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with Generalized Taylor Rule:
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Abstract

This paper examines equilibrium determinacy of a discrete-time AK growth model with a generalized Taylor rule under which interest rate responds to the growth rate of real income as well as to the rate of inflation. We use the standard money-in-the-utility formulation in which money is superneutral on the balanced-growth path. We show that even in such a simple environment, the generalized Taylor rule may yield indeterminacy of equilibrium easily. We also demonstrate that equilibrium determinacy depends on the timing of money holding of households as well.

Keywords and Phrases: equilibrium determinacy, the Taylor rules, endogenous growth, timing of money holdings

JEL Classification Numbers: O42, E52

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1 Introduction

Taylor (1993) proposes a monetary policy rule for economic stabilization under which the central bank adjusts the nominal interest rate in response to real income as well as to the rate of inflation. However, the existing theoretical studies on the interest control rules often assume that the interest rate responds to inflation alone [1]. The purpose of this paper is to explore the efficacy of the original Taylor rule in the context of a model of endogenous growth. We introduce money into the basic AK growth model via the money-in-the-utility-function formulation. In such a simple environment, money is superneutral on the balanced growth path. In our setting, however, money is not superneutral in the transition process and, hence, the selection of monetary policy rule may have relevant effects on determinacy of equilibrium path leading to the balanced-growth equilibrium.

We construct our model in a discrete-time setting, which enables us to consider alternative timings of households' money holdings and of the inflation rate used for controlling nominal interest rate. As for money holding of the household, we can distinguish the cash-in-advance (CIA) timing from the cash-when-I'm-done (CWID) timing. The CIA (resp. CWID) timing means that real money balances in the utility function is the stock of money the household holds before entering (resp. after leaving) the final goods market [2]. Moreover, in our discrete-time model we find that the main results are also sensitive to the assumption whether the central bank's control rule is current-looking or forward-looking. Therefore, in a discrete-time modelling, we can analyze four patterns of formulations: (i) CWID timing with a forward-looking rule, (ii) CIA timing with a forward-looking rule, (iii) CWID

[1] In models of endowment economy as in Leeper (1991) or Benhabib et. al. (2001), real income cannot be used as an index of monetary policy.

[2] The discrete-time monetary models usually assume the CWID timing of the money holdings. However, as Carlstrom and Fuerst (2001) claim, it is difficult to justify CWID timing on theoretical grounds, because this assumption means that the money held at the beginning of $t+1$ reduces transaction costs in period t.

timing with a current-looking rule, and (iv) CIA timing with a current-looking rule.

We obtain two main findings. First, the response of the interest rate to the growth rate of income may play a significant role for equilibrium determinacy. In fact, if the monetary authority controls interest rate in response to inflation alone, we obtain the standard results: equilibrium determinacy holds under the forward-looking and active current-looking monetary rule, while the passive current-looking interest-control rule generates equilibrium indeterminacy. If the interest rate responds to the growth rate of income as well, the possibility of emergence of equilibrium indeterminacy may be enhanced. Second, the efficacy of the generalized Taylor rule for macroeconomic stability depends upon the timings of money holding of the households. These findings demonstrate that the monetary authority should carefully select a specific interest rate control rule in order to attain stability even if the economic environment is simple enough to hold superneutrality of money in the long run.

Several studies are closely related to this paper. As for the equilibrium determinacy in monetary growth model with an AK technology, Suen and Yip (2005) and Chen and Guo (2007) introduce money into the model in the form of cash-in-advance (CIA) constraint. Those authors show that the balanced-growth path may be indeterminate under a constant money growth rule if the CIA constraint applies not only to consumption but also to investment so that money is not superneutral on the balanced growth path [3]. Indeterminacy is generated by this form of the CIA constraint rather than by monetary policy rule.

Li and Yip (2004) and Meng and Yip (2004) investigate the effect of Taylor-type interest rate control in the neoclassical growth (i.e. exogenous growth) models. The main message of these studies is that in the neoclassical growth models equilibrium is mostly determinate regardless of the form of interest rate control rules. Such a conclusion may not hold in endogenous growth models. Fujisaki and Mino (2007)

[3] Chen and Guo (2007) generalize Suen and Yip (2005) in a way that the CIA constraint applies to consumption and to a certain fraction of gross investment.

use an AK growth model with a generalized Taylor rule to demonstrate that equilibrium indeterminacy may emerge more easily than in the exogenous growth models. Since Fujisaki and Mino (2007) use a continuous-time formulation, our discrete-time setting can provide us with a richer set of results concerning equilibrium determinacy.

2 The Model

2.1 Households

The economy consists of a continuum of identical households with a unit mass. The agent maximizes her lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_{t-J}), \quad 0 < \beta < 1, \quad J = 0, 1 \tag{1}$$

subject to the flow budget constraint such that

$$k_{t+1} - (1-\delta)k_t + c_t + m_t + b_t + \tau_t = y_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}, \quad 0 < \delta < 1. \tag{2}$$

Each variable means the following: β=time discounting rate; δ=capital depreciation rate; c_t =real consumption; m_{t-J}=real money balances at the beginning of period $t - J + 1$; k_t=(per capita) stock of capital; b_t=real stock of bonds at the end of period; τ_t=lump-sum tax; y_t=real income; $\pi_t \equiv P_t/P_{t-1}$=gross rate of inflation; P_t=nominal price level; R_{t-1}=gross nominal interest rate in period $t-1$. In this paper, we specify the utility function as follows:

$$u(c_t, m_{t-J}) = \frac{(c_t^{\rho_1} m_{t-J}^{\rho_2})^{1-\sigma}}{1-\sigma}, \quad \rho_1 + \rho_2 = 1, \quad \sigma > 0,$$

where σ is the inverse of intertemporal elasticity of substitution [4]. This felicity function satisfies $\text{sign}(u_{cm}) = \text{sign}(1-\sigma)$, so that consumption and real money

[4] This instantaneous utility function satisfies $u_c > 0$, $u_m > 0$, $u_{cc} < 0$, $u_{mm} < 0$, $u_{cc}u_m - u_{cm}u_c < 0$, and $u_{mm}u_c - u_{cm}u_m < 0$. That is, the utility function is strictly increasing and strictly concave in c and m, and consumption c and real money balances m are both normal goods.

balances are Edgeworth complements if $0 < \sigma < 1$, while they are Edgeworth substitutes if $\sigma > 1$. We define $J = 1$ as cash-in-advance (CIA) timing, and $J = 0$ as cash-when-I'm-done (CWID) timing.

We assume that the production function of the representative firm is given by a simple AK technology, $y_t = Ak_t$. Thus the competitive rate of return to capital is fixed at A.

