_A. \tag{96}$$

We can solve above equation with (2), (5), $\bar{g}_A = B\{(1-u_{Y,c}^{sgp})\xi_c^{sgp}\}^{\phi}$, and $u_{Y,c}^{sgp} + u_{A,c}^{sgp} = 1$ as

$$u_{Y,c}^{sgp} = \frac{\Omega}{1+\Omega}, \quad \text{where} \quad \Omega = \beta\zeta(r - \bar{g}_Y + \bar{g}_A)/(\gamma(\zeta-\gamma)\bar{g}_A). \tag{97}$$

ξ_c^{sgp} follows immediately due to $\bar{g}_A = B\{(1-u_{Y,c}^{sgp})\xi_c^{sgp}\}^{\phi}$ as

$$\xi_c^{sgp} = \left(\frac{\bar{g}_A}{B}\right)^{\frac{1}{\phi}} \frac{1}{(1-u_{Y,c}^{sgp})}. \tag{98}$$

Next, from the resource constraint ($\dot{K} = Y - C$) and (2) we have

$$\frac{C}{K} = \frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y = \frac{\tilde{c}_c^{sgp}}{\tilde{k}_c^{sgp}}. \tag{99}$$

Moreover, from $\frac{Y}{K} = \frac{\bar{r}\zeta}{\gamma^2}$ and (1) we have

$$\eta^{-\gamma} L^{\alpha} (u_{Y,c} H)^{\beta} A^{\zeta-\gamma} K^{\gamma-1} = \frac{\bar{r}\zeta}{\gamma^2}. \tag{100}$$

From the definition of $\tilde{k}_c$ and with a little algebra, we have

$$\tilde{k}_c^{sgp} = \left(\frac{\bar{r}\zeta\eta^{\gamma}}{\gamma^2 (u_{Y,c}^{sgp})^{\beta}}\right)^{\frac{-1}{1-\gamma}}. \tag{101}$$

Finally, with $\tilde{k}_c^{sgp}$, $\tilde{c}_c^{sgp}$ is obtained from (99) as

$$\tilde{c}_c^{sgp} = (\frac{\bar{r}\zeta}{\gamma^2} - \bar{g}_Y)\tilde{k}_c^{sgp}. \tag{102}$$

References

[1] Aghion, Philippe and Howitt, Peter, 1992. A Model of Growth through Creative Destruction. Econometrica 60, 323-351.

[2] Ahituv, Avner, 2001. Be fruitful or multiply: On the interplay between fertility and economic development, Journal of Population Economics 14, 51-71.

[3] Arnold, Lutz G., 1998. Growth, welfare, and trade in an integrated model of human-capital accumulation and research. Journal of Macroeconomics 20, 81-105.

[4] Azariadis, Costas and Drazen, Allan, 1990. Threshold Externalities in Economic Development. The Quarterly Journal of Economics 105, 501-26.

[5] Benhabib, Jess and Perli, Roberto, 1994. Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth, Journal of Economic Theory 63, 113-142.

[6] Benhabib, Jess, Perli, Roberto and Xie, Danyang, 1994. Monopolistic Competition, Indeterminacy and Growth, Ricerche Economiche 48, 279-298.

[7] Dalgaard, Carl-Johan and Kreiner, Claus Thustrup, 2001. Is Declining Productivity Inevitable?, Journal of Economic Growth 6, 187-203,

[8] Easterly, William, 1994. Economic Stagnation, Fixed Factors and Policy Thresholds, Journal of Monetary Economics 33, 525-557.

[9] Funke, Michael and Strulik, Holger, 2000. On Endogenous Growth with Physical Capital, Human Capital and Product Variety, European Economic Review 44, 491-515.

[10] Galor, Oded, 2005. From Stagnation to Growth: Unified Growth Theory, in: Philippe Aghion & Steven Durlauf (ed.) Handbook of Economic Growth, edition 1, volume 1, chapter 4, 171-293, Elsevier.

[11] Galor, Oded and Tsiddon, Daniel, 1997a. The Distribution of Human Capital and Economic Growth, Journal of Economic Growth 2, 93-124.

[12] Galor, Oded and Tsiddon, Daniel, 1997b. From Malthusian Stagnation to Modern Growth, American Economic Review 89, 150-154.

[13] Galor, Oded, David N. Weil, 2000. Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond, American Economic Review 90, 806-828.

