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Abstract

Using a directed search model, modified from random matching, this paper investigates how trading frictions in asset markets affect portfolio choices, asset prices, and welfare. By solving the model numerically, it is demonstrated that the asset price increases (decreases) in the matching efficiency, if the relative risk aversion is smaller (larger) than unity. Efficient asset allocation is achieved in the directed search framework, while random matching is known not to achieve efficient allocation.

JEL Classification: G11, G12, G14, D83
Keywords: directed search, asset market, social welfare, intermediation

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1. Introduction

If there are no frictions in the asset market, investors can trade immediately. In reality, however, it is almost impossible for investors to locate trading partners immediately. Therefore, financial institutions intermediate and conduct transactions for investors in return for payment of intermediation fees. In other words, intermediaries exist to reduce transaction difficulty.[1] Of course, intermediaries cannot be perfect and it takes time to find trading partners. Suppose an investor wants to buy an asset, such as a government bond issued by a developing country whose liquidity is extremely low, and sends an order to an intermediary. If the intermediary happens to have this asset, the investor will obtain it immediately. Otherwise, it may take several hours or days to complete the transaction because of the low liquidity of the asset. Therefore, it is interesting to examine how the ability of the intermediary affects the efficiency of asset markets. That is the aim of this paper.

Researchers have explored how trading delays affect market efficiency using the search theoretic approach. The search theory assumes that: (a) market participants cannot make transactions without "searching" for trading partners and (b) they find trading partners stochastically. The search theory has been applied mainly to explain unemployment and why money is used as a medium of exchange.

Only recently, the search theory was applied to asset markets. Some studies have shown that search friction affects asset prices, market efficiency, risk premiums, etc. Lagos (2006) explains the equity premium puzzle by analyzing the search premium for risky assets. He assumes that search costs are necessary to trade risky assets, while such costs are not necessary to trade riskfree assets. Since risky assets are subject to a search premium, the premium for risky assets becomes larger. Voyanos and Wang (2007), Voyanos and Weill (2008), and Weill (2008) built models in which two assets with identical cash flows were traded at different prices because of different search frictions. Duffie et al. (2007) investigated the impact of search-and-bargaining friction on asset prices in over-the-counter markets.[2] They found that, under certain conditions,

[1] Another reason for the existence of intermediaries is to save monitoring costs. See Diamond (1984).

[2] Over-the-counter (OTC) markets provide the opportunity to trade financial instruments such as stocks, bonds, commodities, and derivatives directly between two parties. This differs from exchange trading, which occurs in facilities such as futures exchanges or stock exchanges which were created especially for the purpose of trading.

asset prices are highly discounted because of search friction.

Lagos and Rocheteau (2007) is especially important to this paper as it inclusively investigates the effects of trade delays on asset markets in a random matching model framework. The random matching model assumes two points: (a) that market participants do not have advance information about their trading partners and (b) they randomly meet their trading partners only after a search. If the partner is not suitable, market participants cease contact and begin a new search. Lagos and Rocheteau (2007) found a condition relating to preferences regarding market participants, i.e., a reduction in trading frictions raises the asset price. They found that, unless investors have all the bargaining power in bilateral negotiations with brokers, asset allocation is inefficient even if the Hosios condition is satisfied.[3]

Lagos and Rocheteau (2007) show that efficient portfolio allocation is not achieved in the random matching framework without particular conditions. This raises the question, "Are there any market mechanisms that realize more efficient allocations than random matching?" In some labor market studies, more efficient outcomes were achieved via the directed search framework than random matching.[4] The directed search model differs from the random matching model in the following: market participants have information about potential trading partners in advance and choose their partners before they search. However, because of search frictions, they may not find the chosen partners. Market participants never meet partners who were not chosen. They meet the chosen partner with some probability, otherwise fail to find any partner. Once they meet their partners, transactions are made immediately. In this paper, I follow Lagos and Rocheteau (2007)'s framework, but modify it from random matching to a directed search model to compare the efficiency of asset markets in these two frameworks.

The modification is not merely an intellectual exercise. I believe the directed search model better describes asset transactions in the real world than the random matching model. First, in contrast to the assumption of the random matching model, investors can actually choose their broker. Second, contrary to the assumption of the random matching model that intermediation

[3]The Hosios condition indicates that the value of a broker's bargaining power is equal to the elasticity of the matching function. See Hosios (1990).

[4]Moen (1997) used a directed search framework to analyze labor markets. He assumed that firms first determine their wage rates and then announce them. Workers apply to firms knowing the wage rates and there is no bargaining.

fees are determined through bilateral negotiations between investors and brokers, in practice brokers announce their fees and there are no negotiations. Thus, the directed search model assumes that investors choose their dealers, and that dealers announce their intermediation fees in advance.

I show that the asset allocation is efficient in the directed search framework. Moreover, by solving the model numerically, I show that the asset price increases (decreases) with the matching efficiency, if the relative risk aversion is smaller (larger) than unity.

The rest of this paper is organized as follows. In section 2, I lay out the model settings. In section 3, I define the competitive equilibrium of this economy and examine how trading frictions affect asset price and portfolio choices numerically. In section 4, I show the relationship between trading frictions and social welfare and discuss the efficiency of asset allocation. Section 5 is the conclusion.