To derive the optimality conditions for the household's consumption plan, set up the following Lagrangian function:

$$\mathcal{L} \equiv \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, m_{t-J}) + \lambda_t \left[-k_{t+1} + (1-\delta)k_t - c_t - m_t - b_t - \tau_t + Ak_t + \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t} \right] \right\}.$$

The first-order conditions for the household's optimization problem are:

$$\lambda_t = u_c(c_t, m_{t-J}) = (c_t^{\rho_1} m_{t-J}^{\rho_2})^{(1-\sigma)} \frac{\rho_1}{c_t}; \tag{3}$$

$$u_m(c_t, m_t) = (c_t^{\rho_1} m_t^{\rho_2})^{(1-\sigma)} \frac{\rho_2}{m_t} = \lambda_t - \frac{\beta\lambda_{t+1}}{\pi_{t+1}} \text{ when } J = 0; \tag{4}$$

$$u_m(c_{t+1}, m_t) = (c_{t+1}^{\rho_1} m_t^{\rho_2})^{(1-\sigma)} \frac{\rho_2}{m_t} = \frac{\lambda_t}{\beta} - \frac{\lambda_{t+1}}{\pi_{t+1}} \text{ when } J = 1; \tag{5}$$

$$\lambda_{t-1} = \beta\lambda_t(A + 1 - \delta); \tag{6}$$

$$\lambda_t = \frac{\beta\lambda_{t+1}R_t}{\pi_{t+1}}; \tag{7}$$

$$\lim_{t\to\infty} \beta^{t+1}\lambda_{t+1}k_{t+1} = 0; \tag{8}$$

$$\lim_{t\to\infty} \beta^t\lambda_t m_t = 0; \tag{9}$$

$$\lim_{t\to\infty} \beta^t\lambda_t b_t = 0. \tag{10}$$

Equations (8), (9) and (10) are the transversality conditions.

From (6) and (7), we obtain the following Fisher equation:

$$\frac{R_t}{\pi_{t+1}} = A + 1 - \delta. \tag{11}$$

This represents the non-arbitrage condition, under which the real interest rate of bond is equal to the net real rate of return on capital. From (3), (4), (5), (7) and

(11), we obtain

$$\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{\rho_2}{\rho_1}\frac{c_t}{m_t} = \frac{1}{A+1-\delta}\frac{R_t - 1}{\pi_{t+1}} \quad \text{when } J = 0, \tag{12}$$

$$\frac{u_m(c_{t+1}, m_t)}{u_c(c_{t+1}, m_t)} = \frac{\rho_2}{\rho_1}\frac{c_{t+1}}{m_t} = \frac{R_t - 1}{\pi_{t+1}} \quad \text{when } J = 1. \tag{13}$$

Equations (12) and (13) show that the marginal rate of substitution between consumption and real money holdings is equal to the opportunity cost of holding money.

2.2 Capital Formation

The government budget constraint is

$$m_t + b_t + \tau_t = \frac{m_{t-1}}{\pi_t} + \frac{R_{t-1}b_{t-1}}{\pi_t}. \tag{14}$$

From (2), (14), and the production function $y_t = Ak_t$, we obtain the goods-market equilibrium condition:

$$k_{t+1} = Ak_t + (1-\delta)k_t - c_t. \tag{15}$$

Denoting $z_t \equiv \dfrac{c_t}{k_t}$, we can rewrite the condition (15) as

$$\frac{k_{t+1}}{k_t} = A + 1 - \delta - z_t. \tag{16}$$

2.3 Policy Rules

We consider the Taylor-type monetary policy rule under which the central bank controls the nominal interest rate in response to the growth rate of income as well as to the rate of either current or expected inflation. Formally, we assume that

$$R_t = R(\pi_{t+i}, g_{t+i}), \quad \frac{\partial R_t}{\partial \pi_{t+i}} \geq 0, \quad \frac{\partial R_t}{\partial g_{t+i}} \geq 0, \quad i = 0 \text{ or } 1, \tag{17}$$

where $g_{t+i} \equiv \dfrac{y_{t+1+i}}{y_{t+i}} = \dfrac{k_{t+1+i}}{k_{t+i}} = A+1-\delta-z_{t+i}$ is the gross rate of real income growth. If $i = 0$ (resp. $i = 1$), the interest rate rule is said to be current-looking (resp.

forward-looking), in which monetary authority uses the current (resp. expected) values of economic variables as indices to stabilize economy [5].

For analytical simplicity, we specify (17) as

$$R_t = \pi^* \left(\frac{\pi_{t+i}}{\pi^*}\right)^{\phi} (A + 1 - \delta) \left(\frac{g_{t+i}}{g^*}\right)^{\eta}, \quad \phi \geq 0, \quad \phi \neq 1, \quad \eta \geq 0. \tag{18}$$

In the above, x^* is the steady-state value of a variable x_t, and π^* is the target rate of inflation. If $\phi > 1$, the nominal interest rate rises more than one for one in response to a change in the rate of inflation. Then, the interest control rule is said to be active as to inflation. Conversely, the rule (18) with $\phi < 1$ is defined as passive monetary policy. From (11) and (18),

$$\phi \frac{d\pi_{t+i}}{\pi_{t+i}} + \eta \frac{dg_{t+i}}{g_{t+i}} = \frac{dR_t}{R_t} = \frac{d\pi_{t+1}}{\pi_{t+1}}. \tag{19}$$

If $i = 1$, (19) becomes $\frac{dR_t}{R_t} = \frac{d\pi_{t+1}}{\pi_{t+1}} = -\frac{\eta}{\phi - 1} \frac{dg_{t+1}}{g_{t+1}}$. When the expected growth rate of income increases, the central bank should raise the nominal interest rate to stabilize economy. However, since the net real rate of return to capital is constant due to the assumption of AK technology, the real interest rate also should be kept constant by controlling the rate of inflation to satisfy non-arbitrage condition. Formally, $(\phi - 1)\frac{d\pi_{t+1}}{\pi_{t+1}} < 0$, that is, an active (resp. a passive) policy lowers (resp. raises) rate of inflation. If $i = 0$, (19) is rewritten as

$$\frac{d\pi_{t+1}}{\pi_{t+1}} - \frac{d\pi_t}{\pi_t} = (\phi - 1)\frac{d\pi_t}{\pi_t} + \eta \frac{dg_t}{g_t}.$$

For a positive value of η, the growth rate of inflation may be higher when the rates of inflation and income growth rise, even if the interest control rule is passive.

From (11), (18) and $g_t = 1 + A - \delta - z_t$, the equilibrium rate of inflation is

$$\pi_{t+1} = \pi_F(z_{t+1}) = \pi^* \left(\frac{1 + A - \delta - z_{t+1}}{1 + A - \delta - z^*}\right)^{-\frac{\eta}{\phi - 1}} \text{ for } i = 1,$$

$$\pi_{t+1} = \pi_C(\pi_t, z_t) = (\pi^*)^{-(\phi-1)} (\pi_t)^{\phi} \left(\frac{1 + A - \delta - z_t}{1 + A - \delta - z^*}\right)^{\eta} \text{ for } i = 0,$$

[5]Since we deal with a growing economy in which real income continues expanding, our formulation of interest-rate control rule is a natural extension of Taylor's (1993) original proposal.

where

$$\mathrm{sign}[{\pi_F}'(z_{t+1})] = \mathrm{sign}(\phi - 1),$$
$$\frac{\partial \pi_C(\pi_t, z_t)}{\partial \pi_t} > 0, \frac{\partial \pi_C(\pi_t, z_t)}{\partial z_t} < 0, \text{and sign}\left[\frac{\partial \pi_C(\pi_t, z_t)/\pi_t}{\partial \pi_t}\right] = \mathrm{sign}(\phi - 1).$$