[14] Grossman, Gene M and Helpman, Elhanan, 1991, Innovation and growth in the global economy, MIT Press, Cambridge, MA.

[15] Howitt, Peter, 1994. Adjusting to Technological Change, Canadian Journal of Economics 27, 763-75

[16] Howitt, Peter, 1999. Steady Endogenous Growth with Population and R & D Inputs Growing. Journal of Political Economy 107, 715-730.

[17] Jones, Charles I., 1995a. Time series tests of endogenous growth models. Quarterly Journal of Economics 110, 495-525.

[18] Jones, Charles I., 1995b. R&D-based models of economic growth. Journal of Political Economy 103, 759-784.

[19] Jones, Charles I., 2001. Population and Ideas: A Theory of Endogeous Growth. mimeo.

[20] Kelley, Allen, 1988. Economic consequence of population change in the third world. Journal of Economic Literature 26, 1685-1728.

[21] Kelley, Allen and Schmidt, Robert, 1995. Aggregate population and economic growth correlations: The role of the components of demographic change. Demography 32, 543-555.

[22] Kortum, Samuel, 1997. Research, Patenting, and Technological Change. Econometrica 65, 1389-1419.

[23] Krugman, Paul R., 1991. History versus Expectations. The Quarterly Journal of Economics 106, 651-67.

[24] Lucas, Robert E, Jr, 1988. On the mechanism of economic development. Journal of Monetary Economics 22, 3-42.

[25] Lucas, Robert E, Jr, 1993. Making a Miracle. Econometrica 61, 251-72.

[26] Quah, Danny, 1996. Empirics for economic growth and convergence. European Economic Review 40, 1353-1375.

[27] Quah, Danny, 1997. Empirics for growth and distribution: Polarization, stratification, and convergence clubs. Journal of Economic Growth 2, 27-59.

[28] Romer, Paul M, 1990. Endogenous Technological Change, Journal of Political Economy 98, S71-S102.

[29] Segerstrom, Paul, 2000. The Long-Run Growth Effects of R&D Subsidies, Journal of Economic Growth 5, 277-305.

[30] Schultz, Theodore W., 1964. Transforming Traditional Agriculture. New Haven: Yale University Press.

[31] Strulik, Holger, 2005. The Role of Human Capital and Population Growth in R&D-based Model of Economic Growth. Review of International Economics 13, 129-145.

[32] Uzawa, Hirofumi, 1965. An aggregative model of optimal technical change. International Economic Review 6, 18-31.

Table 1: Key Economic Variables

Internal Solution Case						
$\delta = 0.9$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	-0.1564	0.0486	0.0868	0.1535	0.6092	0.2373
$\delta = 0.8$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	-0.0564	0.0592	0.0963	0.1762	0.5655	0.2584
Corner Solution Case						
$\delta = 0.9$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	-0.1564	0.0178	0.059	0.0527	0.9473	0
$\delta = 0.8$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	-0.0564	0.0256	0.066	0.098	0.902	0

Table 2: Eigen Values and the Number of Unstable Roots

$\delta = 0.9$		$\delta = 0.8$	
Eigen Values (I)	Eigen Values (C)	Eigen Values (I)	Eigen Values (C)
-4.36E+04	0.5834	-2.368E+04	-0.99921
0.4439	-0.9997	0.4876	0.53974
0.089277 + 0.023664i	0.0013	0.097501 + 0.018529i	0.0019168
0.089277 - 0.023664i	-8.00E-4	0.097501 - 0.018529i	-1.49E-03
$-$ 8.18E-12		1.96E-09	
3	2	4	2
Saddle Stable	Saddle Stable	No Solution	Saddle Stable

The Number of Unstable Roots: (I) stands for the internal solution case and (C) stands for the corner solution case.

Table 3: Key Economic Variables

Internal Solution Case						
$\delta = 1$						
Υ	$\underline{\delta} - \delta$	g_Y	R	u_A	u_Y	u_H
0.3330	-0.1564	0.0381	0.0773	0.1245	0.6593	0.2162

Table 4: Eigen Values and the Number of Unstable Roots

$\delta = 1$
Eigen Values (I)
-9.13E+04 0.4004 0.080887 + 0.027064i 0.080887 - 0.027064i $-$ 3.18E-10
3
Saddle Stable

The Number of Unstable Roots: (I) stands for the internal solution case.