2. Environment

The environment of the model basically follows Lagos and Rocheteau (2007). Time is continuous and infinite. Infinite investors, scaled up to unity, maximize the present value of their lifetime utility. Homogeneous brokers, indexed by $n = 1, 2, ...$, maximize their instantaneous profit. The number of brokers is determined by a free entry condition. There is one durable and perfectly divisible asset. There is one perishable good which we use as the numeraire. Total supply of the asset is fixed to A. The numeraire goods are produced and consumed by all agents, who produce in units of numeraire goods per working hour. Each investor's instantaneous utility function depends on his assets, consumption, and working hours, described as $u_i(a) + c - w$, where a is the quantity of assets that type i investor holds; i specifies the investor's preference, which can be either h(high) or l(low). I assume that $u_h'(a) > u_l'(a)$ for all a. The utility function is bounded, continuously differentiable, strictly increasing, and strictly concave. c is the investor's consumption of numeraire goods, w is his working hours. Each investor's preference type randomly changes from low and high subject to a Poisson process with an arrival rate of λ_h (λ_l). Each investor's time preference rate is always ρ.

There is a competitive asset market that investors cannot access directly but brokers can access at any time. Therefore, investors must use brokers when they want to trade in that asset market. However, they may not be able to meet their broker because of matching frictions between investors and brokers, as discussed below.

Brokers cannot own assets, so they do not engage in dealing. The only source of revenue for brokers is the intermediation fee paid by investors. Brokers have an identical operating function, $c(q)$, which is continuously differentiable and nondecreasing in the transaction volume they intermediate q. Each broker announces the intermediation fee function $\phi_n(q)$ in advance and commits to it, where index n represents the n-th broker. The function is assumed to be continuously differentiable and nondecreasing in q. Investors choose brokers based on information about intermediation fees. After deciding to partner with, say, the n-th broker, the investor must wait to meet the broker. How long the investor waits is determined by a matching function $M(\eta_n)$, where η_n is the measure of investors who choose the n-th broker.The investor meets the n-th broker according to a Poisson process with an arrival rate of $\frac{M(\eta_n)}{\eta_n}$. $\frac{M(\eta_n)}{\eta_n}$ represents the frequency of successful meetings, so that a larger $\frac{M(\eta_n)}{\eta_n}$ implies a shorter waiting time. I assume the matching function, $M(\eta_n)$, is continuously differentiable, strictly increasing, and strictly concave. In addition, $M(\eta_n) = 0$ if and only if $\eta_n = 0$.

3. Competitive equilibrium

In this section, I characterize the steady-state equilibrium of the model defined in the previous section. I focus on the steady-state where the problem is solved recursively. Thus, I omit the time subscript t in this section.

3.1. Broker

I define the n-th broker's instantaneous profit function π_n such that

$$\pi_n = M(\eta_n)[\phi_n(q_n) - c(q_n)], \quad n = 1, 2, \cdots,$$

where $M(\eta_n)$ is the intermediation frequency, $\phi(q_n)$ is the intermediation fee, and $c(q_n)$ is the operating cost.

Brokers earn no profits, i.e., $\pi_n = 0$, because of the free entry condition, implying that

$$M(\eta_n) = 0 \quad \Leftrightarrow \quad \eta_n = 0,$$

or

$$\phi(q_n) - c(q_n) = 0 \quad \Leftrightarrow \quad \phi(q_n) = c(q_n).$$

$\eta_n = 0$ implies that the n-th broker no longer operates in the market, so this case is excluded from the following analysis and I obtain $\phi(q_n) = c(q_n)$. In steady-state equilibrium, the broker's fee function is identically characterized as

$$\phi_n(\cdot) = c(\cdot), \quad n = 1, 2, \cdots, \tag{1}$$

so that

$$\phi_n'(\cdot) = c'(\cdot), \; n = 1, 2, \cdots. \tag{2}$$

Eq. (1) indicates that every dealer charges the same intermediate fee schedule. All brokers look the same, resulting in a situation in which investors are uniformly distributed among brokers, that is

$$\eta_1 = \eta_2 = \cdots. \tag{3}$$

The number of successful matches for all brokers also becomes equal,

$$M(\eta_1) = M(\eta_2) = \cdots.$$

3.2. Investor

Denoting type i investor's value function as $V_i(\cdot)$, the Bellman equation of a high-type investor's problem is described as

$$\begin{aligned}\rho V_h(a) &= u_h(a) + \lambda_l[V_l(a) - V_h(a)] \\ &\quad + \max\left\{\max_{a' \in \mathbf{R}, n \in \{1,2,\cdots\}} \left\{m_n[V_h(a') - V_h(a) - p(a' - a) - \phi_n(|a' - a|)]\right\},\; 0\right\},\end{aligned}$$

where p is the asset price. The first term, $u_h(a)$, represents the instantaneous utility of a high-type investor who owns a units of assets. The second term arises from the possibility that the investor will change to a low preference type at the next moment. The third term reflects the change in asset value stemming from the transaction. Investors must decide in advance whether or not to trade. If they decide not to trade, the value will not change. If they decide to trade, then they need to choose a broker and wait to meet at a Poisson arrival rate of $m_n \equiv \frac{M(\eta_n)}{\eta_n}$. The investor decides the amount of assets they will own in the next period, $a' \in \mathbf{R}$, and pay the intermediation fee, $\phi_n(|a' - a|)$, to the n-th broker.