Using these functions, we obtain the following:

$$\frac{R_t - 1}{\pi_{t+1}} = A + 1 - \delta - \frac{1}{\pi_F(z_{t+1})} = o_F(z_{t+1}) \quad \text{for } i = 1, \tag{20}$$
$$\frac{R_t - 1}{\pi_{t+1}} = A + 1 - \delta - \frac{1}{\pi_C(\pi_t, z_t)} = o_C(\pi_t, z_t) \quad \text{for } i = 0, \tag{21}$$

where

$${o_F}'(z_{t+1}) = \frac{{\pi_F}'(z_{t+1})}{[\pi_F(z_{t+1})]^2} : \mathrm{sign}[{o_F}'(z_{t+1})] = \mathrm{sign}(\phi - 1),$$
$$\frac{\partial o_C(\pi_t, z_t)}{\partial \pi_t} = \frac{\partial \pi_C(\pi_t, z_t)}{\partial \pi_t} \frac{1}{[\pi_C(\pi_t, z_t)]^2} > 0,$$
$$\frac{\partial o_C(\pi_t, z_t)}{\partial z_t} = \frac{\partial \pi_C(\pi_t, z_t)}{\partial z_t} \frac{1}{[\pi_C(\pi_t, z_t)]^2} < 0.$$

Hence, the opportunity cost of holding money is positively related to the equilibrium rate of inflation.

3 Forward-looking Rule

3.1 The CWID timing

When we assume CWID timing of money holding and forward-looking monetary policy rule, a complete dynamic equation is given by the following [6]:

$$z_{t+1} = [\theta^* \theta^{FW}(z_{t+2}, z_{t+1}) - A + \delta + z_t] z_t, \tag{22}$$

where $\theta^* \equiv \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} = 1+A-\delta-z^*$ and $\theta^{FW}(z_{t+2}, z_{t+1}) = \left(\frac{o_F(z_{t+2})}{o_F(z_{t+1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$.
In the following, we focus on the balanced-growth path with a positive growth rate. Linearizing (22) at the steady state where $\theta^* > 1$ and $z^* > 0$, we obtain [7]

$$\hat{z}_{t+2} = \left(1 - \frac{1}{z^* \theta^* \bar{\theta}_z^{FW}}\right) \hat{z}_{t+1} + \frac{1 + z^*}{z^* \theta^* \bar{\theta}_z^{FW}} \hat{z}_t, \tag{23}$$

[6] A derivation of the dynamics of z_t in each case is shown in Appendix 1.

[7] We assume that $0 < z^* < A - \delta$.

Table 1: The Property of Equilibrium Path under Two Variables

	$p(-1) > 0$	$p(-1) < 0$
$p(1) > 0$	1)Unstable: $p(0) > 1$ 2)Stable: $p(0) < 1$	Saddle
$p(1) < 0$	Saddle	Unstable

where $\hat{z}_t \equiv z_t - z^*$ and

$$\bar{\theta}_z^{FW} \equiv \left.\frac{\partial \theta^{FW}}{\partial z_{t+1}}\right|_{ss} = -\left.\frac{\partial \theta^{FW}}{\partial z_{t+2}}\right|_{ss} = \frac{1-\sigma}{\sigma} \frac{\rho_2 \eta}{[\pi^*(1+A-\delta)-1](\phi-1)\theta^*}.$$

Equation (23) is derived from $\hat{z}_{t+1} = z^*\theta^*\bar{\theta}_z^{FW}(\hat{z}_{t+1} - \hat{z}_{t+2}) + (1+z^*)\hat{z}_t$. If $\bar{\theta}_z^{FW} = 0$, that is, if $\frac{(1-\sigma)\rho_2\eta}{\phi - 1} = 0$, this becomes $\hat{z}_{t+1} = (1+z^*)\hat{z}_t$, which implies that there is a unique equilibrium path.

When $\bar{\theta}_z^{FW} \neq 0$, the corresponding characteristic equation is

$$p^{FW}(\mu) = \mu^2 - \left(1 - \frac{1}{z^*\theta^*\bar{\theta}_z^{FW}}\right)\mu - \frac{1+z^*}{z^*\theta^*\bar{\theta}_z^{FW}} = 0. \tag{24}$$

The properties of equilibrium path are summarized in Table 1. In this dynamic system, there are two jump variables, z_{t+1} and z_t. Thus equilibrium determinacy holds if the two roots of (24) are out of the unit circle. The critical equations for checking the characteristic roots are the following:

$$p^{FW}(1) = -\frac{1}{\theta^*\bar{\theta}_z^{FW}} = -\frac{\sigma}{1-\sigma}\frac{[\pi^*(1+A-\delta)-1](\phi-1)}{\rho_2\eta},$$

$$p^{FW}(-1) = 2 - \frac{z^*+2}{z^*}\frac{1}{\theta^*\bar{\theta}_z^{FW}} = 2 - \frac{z^*+2}{z^*}\frac{\sigma}{1-\sigma}\frac{[\pi^*(1+A-\delta)-1](\phi-1)}{\rho_2\eta},$$

$$p^{FW}(0) = -\frac{1+z^*}{z^*}\frac{1}{\theta^*\bar{\theta}_z^{FW}} = -\frac{1+z^*}{z^*}\frac{\sigma}{1-\sigma}\frac{[\pi^*(1+A-\delta)-1](\phi-1)}{\rho_2\eta}.$$

Let us focus on the case of $0 < \sigma < 1$. We can discuss the case of $\sigma > 1$ in a similar manner. When $\phi > 1$, $p^{FW}(1) < 0$ is satisfied. Note that if η is high $p^{FW}(-1) > 0$ so that the steady state is a saddle point, otherwise the steady state equilibrium becomes unstable. If $\phi < 1$, then $p^{FW}(1) > 0$, $p^{FW}(-1) > 0$, and

$p^{FW}(0) > 1$ with low η. The following proposition and Figure 1 summarize our result [8].

Proposition 1 *Consider the economy with the CWID timing under the forward-looking interest rate rule. Then, regardless of the sign of* $(1-\sigma)$*, equilibrium indeterminacy tends to hold if* $\dfrac{\eta}{|\phi - 1|}$ *is high.*

3.2 The CIA timing

In this case, we can derive a complete dynamic system as a single equation such that

$$z_{t+1} = [\theta^* \theta^{FI}(z_{t+1}, z_t) - A + \delta + z_t] z_t, \tag{25}$$

where $\theta^{FI}(z_{t+1}, z_t) = \left(\dfrac{o_F(z_{t+1})}{o_F(z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$. Linearizing the system around the steady state, we obtain

$$\hat{z}_{t+1} = \left(1 + \frac{z^*}{1 + z^* \theta^* \bar{\theta}_z^{FW}}\right) \hat{z}_t. \tag{26}$$

To derive (26), we use $\hat{z}_{t+1} = (1+z^*)\hat{z}_t - z^* \theta^* \bar{\theta}_z^{FW} (\hat{z}_{t+1} - \hat{z}_t)$.

