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Abstract

In a small open economy model of endogenous growth with public capital accumulation, we examine the effects of a debt policy rule under which the government must reduce its debt–GDP ratio if it exceeds the criterion level. To sustain public debt at a finite level, the government should adjust public spending rather than the income tax rate. The long run debt–GDP ratio should be kept sufficiently low to avoid equilibrium indeterminacy. Under sustainability and determinacy, a tighter (looser) debt rule brings welfare gains when the world interest rate is relatively high (low).

JEL classification: E62; H54; H63

Keywords: Fiscal policy, Public debt, Welfare, Small open economy, Indeterminacy, Limit cycles

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1 Introduction

Discretionary fiscal policies during the 2008–2009 world crisis resulted in serious increases in government debt in the Euro area. In 2011, the average debt–GDP ratio in the Euro area reached 88 percent of GDP, some 20 percentage points higher than at the start of the crisis in 2007. Public debt as a share of GDP in Greece equaled 166.1 percent in 2012. Debt–GDP ratios in Italy, Ireland, and Portugal also exceeded 100 percent. These weak fiscal conditions raised doubts about these countries' abilities to finance their increased debt. As a response to the crisis, the EU has introduced strong fiscal consolidations under the surveillance of the European Commission. Overall public deficits were reduced thanks to expenditure cuts, especially lower public investments, as stated in Public finances in EMU (2012).[1] According to the Stability and Convergence Programmes submitted to the Commission and Council in Spring 2012, EU member states plan to base further fiscal consolidation on expenditure cuts that include reductions in public investment. According to the debt reduction benchmark introduced by the reform of the Stability and Growth Pact (SGP), the so-called Six-Pack in December 2011, member states whose current debt-to-GDP ratio is above the 60% threshold have to reduce the distance to 60% by an average rate of one-twentieth per year.[2] It is important to investigate the effects of the debt-reduction rule proposed by the SGP under its requirements.

Some authors have examined the effects of such a debt-reduction rule. In an endogenous growth model whose growth engine is the flow of public service as in Barro (1990), Futagami et al. (2008) investigate the effects of a government bond-issuance rule that requires the government to reduce its debt at a steady pace if its debt is beyond the criterion level. Maebayashi et al. (2013) uses an endogenous growth model whose engine of growth is public capital accumulation to study the same issue. These authors provide interesting results, but their investigations are confined to closed economies; accordingly,

[1]In the Euro area, the average general government deficit fell from 6.2 percent of GDP in 2010 to 4.1 percent of GDP in 2011.

[2]The Maastrichit Treaty asks EU countries to keep their deficit and debt levels below 3 and 60 percent, respectively, to ensure compliance with budgetary discipline.

transactions in foreign capital markets are removed. In reality, both the government and private sector can borrow and lend their assets in the foreign capital market. Countries holding large levels of debt such as Greece, Italy, Ireland, and Portugal hold large external debt as well. This shows the significance of studying the debt policy rule described here in a model of an open economy.

For our purpose, we consider an endogenously growing small open economy where the government adopts a debt-reduction rule. As in Futagami et al. (1993) and Turnovsky (1997), public capital accumulated through public investment has positive effects on private goods production. The government finances its spending on public investment by imposing a tax on income and by issuing bonds. Public bond-issuance is under the restriction of the same debt policy rule as that in Futagami et al. (2008). We consider two types of public finance budget regimes. In budgetary regime (I), if the debt–GDP ratio exceeds the criterion level, the government adjusts its expenditure with a fixed tax rate to reduce this ratio. In budgetary regime (II), if the debt–GDP ratio exceeds the criterion level, the government controls the tax rate to reduce its debt with a fixed expenditure ratio. In both regimes, the debt–GDP ratio tends to the criterion level in the long run. The criterion level can be considered as the long-run debt–GDP ratio.

In budgetary regime (I), there exists a unique steady-state equilibrium. The long-run debt–GDP ratio is a crucial determinant of the steady-state stability and equilibrium (in)determinacy. When the long-run debt–GDP ratio is sufficiently low, the steady state is saddle stable and hence exhibits equilibrium determinacy. However, if the government sets a high criterion debt–GDP ratio, equilibrium indeterminacy arises because the steady state is a sink or there exists a limit cycle around the steady state.[3] The data in countries in the Euro area show that the 60% criterion level of the debt–GDP ratio proposed by the SGP may not be sufficiently low to ensure equilibrium determinacy.

[3]Some authors study the relation between fiscal policy and indeterminacy. Focusing on balanced budget rules, Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004, 2008) discuss the effects of fiscal policies on equilibrium indeterminacy in closed economies. Farmer (1986) and Greiner (2007) show that in closed economies, limit cycles emerge and equilibrium indeterminacy arises when the government controls the value of its deficit rather than the value of its debt.

We also examine the welfare effects of debt reduction under budgetary regime (I), assuming that the long-run debt–GDP ratio is sufficiently low to ensure determinacy. The welfare effects of debt reduction depend on the rates of returns from private savings and public investment. In our small open economy, the rate of return from private savings is equal to the world interest rate. When the world interest rate is higher (lower) than the rate of return from public investment, reductions in government debt improve (deteriorate) welfare. Furthermore, the pace of debt reduction is an important determinant of the magnitude of welfare gains (losses). When the world interest rate is higher than the rate of return from public savings, the government can further improve welfare by reducing the debt at a faster pace. In contrast, if the world interest rate is relatively low, the government can mitigate welfare losses by reducing the debt at a slower pace.