Similarly, the Bellman equation of a low-type investor's problem is written as

$$\begin{aligned}\rho V_l(a) &= u_l(a) + \lambda_h[V_h(a) - V_l(a)] \\ &\quad + \max\left\{\max_{a' \in \mathbf{R}, n \in \{1,2,\cdots\}} \left\{m_n[V_l(a') - V_l(a) - p(a' - a) - \phi_n(|a' - a|)]\right\},\ 0\right\}.\end{aligned}$$

The following lemma is necessary for subsequent arguments. All proofs are given in the Appendix.

Lemma 1. $V_i(\cdot)$ *is strictly increasing, strictly concave, and continuously differentiable for* $i = h, l$.

Once investor i decides to trade, his maximization problem is reduced to

$$\max_{a' \in \mathbf{R}, n \in \{1,2,\cdots\}} m_n[V_i(a') - V_i(a) - p(a' - a) - \phi_n(|a' - a|)].$$

The first order condition (henceforth FOC) with respect to a' is

$$V_i'(a') - p - I_{(a,a')}\phi_n'(|a' - a|) = 0, \quad I_{(a,a')} = \begin{cases} 1 & a' > a, \\ -1 & a' < a, \end{cases} \quad i = h, l, \tag{4}$$

where $V_i'(\cdot)$ is the first differential of the value function. The existence of $V_i'(\cdot)$ is warranted by Lemma 1; $\phi_n'(|a' - a|)$ exists by assumption. Eq. (4) indicates that the optimal portfolio depends on the investor's type (i) and the portfolio held by the investor before trading (a). Although Eq. (4) also depends on n, the optimal portfolio does not depend on the broker, because every broker charges the same fee.

Thus, denoting the optimal portfolio for the high-type investor as a_h and for the low-type as a_l, the Bellman equations are rewritten as:

$$\rho V_h(a_h) = u_h(a_h) + \lambda_l[V_l(a_h) - V_h(a_h)], \tag{5}$$

$$\begin{aligned}\rho V_h(a_l) =& u_h(a_l) + \lambda_l[V_l(a_l) - V_h(a_l)] \\ &+ m_n[V_h(a_h) - V_h(a_l) - p(a_h - a_l) - \phi_n(|a_h - a_l|)],\end{aligned} \tag{6}$$

$$\begin{aligned}\rho V_l(a_h) =& u_l(a_h) + \lambda_h[V_h(a_h) - V_l(a_h)] \\ &+ m_n[V_l(a_l) - V_l(a_h) - p(a_l - a_h) - \phi_n(|a_l - a_h|)],\end{aligned} \tag{7}$$

$$\rho V_l(a_l) = u_l(a_l) + \lambda_h[V_h(a_l) - V_l(a_l)]. \tag{8}$$

I exclude the case of an investor who does not want to trade because that investor's portfolio will not change from the beginning, i.e., the steady-state of the economy coincides with the initial state. In other words, I assume that the intermediation fees are not so large that such a case exists.

Now, I demonstrate the following lemma.

Lemma 2. *The optimal portfolio of the high type investor is greater than that of the low type investor, i.e.,* $a_h \geq a_l$.

Using lemma 2, the following proposition can be proved.

Proposition 1. *The demand function for the assets of the high-type investor is*

$$\begin{aligned}p =& \frac{(\rho + \lambda_h + m_n)u_h'(a_h) + \lambda_l u_l'(a_h)}{\rho(\rho + \lambda_h + \lambda_l + m_n)} \\ &- \left[1 + \frac{2m_n\lambda_l}{\rho(\rho + \lambda_h + \lambda_l + m_n)}\right]\phi_n'(a_h - a_l),\end{aligned} \tag{9}$$

and for that of the low-type investor is

$$\begin{aligned}p =& \frac{(\rho + \lambda_l + m_n)u_l'(a_l) + \lambda_h u_h'(a_l)}{\rho(\rho + \lambda_h + \lambda_l + m_n)} \\ &+ \left[1 + \frac{2m_n\lambda_h}{\rho(\rho + \lambda_h + \lambda_l + m_n)}\right]\phi_n'(a_h - a_l),\end{aligned} \tag{10}$$

These demand functions warrant that when trading frictions do not exist and no intermediation fees are levied, the asset price of this model decays into the Walrasian equilibrium price. To confirm this, I substitute $\lambda_h = \lambda_l = 0$ and $\phi_n(a_h - a_l) = 0$ in Eqs. (9) and (10) to get $p^W \equiv \frac{u_h'(a_h)}{\rho} = \frac{u_l'(a_l)}{\rho}$

3.3. Law of transition

I classify investors' states into four classes, $\{hh, hl, lh, ll\}$, whose definitions are given in Table 1.

Using these variables, I can describe the laws of transition among the states of investors, which should be followed by steady states as follows.

$$m_n\xi_{hl} + \lambda_h\xi_{lh} - \lambda_l\xi_{hh} = 0, \tag{11}$$

$$\lambda_h\xi_{ll} - m_n\xi_{hl} - \lambda_l\xi_{hl} = 0, \tag{12}$$

$$\lambda_l\xi_{hh} - m_n\xi_{lh} - \lambda_h\xi_{lh} = 0, \tag{13}$$

$$m_n\xi_{lh} + \lambda_l\xi_{hl} - \lambda_h\xi_{ll} = 0, \tag{14}$$

$$\xi_{hl} + \xi_{lh} = \sum_n \eta_n, \tag{15}$$

$$\xi_{hh} + \xi_{hl} + \xi_{lh} + \xi_{ll} = 1. \tag{16}$$

Eq. (11) means that inflow and outflow of *hh* are equal in the steady-state. The first term, $m_n\xi_{hl}$, represents the movement of investors from *hl* to *hh* during one period because of transactions. The second term, $\lambda_h\xi_{lh}$, indicates movement of investors from *lh* to *hh* because of a change in type. The third term, $\lambda_l\xi_{hh}$, are investors who move from *hh* to *lh* because of a change in type. Eqs. (12), (13), and (14), describe the conditions associated with states *hl*, *lh*, and *ll*, respectively. Eq. (15) shows that investors who tend to trade are identical to investors whose types and portfolios are not coincidental. Eq. (16) assumes that investors are scaled up to unity.