Since z_t is a jump variable, the condition for indeterminacy is $\left(1 + \dfrac{z^*}{1 + z^* \theta^* \bar{\theta}_z^{FW}}\right)^2 < 1$, that is,

$$\frac{z^*(z^* + 2 + 2z^* \theta^* \bar{\theta}_z^{FW})}{(1 + z^* \theta^* \bar{\theta}_z^{FW})^2} < 0.$$

Since $z^* > 0$, the condition can be rewritten such that $z^* + 2 + 2z^* \theta^* \bar{\theta}_z^{FW} < 0$. This can be satisfied when $\bar{\theta}_z^{FW} < 0$. We summarize the result in the following proposition and Figure 2.

Proposition 2 *In the economy with the CWID timing under the forward-looking interest rate rule, equilibrium path is determinate if* $\dfrac{(1-\sigma)\rho_2 \eta}{\phi - 1} \geq 0$*. Otherwise, equilibrium indeterminacy may emerge* [9]*.*

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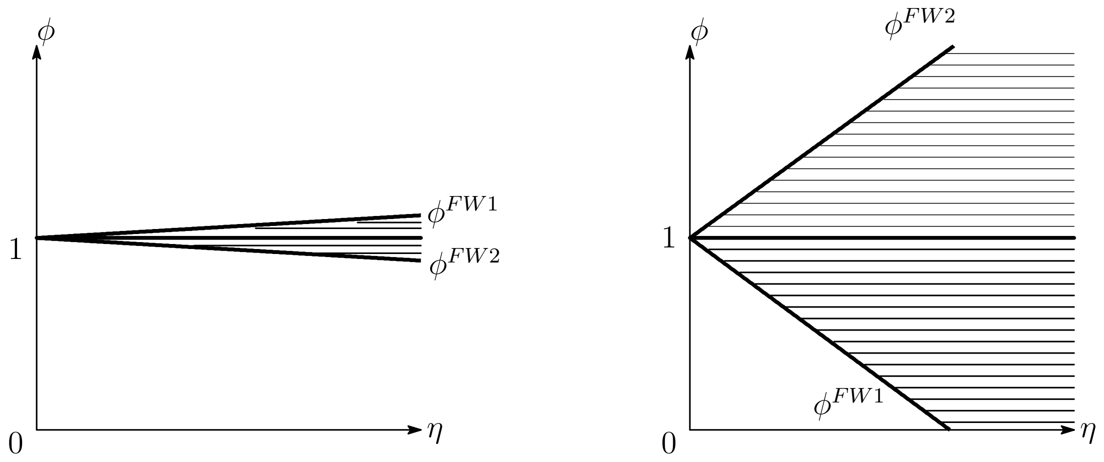
[8] Section 5 and Appendix 2 show how to draw Figures 1 to 4.

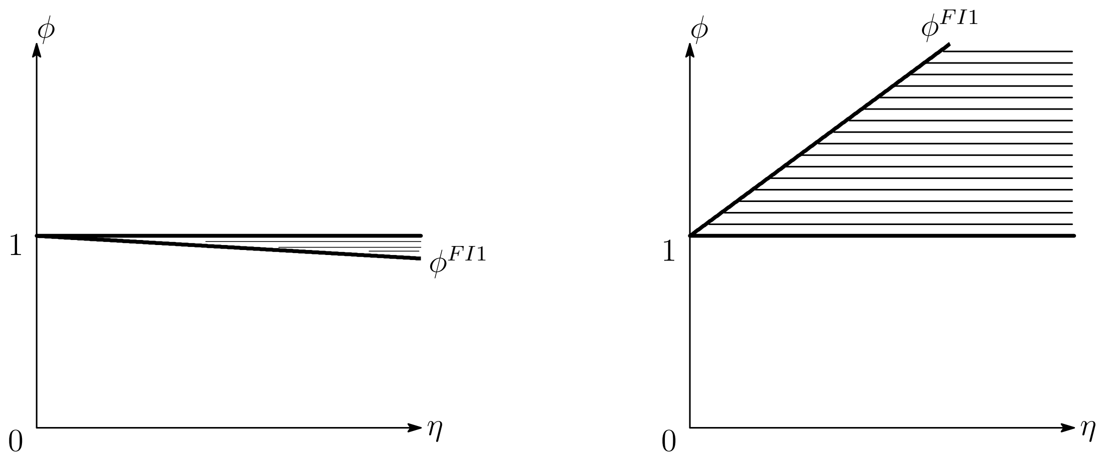
[9] This result is close to the finding in Fujisaki and Mino (2007) who use a continuous-time formulation.



(1a)$0 < \sigma < 1$ (1b)$1 < \sigma$

Figure 1. The CWID timing with forward-looking rule

determinacy indeterminacy



(2a)$0 < \sigma < 1$ (2b)$1 < \sigma$

Figure 2. The CIA timing with forward-looking rule

determinacy indeterminacy

4 Current-looking Rule

4.1 The CWID timing

A complete dynamic system in this case consists of the following difference equations:

$$\pi_{t+1} = (\pi^*)^{-(\phi-1)}(\pi_t)^{\phi}\left(\frac{1+A-\delta-z_t}{1+A-\delta-z^*}\right)^{\eta}, \tag{27}$$

$$z_{t+1} = [\theta^*\theta^{CW}(\pi_t, z_{t+1}, z_t) - A + \delta + z_t]z_t, \tag{28}$$

where $\theta^{CW}(\pi_t, z_{t+1}, z_t) = \left(\dfrac{o_C(\pi_{t+1}, z_{t+1})}{o_C(\pi_t, z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$, because $\pi_{t+1} = \pi^C(\pi_t, z_t)$. The system linearized at the steady state is

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \phi & -\dfrac{\eta\pi^*}{\theta^*} \\ X_\pi & X_z \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{z}_t \end{bmatrix}, \tag{29}$$

where

$$X_\pi = -\frac{\phi}{\eta}\frac{z^*}{\pi^*}\frac{(\theta^*)^2\bar{\theta}_z^{CW}(\phi-1)}{(\phi-1)-z^*\theta^*\bar{\theta}_z^{CW}} \text{ and } X_z = \frac{(1+z^*+z^*\theta^*\bar{\theta}_z^{CW})(\phi-1)}{(\phi-1)-z^*\theta^*\bar{\theta}_z^{CW}}.$$

The linearized dynamic equation of z_t in (29) is derived from

$$\left(1-\frac{z^*}{\phi-1}\theta^*\bar{\theta}_z^{CW}\right)\hat{z}_{t+1} = (1+z^*+z^*\theta^*\bar{\theta}_z^{CW})\hat{z}_t - \frac{\phi}{\eta}\frac{z^*}{\pi^*}(\theta^*)^2\bar{\theta}_z^{CW}\hat{\pi}_t,$$

where

$$\begin{aligned}\bar{\theta}_z^{CW} \equiv \left.\frac{\partial\theta^{CW}}{\partial z_t}\right|_{ss} &= \frac{1-\sigma}{\sigma}\frac{\eta(\phi-1)}{(\pi^*(1+A-\delta)-1)\theta^*} \\ &= (\phi-1)\left.\frac{\partial\theta^{CW}}{\partial z_{t+1}}\right|_{ss} = -\frac{\rho_2\eta}{\phi}\frac{\pi^*}{\theta^*}\left.\frac{\partial\theta^{CW}}{\partial\pi_t}\right|_{ss} \\ &= (\phi-1)^2\bar{\theta}_z^{FW}.\end{aligned}$$

The characteristic equation is

$$p^{CW}(\mu) = \mu^2 - (\phi+X_z)\mu + \phi X_z + \frac{\eta\pi^*}{\theta^*}X_\pi = 0. \tag{30}$$

There are two jump variables, π_t and z_t, in this system so that equilibrium determinacy emerges when two roots of (30) are out of the unit circle. From (30), we find the following:

$$p^{CW}(1) = \frac{(\phi-1)z^*}{Q(\eta;\pi^*,\sigma)},$$

$$p^{CW}(-1) = \frac{\phi(2+z^*) + 1 + z^* + Q(\eta;\pi^*,\sigma)}{Q(\eta;\pi^*,\sigma)},$$
$$p^{CW}(0) = \frac{\phi(1+z^*)}{Q(\eta;\pi^*,\sigma)},$$
where $Q(\eta;\pi^*,\sigma) \equiv 1 - z^*\theta^*(\phi-1)\bar{\theta}_z^{FW} = 1 - \dfrac{1-\sigma}{\sigma}\dfrac{\rho_2\eta z^*}{\pi^*(A+1-\delta)-1}$.

We examine the properties of $p^{CW}(\mu)$ when monetary policy is active ($\phi > 1$). Consider the case of $0 < \sigma < 1$. If η is low enough that $Q(\cdot) > 0$, the equilibrium path is determinate, since $p^{CW}(1) > 0$, $p^{CW}(-1) > 0$ and $p^{CW}(0) > 1$. When $Q(\cdot) < 0$ is satisfied, $p^{CW}(1) < 0$ and $p^{CW}(0) < 0$. Therefore, equilibrium path is unstable or saddle. Under $\sigma \geq 1$, $Q(\cdot) > 0$ always holds. Then, $p^{CW}(1) > 0$ and $p^{CW}(-1) > 0$ if $\phi > 1$. Determinacy holds if $p^{CW}(0) > 1$, which requires that ϕ is large enough relative to η.

We can discuss the case of passive policy rule ($\phi < 1$) in the same way. These results are summarized in the propositions below and in Figure 3.

Proposition 3 *Suppose that money holding satisfies the CWID timing and that the interest-rate control is active ($\phi > 1$) and current-looking. Then, the equilibrium path is determinate either if η is small or if $(1-\sigma)\rho_2\eta = 0$. If η is sufficiently large, indeterminacy may emerge.*

Proposition 4 *In the case of the CWID timing and the passive current-looking monetary policy rule ($\phi < 1$), equilibrium indeterminacy is generated.*

4.2 The CIA timing

Since $o_C(\pi_{t-1}, z_{t-1}) = A + 1 - \delta - \dfrac{1}{\pi_t}$, a complete dynamic system in this case consists of (27) and

$$z_{t+1} = [\theta^*\theta^{CI}(\pi_t, z_t) - A + \delta + z_t]z_t, \tag{31}$$

where $\theta^*\theta^{CI}(\pi_t, z_t) = \left(\dfrac{o_C(\pi_t, z_t)}{o_C(\pi_{t-1}, z_{t-1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2}$. Linearizing (27) and (31) around the steady state yields

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \phi & -\dfrac{\eta\pi^*}{\theta^*} \\ -\dfrac{\phi-1}{\eta\pi^*}(\theta^*)^2\bar{\theta}_z^{CI}z^* & 1+z^*+z^*\theta^*\bar{\theta}_z^{CI} \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{z}_t \end{bmatrix}, \tag{32}$$

where

$$\begin{aligned} \bar{\theta}_z^{CI} \equiv \left.\frac{\partial\theta^{CI}}{\partial z_t}\right|_{ss} &= \frac{1-\sigma}{\sigma}\rho_2\frac{\eta}{(\pi^*(1+A-\delta)-1)\theta^*} \\ &= -\frac{\eta}{\phi-1}\frac{\pi^*}{\theta^*}\left.\frac{\partial\theta^{CI}}{\partial\pi_t}\right|_{ss} = \frac{\bar{\theta}_z^{CW}}{\phi-1} \\ &= (\phi-1)\bar{\theta}_z^{FW}. \end{aligned}$$

The characteristic equation is

$$p^{CI}(\mu) = \mu^2 - (\phi+1+z^*+z^*\theta^*\bar{\theta}_z^{CI})\mu + \phi(1+z^*) + z^*\theta^*\bar{\theta}_z^{CI} = 0. \tag{33}$$

There are two jump variables π_t and z_t in this system so local equilibrium determinacy requires that the steady state equilibrium is a source. From (33),

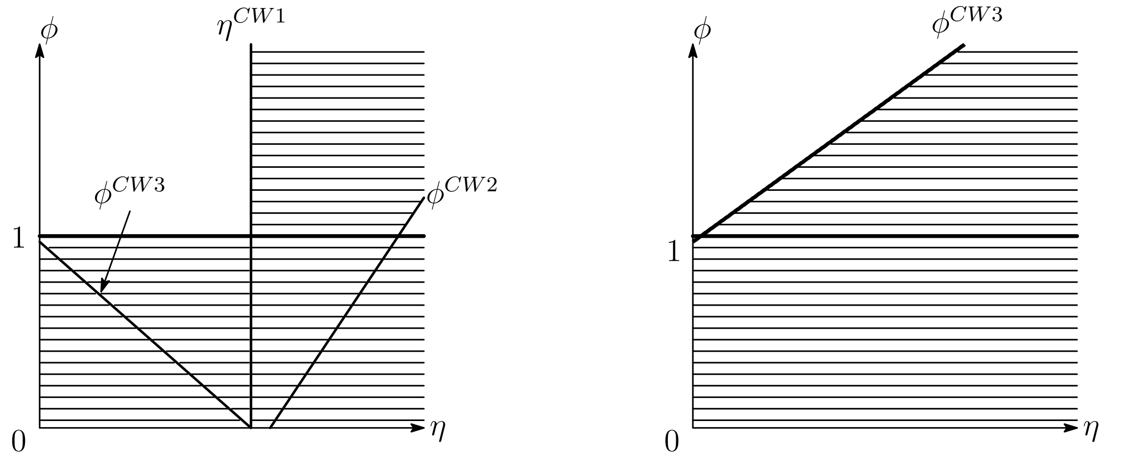
$$p^{CI}(1) = z^*(\phi-1) \gtrless 0 \text{ if } \phi \gtrless 1,$$

$$\begin{aligned} p^{CI}(-1) &= (2+z^*)(\phi+1) + 2z^*\theta^*(\phi-1)\bar{\theta}_z^{FW} \\ &= (2+z^*)(\phi+1) + 2\frac{1-\sigma}{\sigma}\frac{\rho_2\eta z^*}{\pi^*(A+1-\delta)-1}, \end{aligned}$$

$$p^{CI}(0) = \phi(1+z^*) + z^*\theta^*(\phi-1)\bar{\theta}_z^{FW} = \phi(1+z^*) + \frac{1-\sigma}{\sigma}\frac{\rho_2\eta z^*}{\pi^*(A+1-\delta)-1}.$$

We consider the case of $\phi < 1$ in which the result seems to be interesting. When $0 < \sigma \leq 1$, $p^{CI}(-1) > 0$ is satisfied, and therefore equilibrium indeterminacy holds. If $\sigma > 1$, equilibrium determinacy is generated if η is large enough to satisfy $p^{CI}(-1) < 0$.