3.4. Asset market

The market clearing condition is:

$$(\xi_{hh} + \xi_{lh})a_h + (\xi_{hl} + \xi_{ll})a_l = A. \tag{17}$$

The left side is demand for the asset. $\xi_{hh}+\xi_{lh}$ ($\xi_{hl}+\xi_{ll}$) is the sum of investors who have a_h (a_l). The total supply of the asset is fixed to A by assumption.

3.5. Equilibrium

I define the competitive equilibrium of this economy as follows.

Definition 1. *The steady-state competitive equilibrium is characterized by* $\{p, a_h, a_l, \phi_1(\cdot), \phi_2(\cdot), \cdots, \eta_1, \eta_2, \cdots, \xi_{hh}, \xi_{hl}, \xi_{lh}, \xi_{ll}\}$ *that satisfies Eqs. (1), (3), and (9)-(17).*

3.6. Numerical Example

3.6.1 Specifications

The equilibrium system defined above is too complicated to solve analytically. Therefore, let me demonstrate a numerical solution instead. To calibrate the system, I specified the matching, utility, and cost functions, and the value of parameters in the functions, as shown in Table 2. The matching function is specified as continuous, strictly increasing, and strictly convex. A constant relative risk aversion (CRRA) type of utility function and a linear cost function are assumed. I determined parameters that are appropriate for fitting quarterly data. The value adopted for the time preference rate, ρ, is frequently used in macroeconomics studies such as the real business cycle theory. I set the Poisson arrival rate for the transition of investor type, λ_h and λ_l, at approximately every week and parameters of the utility functions, ϵ_h and ϵ_l, to make the marginal utility of low-type investors be half that of high-type. The cost function parameters, c_1 and c_2, were set to be small enough to induce the incentive to trade. With these specifications, a larger M implies more opportunity for transactions. Thus, we can regard the parameter M as the matching efficiency.

3.6.2. Results

Figure 1 shows how the matching efficiency (M) affects price (p), the high-type investor's optimal portfolio (a_h), and the low-type investor's optimal portfolio (a_l) when the relative risk aversion is less than unity. When M equals 10 (20), the investor meets the broker every few days (several hours). In this case, the asset price increases with the matching efficiency. In the reverse case, when relative risk aversion is larger than unity, the opposite result is obtained (Figure 2). In both cases, the optimal portfolio of the high-type investor increases and of the low-type investor decreases with the matching efficiency. In other words, $a_h - a_l$ increases with the matching efficiency.

3.6.3. Intuition

Let me explain why the above results were obtained. First, suppose that if the matching efficiency (M) is extremely low, investors know that transactions are hardly possible; they will have to keep their portfolios as they are even though their portfolios are not optimal. The larger the gap between a_h and a_l, the more opportunity costs. Thus, when M is low, $a_h - a_l$ has a low value. Second, when the relative risk aversion is smaller than unity, investors are close to neutral risk and their elasticity of intertemporal substitution is larger than unity. In this case, the substitution effect exceeds the income effect, so that the price elasticity of an asset is larger than unity. The demand functions corresponding to this case are described in Figure 3. Eqs. (9) and (10) indicate that an improvement of matching efficiency shifts the high-type investor's demand function upward and the low-type investor's downward. If the asset price remains at p^*, expansion of the high-type investor's demand $a'_h - a_h$ dominates the reduction of the low-type investors' $a_l - a'_l$, resulting in expansion of the aggregate demand. Since the total asset supply is fixed at A, expansion of the aggregate demand causes excess demand for the asset. As a result, the asset price rises to clear the market. When the relative risk aversion coefficient is larger than unity, the argument is opposite. These effects of matching efficiency on asset price and optimal portfolios are consistent with Lagos and Rocheteau (2007).

4. Efficiency

4.1. Efficient allocation

In this section, I analyze a social planner's problem. I assume that the social planner manipulates investors' portfolios. The free entry condition is assumed in this section to compare the social optimum with the market equilibrium under the same exogenous condition.

For analytical tractability, I consider the case where the time preference rate is close to 0.