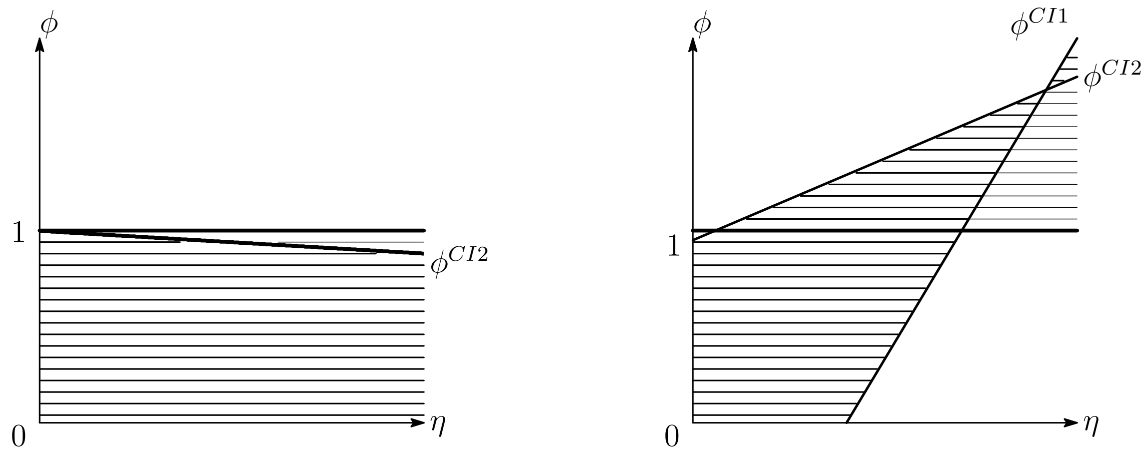
The main results obtained in this system are summarized as the following propositions and Figure 4.



(3a) $0 < \sigma < 1$ (3b) $1 < \sigma$

Figure 3. The CWID timing with current-looking rule

determinacy indeterminacy



(4a) $0 < \sigma < 1$ (4b) $1 < \sigma$

Figure 4. The CIA timing with current-looking rule

determinacy indeterminacy

Proposition 5 *Assume that the money holdings satisfies the CIA timing and that the interest-rate control is active current-looking ($\phi > 1$). Then equilibrium determinacy holds if $(1 - \sigma)\rho_2\eta \geq 0$. Otherwise, the equilibrium path can be indeterminate when η is large.*

Proposition 6 *Assume that the economy with the CIA timing under the passive current-looking interest-rate control rule ($\phi < 1$). If $(1-\sigma)\rho_2\eta \geq 0$, balanced growth path is a saddlepoint so that indeterminacy emerges. Otherwise, the equilibrium path is determinate when η is large.*

5 A Numerical Example

As shown above, whether the equilibrium path is determinate or not critically depends on the magnitudes of σ, η, and ϕ. To check the plausibility of our analytical results, we examine a numerical example. Let us set:

$$A = 0.08, \beta = 0.98, \delta = 0.04, \pi^* = 1.02, \rho_1 = 0.7, \rho_2 = 0.3.$$

In addition, in order to draw the figures in (η, ϕ)-plane, we set $\sigma = 0.5$ or $\sigma = 2$, which respectively implies $(\theta^*, z^*) = (1.0388, 0.0012)$ or $(\theta^*, z^*) = (1.0096, 0.0304)$ From the critical equations for equilibrium determinacy in Sections 3 and 4, we derive loci and substitute the numerical example into these loci as shown in Appendix 2.

According to Taylor (1993), $\phi = 1.5$ and $\eta = 0.5$ are empirically plausible values. Since ϕ and η are respectively the elasticity of nominal interest rate to the inflation rate and to the rate of income growth, these values cannot be extremely high. We calculate percentage of the area in which equilibrium determinacy holds within the range $0 < \phi \leq 3$ and $0 \leq \eta \leq 3$. We denote the value of this range by $L(\sigma)$. For example, $L_F(1) = 100$ and $L_C(1) = \dfrac{2 * 3}{9} * 100 = 66.67$ in the forward-looking and current-looking rules respectively regardless of the timing of real money balances in the MIUF. These values are the criterions to compare the numerical results. The

Table 2: The Values of $L(\sigma)$

	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 2$
$L_{FW}(\sigma)$	99.41	100.00	92.67
$L_{FI}(\sigma)$	99.71	100.00	96.31
$L_{CW}(\sigma)$	66.67	66.67	63.95
$L_{CI}(\sigma)$	66.67	66.67	63.95

Table 3: Equilibrium Determinacy (1)

	$\bar{\theta}_z^{FW} < 0$	$\bar{\theta}_z^{FW} = 0$	$\bar{\theta}_z^{FW} > 0$
CWID, FL	D, I	D	D, I
CIA, FL	D, I	D	D
CWID, CL ($\phi > 1$)	D, I	D	D, I
CWID, CL ($\phi < 1$)	I	I	I
CIA, CL ($\phi > 1$)	D, I	D	D
CIA, CL ($\phi < 1$)	I	I	D, I

FL=forward-looking rule, CL=current-looking rule

D=determinate, I=indeterminate

values of $L(\sigma)$ are calculated in Appendix 2 and summarized in Table 2, which show that the positive response of the interest rate to the income growth and the timing of money holdings have small impacts on equilibrium determinacy.

6 Discussion

Tables 3 and 4 summarize the results shown in Sections 3 and 4. In this section, we discuss the intuitive implication of our findings.

Table 4: Equilibrium Determinacy (2)

CWID, FL	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D, I	D	D, I
$\phi < 1$	D, I	D	D, I

FL with $\eta = 0$	CWID	CIA
$\phi > 1$	D	D
$\phi < 1$	D	D

CIA, FL	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D	D	D, I
$\phi < 1$	D, I	D	D

CL with $\eta = 0$	CWID	CIA
$\phi > 1$	D	D
$\phi < 1$	I	I

CWID, CL	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D, I	D	D, I
$\phi < 1$	I	I	I

$\phi = \eta = 0$	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
CWID	D	D	D
CIA	D	D	D

CIA, CL	$\sigma < 1$	$\sigma = 1$	$1 < \sigma$
$\phi > 1$	D	D	D, I
$\phi < 1$	I	I	I, D

FL=forward-looking rule, CL=current-looking rule

D=determinate, I=indeterminate

6.1 $\bar{\theta}_z^{FW} = 0$

If $(1-\sigma)\rho_2\eta = 0$ is satisfied, $\bar{\theta}_z^{FW}$ becomes zero. Then, we obtain the standard results in the AK growth model with the Taylor rule under which the nominal interest rate responds to inflation alone. Namely, both the forward-looking and active current-looking rules generate equilibrium determinacy, while indeterminacy holds under the passive current-looking rule, regardless of the timings of agent's money holdings. Two factors neutralizing the effect of the opportunity cost of holding money eliminate the efficacy of the generalized Taylor rule. The first is

$\rho_2 = 0$, which means no need for money. Secondly, when the utility is additively separable ($\sigma = 1$), the optimal consumption is independent from the demand for real money holdings.