Under this assumption, the social planner's problem decays to the maximization of the steady-state instantaneous welfare. The planner solves

$$\max_{a_h^s, a_l^s} \quad W = \xi_{hh}^s u_h(a_h^s) + \xi_{hl}^s u_h(a_l^s), \qquad (18)$$
$$+ \xi_{lh}^s u_l(a_h^s) + \xi_{ll}^s u_l(a_l^s) - \sum_n M(\eta_n^s) c(a_h^s - a_l^s),$$

subject to

$$(\xi_{hh}^s + \xi_{lh}^s) a_h^s + (\xi_{hl}^s + \xi_{ll}^s) a_l^s = A, \qquad (19)$$

$$m_n^s \xi_{hl}^s + \lambda_h \xi_{lh}^s - \lambda_l \xi_{hh}^s = 0, \qquad (20)$$

$$\lambda_h \xi_{ll}^s - m_n^s \xi_{hl}^s - \lambda_l \xi_{hl}^s = 0, \qquad (21)$$

$$\lambda_l \xi_{hh}^s - m_n^s \xi_{lh}^s - \lambda_h \xi_{lh}^s = 0, \qquad (22)$$

$$m_n^s \xi_{lh}^s + \lambda_l \xi_{hl}^s - \lambda_h \xi_{ll}^s = 0, \qquad (23)$$

$$\xi_{hh}^s + \xi_{hl}^s + \xi_{lh}^s + \xi_{ll}^s = 1, \qquad (24)$$

$$\sum_n \eta_n^s = \xi_{hl}^s + \xi_{lh}^s, \qquad (25)$$

$$\eta_1^s = \eta_2^s = \cdots . \qquad (26)$$

where s denotes the social planner's problem and $m_n^s \equiv \frac{M(\eta_n^s)}{\eta_n^s}$ is common for all brokers because of Eq. (26). The social planner's objective function appeared in Eq. (18). It is composed of the weighted sum of the investors' utility and the brokers' operating costs. It does not include intermediation fees, because the social planner is not interested in these fees that are merely transferred from investors to brokers.

FOCs with respect to a_h^s and a_l^s are

$$\xi_{hh}^s u_h'(a_h^s) + \xi_{lh}^s u_l'(a_h^s) - \sum_n M(\eta_n^s) c'(a_h^s - a_l^s) - \mu(\xi_{hh}^s + \xi_{lh}^s) = 0, \qquad (27)$$

$$\xi_{hl}^s u_h'(a_l^s) + \xi_{ll}^s u_l'(a_l^s) + \sum_n M(\eta_n^s) c'(a_h^s - a_l^s) - \mu(\xi_{hl}^s + \xi_{ll}^s) = 0, \qquad (28)$$

where μ is a Lagrange multiplier for the constraint (19). Eliminating the Lagrange multiplier in Eqs. (27) and (28), I obtain

$$\frac{\xi_{hh}^s u_h'(a_h^s) + \xi_{lh}^s u_l'(a_h^s) - \sum_n M(\eta_n^s) c'(a_h^s - a_l^s)}{\xi_{hh}^s + \xi_{lh}^s} = \frac{\xi_{hl}^s u_h'(a_l^s) + \xi_{ll}^s u_l'(a_l^s) + \sum_n M(\eta_n^s) c'(a_h^s - a_l^s)}{\xi_{hl}^s + \xi_{ll}^s}. \qquad (29)$$

From Eqs. (20)-(24), I obtain

$$\xi_{hh}^s = \frac{\lambda_h(m_n^s + \lambda_h)}{(\lambda_h + \lambda_l)(m_n^s + \lambda_h + \lambda_l)}, \tag{30}$$

$$\xi_{hl}^s = \xi_{lh}^s = \frac{\lambda_h\lambda_l}{(\lambda_h + \lambda_l)(m_n^s + \lambda_h + \lambda_l)}, \tag{31}$$

$$\xi_{ll}^s = \frac{\lambda_l(m_n^s + \lambda_l)}{(\lambda_h + \lambda_l)(m_n^s + \lambda_h + \lambda_l)}. \tag{32}$$

Substituting Eqs. (25), (26), and (30)-(32) into Eq. (29), I obtain

$$\frac{(m_n^s + \lambda_h)u_h'(a_h^s) + \lambda_l u_l'(a_h^s) - 2m_n^s\lambda_l c'(a_h^s - a_l^s)}{(m_n^s + \lambda_l)u_l'(a_l^s) + \lambda_h u_h'(a_l^s) + 2m_n^s\lambda_h c'(a_h^s - a_l^s)} = 1. \tag{33}$$

Eq. (33) is interpreted as follows. If the social planner decreases a_h^s, the instantaneous utility of *hh* and *lh* investors decreases and a_l^s increases to satisfy market clearing conditions. In response, *ll* and *hl* investors' utility increases. The changes in a_h^s and a_l^s also alter the broker's operating costs. At the optimum, the weighted sum of these effects is canceled out.

I now define the efficient allocation of this economy.

Definition 2. *The steady-state efficient allocation is characterized by* $\{a_h^s, a_l^s, \eta_1^s, \eta_2^s, \cdots, \xi_{hh}^s, \xi_{hl}^s, \xi_{lh}^s, \xi_{ll}^s\}$ *that satisfies Eqs. (19)-(26) and (33).*

Figure 4 is a numerical example of the relationship between the matching efficiency and social welfare in efficient allocations. The vertical axis W is social welfare, defined in Eq. (18). The horizontal axis represents matching efficiency, parameters of which are shown in Table 2. Social welfare monotonically increases with matching efficiency, implying that smaller trading frictions results in higher social welfare, consistent with our intuition.

4.2. Efficiency of competitive equilibrium

In this subsection, I discuss the efficiency of competitive equilibrium. Specifically, I examine whether competitive equilibrium achieves the social optimum.