Moreover, let us consider the case in which the nominal interest rate is pegged ($\phi = \eta = 0$). From the non-arbitrage condition (11), the rate of inflation is also fixed in the case of AK technology. Therefore, the dynamics of z_t is the same as in the standard AK model and, hence, equilibrium determinacy around the balanced growth path always holds.

6.2 Intuitive Implication

From Table 3, we obtain two significant messages. First, under the forward-looking and active current-looking interest control rules, $\bar{\theta}_z^{FW} > 0$ is a sufficient condition for equilibrium determinacy in the case of CIA timing, while it is not in the case of CWID timing. Second, if the passive current-looking rule is adopted, determinacy does not hold in the CWID timing and it may emerge in the CIA timing when $\bar{\theta}_z^{FW} > 0$.

For example, we consider the case under which agents have a preference with $0 < \sigma < 1$ and monetary policy rule is active and forward-looking so that $\bar{\theta}_z^{FW} > 0$ is satisfied. Suppose that the economy initially stays in the balanced-growth equilibrium and that a rise of the growth rate of consumption is anticipated. According to this anticipation, each agent increases consumption and thus the ratio of consumption to capital z becomes larger ($z_{t+1} > z_t$). Under the active interest control rule, this means an increase of the growth rate of inflation to satisfy the non-arbitrage condition. As shown in Section 2.3, this effect results from the generalization of the interest control rule.

When the timing of money holdings is CIA, the growth rate of the opportunity cost of holding money becomes higher and that of consumption falls, because consumption and real money balances are complements. It contradicts to the above anticipation, so that determinacy holds. In the CWID timing, this mechanism is not effective, because the timing of the growth rate of the opportunity cost of holding money which affects the rate of consumption growth is different from the case of CIA. Therefore, a larger z can be realized and indeterminacy may be generated. This argument can be also applied in other cases.

6.3 Endogenous vs Exogenous Growth

Meng and Yip (2004) claim that a generalized Taylor rule may not yield indeterminacy in the standard neoclassical growth model. In contrast, we have shown that the generalized Taylor rule has a pivotal effect on economic stability in the AK growth model. To see the reason for the presence of such a difference, we consider a continuous-time model [10]. Substituting the interest-rate control rule $R = \mathcal{R}(\pi, f(k), g)$ into the non-arbitrage condition, $R - \pi = f'(k) - \delta$, and linearizing it around the steady state, we obtain:

$$(\mathcal{R}_1 - 1)\hat{\pi} + \mathcal{R}_2 f'\hat{k} + \mathcal{R}_3\hat{g} = f''\hat{k}. \tag{34}$$

Suppose that $\mathcal{R}_1 > 1$. In an exogenous growth model, it holds that $\mathcal{R}_3 = 0$ and $f'' < 0 < f'$, so that (34) becomes $\hat{\pi} = \dfrac{f'' - \mathcal{R}_2 f'}{\mathcal{R}_1 - 1}\hat{k}$, which satisfies $\dfrac{d\hat{\pi}}{d\hat{k}} < 0$, regardless whether $\mathcal{R}_2$ is zero or positive. When the AK technology is assumed ($\mathcal{R}_2 = 0$ and $f'' = 0 < f'$), (34) is rewritten as $\hat{\pi} = \dfrac{-\mathcal{R}_3}{\mathcal{R}_1 - 1}\hat{g}$. Hence, $\hat{\pi} = 0$ if $\mathcal{R}_3 = 0$ and $\dfrac{d\hat{\pi}}{d\hat{g}} < 0$ if $\mathcal{R}_3 > 0$.

Therefore, when the Taylor rule is generalized, the property of equilibrium rate of inflation is dramatically changed in the AK growth model. Such a difference is the main reason for a stark contrast in equilibrium determinacy conditions between the neoclassical and AK growth models.

[10] Notations are the same, and time index is omitted.

7 Conclusion

By use of a discrete-time AK growth model with money, we have investigated the stabilization effect of a generalized Taylor rule under which the nominal interest rate responds to the growth rate of income as well as to the rate of inflation. The central messages of our study are as follows. First, if the interest-rate control is sensitive to the growth rate of income, monetary policy rule may play a pivotal role for economic stability even in a simple environment in which money is superneutral in the balanced growth equilibrium. Second, our discrete-time modelling clearly demonstrates that the timings of money holding of the households and the time perspective of the monetary authority critically affect the efficacy of interest control rules. This aspect cannot be considered in the foregoing studies on equilibrium determinacy of monetary AK growth models in continuous-time settings.

Appendix 1: Step for Deriving the Euler Equation

In all four cases of Sections 3 and 4, we use the same step for obtaining the reduced dynamic system. First, using (12), (13), (20) and (21), we derive the demand for real money balances in each case. Second, we substitute this money demand function into (3), which gives the Euler equation. The growth rate of consumption in each case consists of two parts: a common balanced growth rate of consumption obtained in the standard AK growth model, $\theta^* \equiv \{\beta(1 + A - \delta)\}^{\frac{1}{\sigma}}$, and the part related to the growth rate of the opportunity cost of holding money.

As an example, we show this step formally in the case of CWID with forward-looking rule (Section 3.1). From (12) and (20),

$$m_t = \frac{\rho_2}{\rho_1}(1 + A - \delta)\frac{c_t}{o_F(z_{t+1})}. \tag{35}$$

Substituting this into (3), we obtain

$$\lambda_t = \rho_1 \left\{ \frac{\rho_2}{\rho_1}(1 + A - \delta) \right\}^{\rho_2(1-\sigma)} \frac{o_F(z_{t+1})^{-\rho_2(1-\sigma)}}{c_t^{\sigma}}. \tag{36}$$

Thus the Euler equation can be expressed as

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_F(z_{t+2})}{o_F(z_{t+1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{FW}(z_{t+2}, z_{t+1}). \tag{37}$$

The Euler equations in Sections 3.2, 4.1, and 4.2 are respectively given by:

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_F(z_{t+1})}{o_F(z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{FI}(z_{t+1}, z_t), \tag{38}$$

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_C(\pi_{t+1}, z_{t+1})}{o_C(\pi_t, z_t)}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{CW}(\pi_t, z_{t+1}, z_t), \tag{39}$$

$$\frac{c_{t+1}}{c_t} = \{\beta(1+A-\delta)\}^{\frac{1}{\sigma}} \left(\frac{o_C(\pi_t, z_t)}{o_C(\pi_{t-1}, z_{t-1})}\right)^{-\frac{1-\sigma}{\sigma}\rho_2} = \theta^* \theta^{CI}(\pi_t, z_t). \tag{40}$$

We use $\pi_{t+1} = \pi^C(\pi_t, z_t)$ in (39) and $o_C(\pi_{t-1}, z_{t-1}) = A + 1 - \delta - \dfrac{1}{\pi_t}$ in (40). Note that the timing of the growth rate of the opportunity cost of holding money is one period ahead in the case of CIA than that of CWID. Using these Euler equations and the capital dynamics (16), we obtain the dynamics of z_t in each case.