It is obvious that Eqs. (19)-(26) of the social optimum are parallel to Eqs. (3) and (11)-(17) in the competitive equilibrium. Thus, what is necessary to show is that Eq. (33) is fulfilled in the competitive equilibrium. To confirm this, I incorporate Eqs. (9) and (10) to obtain:

$$\begin{aligned}&\frac{(\rho+\lambda_h+m_n)u_h'(a_h)+\lambda_l u_l'(a_h)}{\rho(\rho+\lambda_h+\lambda_l+m_n)}-\left[1+\frac{2m_n\lambda_l}{\rho(\rho+\lambda_h+\lambda_l+m_n)}\right]\phi_n'(a_h-a_l)\\&=\frac{(\rho+\lambda_l+m_n)u_l'(a_l)+\lambda_h u_h'(a_l)}{\rho(\rho+\lambda_h+\lambda_l+m_n)}+\left[1+\frac{2m_n\lambda_h}{\rho(\rho+\lambda_h+\lambda_l+m_n)}\right]\phi_n'(a_h-a_l).\end{aligned}\tag{34}$$

To derive Eq. (33), I substitute Eq. (2) into Eq. (34), multiply both sides of Eq. (34) by $\rho(\rho+\lambda_h+\lambda_l+m_n)$, and take $\rho\to 0$. Thus, the following proposition is established.

Proposition 2. *The steady-state competitive equilibrium coincides with the efficient allocation when each investor's time preference rate is close to 0.*

This proposition ascertains that trading frictions do not distort efficient allocation when modeled in a directed search framework. Why is this efficient proposition derived? Close inspection of the above proof reveals that Eq. (2), which forces marginal intermediation fees to be equal to the broker's marginal costs, is the key for the competitive equilibrium to be socially optimal. The social planner is concerned not with intermediation fees but with brokers' operating costs, while investors are concerned with the contrary. Eq. (2) makes these two costs coincide, so that both the social planner and the investors make the same decision. Since Eq. (2) comes from competition among brokers, the free entry condition, as well as the obligation to announce intermediation fees in advance, is inevitable for the intermediaries to be socially optimal.

5. Conclusion

In this paper, I investigated how trading frictions of asset markets affect portfolio choices, asset prices, and welfare, essentially following Lagos and Rocheteau (2007)'s framework, but modifying it from random matching to a directed search model. Solving the model numerically, I show that the asset price increases (decreases) with matching efficiency if the relative risk aversion is smaller (larger) than unity. Lagos and Rocheteau (2007), using the random matching framework, found that asset allocation in the competitive equilibrium is not efficient. Contrary to their results and using the directed search framework, I show that asset allocation in the competitive equilibrium is efficient. The key condition for the optimum is that intermediation fees be equal to brokers' operating costs. If the intermediation fees are larger than the brokers' operating costs, the transaction volume of the asset is smaller than optimal. On the other hand, if the intermediation fees are smaller than the brokers' operating costs, the transaction volume is larger than optimal. Note, this condition is not achieved only by Hosios condition but also because the brokers' bargaining power takes a zero value in the random matching and bargaining framework.

In this paper, I assumed that investors' preference type changes according to an identical Poisson process, and that all brokers have a homogeneous cost function. Thus, a possible extension of this paper would be introduction of heterogeneity to reflect the real world, where several types of investors and brokers exist, at least ex-ante.

Appendix

Proof of Lemma 1.

I prove Lemma 1 using theorems 4.7 and 4.8 from Stokey and Lucas (1989). It is sufficient to check the following three items, which are clearly satisfied in this economy.

1. **R** is nonempty, compact-valued, continuous, monotone, and convex.

2. $u(\cdot)$ is bounded, continuous, strictly increasing, and strictly concave.

3. $\rho > 0$.

□

Proof of Lemma 2.

Suppose $a_h < a_l$, then

$$I_{(a_l,a_h)} = -1,$$

$$I_{(a_h,a_l)} = 1,$$

and

$$V_h'(a_h) = p - \phi_n'(a_l - a_h) \leq p + \phi_n'(a_l - a_h) = V_l'(a_l),$$

because $\phi'(\cdot)$ is nonnegative by assumption. The strict concavity of value functions indicates that

$$V_l'(a_h) > V_l'(a_l) \geq V_h'(a_h) > V_h'(a_l).$$

Then note

$$\begin{aligned}
\rho V_h'(a_h) &= u_h'(a_h) + \lambda_l[V_l'(a_h) - V_h'(a_h)], \\
&= u_h'(a_h) + \lambda_l[\text{sign}+], \qquad (35)\\
\rho V_l'(a_h) &= u_l'(a_h) + \lambda_h[V_h'(a_h) - V_l'(a_h)] + m_n[-V_l'(a_h) + p + \phi_k'(a_l - a_h)], \\
&= u_l'(a_h) + \lambda_h[V_h'(a_h) - V_l'(a_h)] + m_n[-V_l'(a_h) + V_l'(a_l)], \\
&= u_l'(a_h) + \lambda_h[\text{sign}-] + m_n[\text{sign}-], \qquad (36)
\end{aligned}$$

where [sign+] and [sign−] mean a positive term and a negative term, respectively. Subtract Eq. (36) from Eq. (35), then

$$\rho[V_h'(a_h) - V_l'(a_h)] = [u_h'(a_h) - u_l'(a_h)] + [\text{sign+}],$$
$$\Rightarrow \quad \rho[\text{sign}-] = [\text{sign+}],$$

where I assume that $u_h'(a) \geq u_l'(a)$ for all a. This is a contradiction because $\rho > 0$.

□

Proof of Proposition 1.

Lemma 2 indicates that

$$I_{(a_l,a_h)} = 1, \tag{37}$$
$$I_{(a_h,a_l)} = -1. \tag{38}$$

Eqs. (37) and (38) indicate that the investors' FOCs are

$$V_h'(a_h) = p + I_{(a_l,a_h)}\phi_n'(a_h - a_l) = p + \phi_n'(a_h - a_l), \tag{39}$$
$$V_l'(a_l) = p + I_{(a_h,a_l)}\phi_n'(a_h - a_l) = p - \phi_n'(a_h - a_l). \tag{40}$$

First, transforming Eqs. (5) and (8), then

$$V_h(a_h) = \frac{u_h(a_h) + \lambda_l V_l(a_h)}{\rho + \lambda_l}, \tag{41}$$
$$V_l(a_l) = \frac{u_l(a_l) + \lambda_h V_h(a_l)}{\rho + \lambda_h}. \tag{42}$$

Next, differentiating both sides of Eq. (41) with respect to a_h and Eq. (42) with respect to a_l, then

$$V_h'(a_h) = \frac{u_h'(a_h) + \lambda_l V_l'(a_h)}{\rho + \lambda_l}, \tag{43}$$
$$V_l'(a_l) = \frac{u_l'(a_l) + \lambda_h V_h'(a_l)}{\rho + \lambda_h} \tag{44}$$

Substituting Eqs. (43) and (44) into Eqs. (39) and (40),

$$p = \frac{u_h'(a_h) + \lambda_l V_l'(a_h)}{\rho + \lambda_l} - \phi_n'(a_h - a_l), \tag{45}$$
$$p = \frac{u_l'(a_l) + \lambda_h V_h'(a_l)}{\rho + \lambda_h} + \phi_n'(a_h - a_l). \tag{46}$$

Transforming Eqs. (6) and (7),

$$V_h(a_l) = \frac{u_h(a_l) + \lambda_l V_l(a_l) + \alpha_n[V_h(a_h) - p(a_h - a_l) - \phi_n(a_h - a_l)]}{\rho + \lambda_l + \alpha_n}, \tag{47}$$

$$V_l(a_h) = \frac{u_l(a_h) + \lambda_h V_h(a_h) + \alpha_n[V_l(a_l) - p(a_l - a_h) - \phi_n(a_h - a_l)]}{\rho + \lambda_h + \alpha_n}. \tag{48}$$

Differentiating Eqs. (47) and (48) with respect to a_l and a_h, respectively, and substituting Eqs. (41) and (42) into them, I obtain

$$V_h'(a_l) = \frac{u_h'(a_l) + (\lambda_l + \alpha_n)p - (\lambda_l - \alpha_n)\phi_n'(a_h - a_l)}{\rho + \lambda_l + \alpha_n}, \tag{49}$$

$$V_l'(a_h) = \frac{u_l'(a_h) + (\lambda_h + \alpha_n)p + (\lambda_h - \alpha_n)\phi_n'(a_h - a_l)}{\rho + \lambda_h + \alpha_n}. \tag{50}$$

Substituting Eqs. (49) and (50) for Eqs. (47) and (48), I can derive proposition 1.

□

Proof of Proposition 2.

See text.

□

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Table 1: Four classes of investors

state	type	portfolio	measure
hh	high	a_h	ξ_{hh}
hl	high	a_l	ξ_{hl}
lh	low	a_h	ξ_{lh}
ll	low	a_l	ξ_{ll}

Table 2: Values of parameters used in functions

$M(\eta_n)$	$u_i(a)$	$c(q)$	β	ρ	λ_h	λ_l	ϵ_h	ϵ_l	A	c_1	c_2
$M\eta_n^{\beta}$	$\epsilon_i \frac{a^{1-\gamma}}{1-\gamma}$	$c_1 + c_2 q$	0.5	0.01	12	12	20	10	10	0.01	0.001

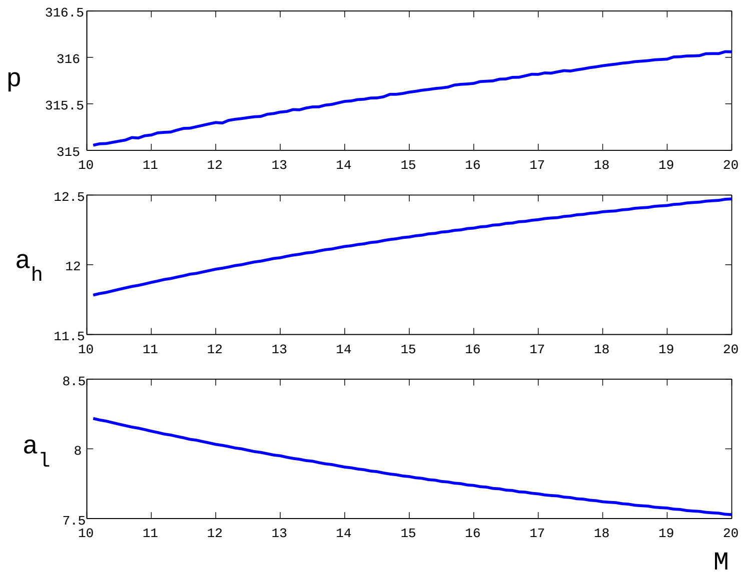

Figure 1: **Asset price and portfolio choices when the relative risk aversion is less than unity ($\gamma = 0.6$).** This figure shows how matching efficiency (M) relates to asset price (p), the high-type investor's optimal portfolio (a_h), and the low-type investor's optimal portfolio, a_l.