Appendix 2: Preparation for Drawing Figures

The CWID timing with forward-looking rule

The loci of $p^{FW}(-1) = 0$ and $p^{FW}(0) = 1$ are respectively given by

$$\phi^{FW1}(\eta; \sigma) = 1 + \frac{1-\sigma}{\sigma} \frac{2z^*}{z^*+2} \frac{\rho_2}{\pi^*(1+A-\delta)-1}\eta. \tag{41}$$

$$\phi^{FW2}(\eta; \sigma) = 1 - \frac{1-\sigma}{\sigma} \frac{z^*}{z^*+1} \frac{\rho_2}{\pi^*(1+A-\delta)-1}\eta. \tag{42}$$

Figure (1a) displays the general loci in the case of $0 < \sigma < 1$. According to a numerical example in Section 5, we obtain

$$\phi^{FW1}(\eta; 0.5) = 0.0059\eta + 1,$$

$$\phi^{FW2}(\eta; 0.5) = -0.0059\eta + 1.$$

Then, we can calculate $L_{FW}(0.5)$ in such a way that

$$L_{FW}(0.5) = \left(9 - 2 * \frac{(0.0059 * 3) * 3}{2}\right) * \frac{1}{9} * 100 = 99.41.$$

When $\sigma > 1$, we can draw the loci as in Figure (1b). Substituting the numerical example into these loci, we obtain

$$\phi^{FW1}(\eta; 2) = -0.0739\eta + 1,$$

$$\phi^{FW2}(\eta; 2) = 0.0728\eta + 1.$$

In this case, $L_{FW}(2)$ is as follows;

$$L_{FW}(2) = \left(9 - \left\{\frac{(0.0739 * 3) * 3}{2} + \frac{(0.0728 * 3) * 3}{2}\right\}\right) * \frac{1}{9} * 100 = 92.67.$$

The CIA timing with forward-looking rule

The locus of $z^* + 2 + 2z^*\theta^*\bar{\theta}_z^{FW} = 0$ which is significant for equilibrium determinacy is

$$\phi^{FI1}(\eta; \sigma) = 1 - \frac{1-\sigma}{\sigma} \frac{2z^*}{z^*+2} \frac{\rho_2}{\pi^*(1+A-\delta)-1}\eta. \tag{43}$$

ϕ^{FI1} is a mirror image of ϕ^{FW1}. We draw this locus in Figure 2.

Using the numerical example, we see that the locus (43) becomes

$$\phi^{FI1}(\eta; 0.5) = -0.0059\eta + 1, \text{ or}$$

$$\phi^{FI1}(\eta; 2) = 0.0739\eta + 1.$$

We can calculate $L_{FI}(\sigma)$ in the following manner:

$$L_{FI}(0.5) = 100 - \frac{(0.0059 * 3) * 3}{2} * \frac{1}{9} * 100 = 99.705, \text{ or}$$

$$L_{FI}(2) = 100 - \frac{(0.0739 * 3) * 3}{2} * \frac{1}{9} * 100 = 96.305.$$

The CWID timing with current-looking rule

The loci of $Q(\cdot) = 0$, $p^{CW}(-1) = 0$, and $p^{CW}(0) = 1$ are respectively given by

$$\eta^{CW1}(\sigma) = \frac{\sigma}{1-\sigma} \frac{\pi^*(A+1-\delta)-1}{\rho_2 z^*}, \tag{44}$$

$$\phi^{CW2}(\eta;\sigma) = \frac{1-\sigma}{\sigma} \frac{2z^*}{2+z^*} \frac{\rho_2}{\pi^*(A+1-\delta)-1}\eta - 1, \tag{45}$$

$$\phi^{CW3}(\eta;\sigma) = \frac{1}{1+z^*}\left[1 - \frac{1-\sigma}{\sigma} \frac{\rho_2 z^*}{\pi^*(A+1-\delta)-1}\eta\right]. \tag{46}$$

In the case of $0 < \sigma < 1$. these loci are generally shown in Figure (3a). (44)-(46) with the numerical example are the following:

$$\eta^{CW1}(0.5) = 168.888,$$

$$\phi^{CW2}(\eta; 0.5) = 0.0059\eta - 1,$$

$$\phi^{CW3}(\eta; 0.5) = -0.0059\eta + 0.9988.$$

$\eta^{CW1}(0.5)$ and $\phi^{CW2}(\eta; 0.5)$ do not appear in the area $0 \leq \eta \leq 3$. This means that $L_{CW}(0.5) = 66.67$.

Figure (3b) displays the loci when $\sigma > 1$. Substituting the numerical example into (44)-(46), we obtain

$$\eta^{CW1}(2) = -13.3351,$$

$$\phi^{CW2}(\eta; 2) = -0.0739\eta - 1,$$

$$\phi^{CW3}(\eta; 2) = 0.0728\eta + 0.9704.$$

Within the area $0 \leq \eta \leq 3$, $\eta^{CW1}(2)$ and $\phi^{CW2}(\eta; 2)$ are not seen. Then, $L_{CW}(2)$ is calculated as

$$L_{CW}(2) = 100 - 100 * \frac{1}{9} * \left\{ 1 * 0.4066 + \frac{(1 + 0.0728 * 3 + 0.9704) * 2.5934}{2} \right\} = 63.95.$$

The CIA timing with current-looking rule

We obtain the loci of $p^{CI}(-1) = 0$ and $p^{CI}(0) = 1$:

$$\phi^{CI1}(\eta; \sigma) = -\frac{1-\sigma}{\sigma}\frac{2z^*}{2+z^*}\frac{\rho_2}{\pi^*(A+1-\delta)-1}\eta - 1, \tag{47}$$

$$\phi^{CI2}(\eta; \sigma) = \frac{1}{1+z^*}\left[1 - \frac{1-\sigma}{\sigma}\frac{\rho_2 z^*}{\pi^*(A+1-\delta)-1}\eta\right]. \tag{48}$$

(47) is a mirror image of (45), and (48) is the same equation as (46).

These loci in the case of $0 < \sigma < 1$ are generally drawn in Figure (4a). Substituting the numerical example into these loci, we find

$$\phi^{CI1}(\eta; 0.5) = -0.0059\eta - 1,$$

$$\phi^{CI2}(\eta; 0.5) = -0.0059\eta + 0.9988.$$

$\phi^{CI1}(\eta; 0.5)$ does not exist in the area $0 \leq \eta \leq 3$. This means that $L_{CI}(0.5) = 66.67$.

If $\sigma > 1$, Figure (4b) generally represents these loci. Using the numerical example, we obtain

$$\phi^{CI1}(\eta; 2) = 0.0739\eta - 1,$$

$$\phi^{CI2}(\eta; 2) = 0.0728\eta + 0.9704.$$

$\phi^{CI1}(2)$ does not appear in the area $0 \leq \eta \leq 3$. Then, $L_{CI}(2)$ is calculated in the same way as $L_{CW}(2)$ so that $L_{CI}(2) = 63.95$. In sum, $L_{CI}(\sigma) = L_{CW}(\sigma)$ when η takes plausible values.

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