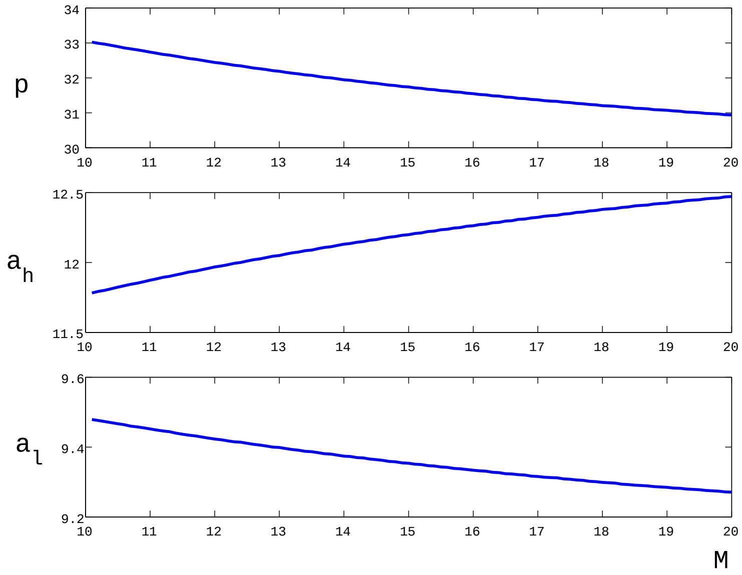

Figure 2: **Asset price and portfolio choices when the relative risk aversion is larger than unity (γ = 1.5).** This figure shows how matching efficiency (M) relates to asset price (p), the high-type investor's optimal portfolio (a_h), and the low-type investor's optimal portfolio, a_l.

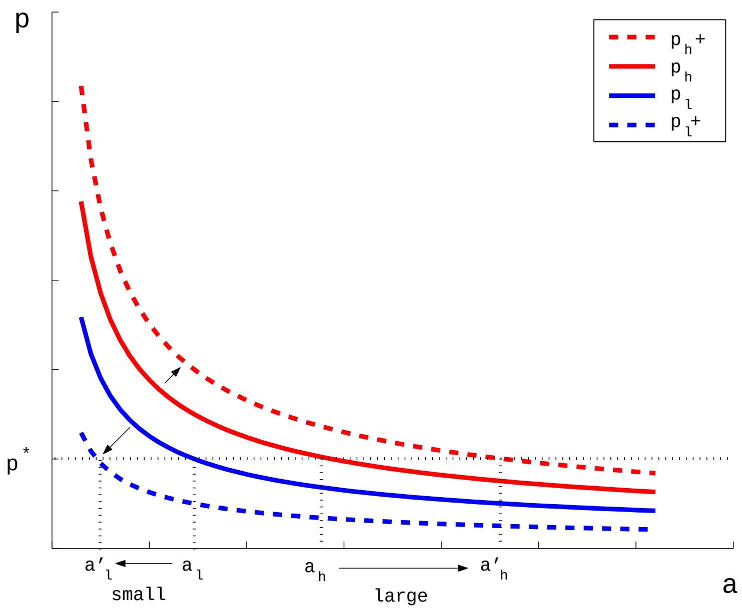

Figure 3: **Demand functions (γ = 0.6).** p_h (p_l) is the demand function of the high-type (low-type) investor. As the matching efficiency improves, demand functions shift from solid lines (p_h, p_l) to dotted lines (p_h^+, p_l^+). If the price remains at p^*, the high-type (low-type) investor's optimal portfolio shifts from a_h (a_l) to a'_h (a'_l).

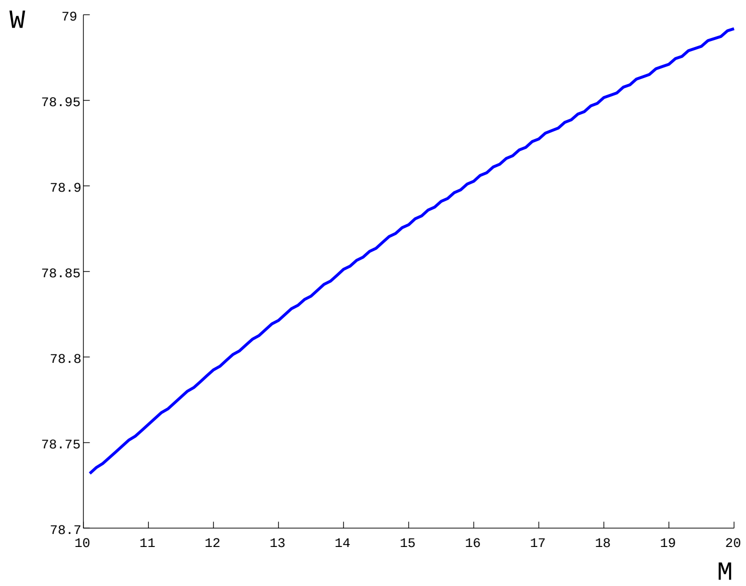

Figure 4: **Welfare.** The relationship between matching efficiency (M) as defined in Table 2 and social welfare (W) as defined in Eq. (18).