In budgetary regime (II), there exists a unique steady-state equilibrium. However, the steady state is always unstable under budgetary regime (II). Because the economy cannot reach the unstable steady state, it makes little sense to study the characteristics of the unstable steady state, and hence, we do not examine the welfare effects of debt reduction in regime (II). Nevertheless, our model provides the following important implication. Suppose that the initial private domestic savings cannot afford to absorb the initial outstanding government debt and the government then borrows from foreign investors. In such a situation, under regime (II), households eventually become overextended with foreign debt, and there exists no equilibrium such that the government can follow the debt-reduction rule. Then, regime (II) is unsuitable for sustaining public debt.

There exist studies on public debt finance in endogenous growth models where government services or public capital are inputs for private goods production (see, e.g., Bruce and Turnovsky (1999), Greiner and Semmler (2000), Ghosh and Mourmouras (2004), Greiner (2007, 2012), and Yakita (2008)). These studies explore the policy implications of public debt finance for equilibrium dynamics, long-run growth, and welfare. However, few studies investigate the debt-reduction rule found in the Maastricht Treaty and the SGP, except Futagami et al. (2008), Minea and Villieu (2013), and Maebayashi et al.

(2013). The present study differs from these studies on the debt-reduction rule in the following three points.

First, although these studies focus only on closed economies, we consider an open economy and show that the openness of the economy provides important implications for equilibrium (in)determinacy. Assuming that the public debt-to-private-capital ratio is constrained by the debt-reduction rule and focusing only on regime (I), Futagami et al. (2008) show that the debt-reduction rule may be a source of indeterminacy. However, Minea and Villieu (2013) indicate that this result crucially depends on how to construct the dynamic system, showing that indeterminacy never arises if the debt–GDP ratio is used as the policy target as in the present model. Maebayashi et al. (2013) show that indeterminacy does not arise in a closed economy version of our model. In contrast, we show that the debt-reduction rule found in the SGP may be a source of equilibrium indeterminacy in a small open economy.

Second, we provide sharp insights on the welfare effects of debt-reduction rules. Although Futagami et al. (2008) and Minea and Villieu (2013) compare multiple balanced growth paths in terms of growth rate, they do not conduct welfare analysis explicitly. Maebayashi et al. (2013) conduct welfare analysis numerically in a closed economy model, but they do not show the exact conditions under which debt reduction improves welfare. We derive the analytical expression of the welfare effects and provide an intuitive interpretation of them.

Finally, we shed light on how different budgetary regimes generate different macroeconomic consequences in a small open economy. In contrast to our study, Futagami et al. (2008) and Minea and Villieu (2013) focus only on regime (I). In a closed version of our model, Maebayashi et al. (2013) show that under both regimes (I) and (II), a saddle stable steady state exists. In our small open model, the stability of the steady state under regime (I) crucially depends on the long-run debt–GDP ratio, whereas the steady state is always unstable under regime (II), which implies that regime (II) is unsuitable for sustaining public debt.

Furthermore, our study is related to the literature on the relationship between fiscal policy and sustainability of economies. It can be compared to the existing results on fiscal sustainability. Assuming that the government adjusts the income tax rate in closed economy models, Bräuninger (2005) and Yakita (2008) show that when the initial debt is too large, the debt–GDP ratio grows unboundedly; hence, public debt is not sustainable.[4] These papers explore the conditions under which public debt can be sustained. In contrast, we investigate which budgetary regime the government should adopt to sustain its debt. Under regime (I), the government can sustain its debt on the condition that the criterion level of the debt–GDP ratio is sufficiently low. However, if the government adopts regime (II) when it borrows from foreign investors, households eventually become overextended with foreign debt, and the economy loses the ability to pay back its debt.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 investigates the local stability and social welfare under budgetary regime (I). Section 4 examines budgetary regime (II). Section 5 concludes.

2 Model

We consider a small open economy. A single final good is produced using labor, private capital, and public capital (infrastructure). It is assumed that the final good and private capital are freely traded beyond the country's borders. However, individuals cannot migrate, and public capital (social infrastructure) cannot cross borders. To construct social infrastructure, the government must make public investments. [5]

[4]Diamond (1965) and Chalk (2000) show that permanent budget deficits cannot be sustainable unless the interest rate is less than the growth rate in closed economies.

[5]The benchmark model assumes that the agents are faced with the constant world interest rate. Following Chatterjee et al. (2003), Appendix G extends the benchmark model assuming that the interest rate varies depending on the fiscal condition of the economy. This extension does not affect our main results.

2.1 Households

We consider a representative household. The size of the population is normalized to one. Let C_t be consumption at time t. The utility of the representative household is given by

$$U_0 = \int_0^{+\infty} [\ln C_t] e^{-\rho t} dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate. The household inelastically supplies one unit of labor at each moment of time. The government imposes a tax on the household's income, $I_t \equiv rA_t + w_t$, where r is the (constant and exogenously given) world interest rate, w_t is the wage rate at time t, and A_t is the asset holdings of the household at time t. In this small open economy, there are three types of assets, private capital (K_t), government bonds (B_t), and foreign assets (FA_t). Hence, we have $A_t = K_t + B_t + FA_t$. If $FA_t < 0$, some fractions of private capital or government bonds are owned by foreign agents. We assume that the tax takes the residence base form. Then, residents' income is taxed at a uniform rate regardless of its source country, while non-residents' income is not taxed. Thus, the flow budget constraint of the representative household is

$$\dot{A}_t = (1 - \tau_t) I_t - C_t, \tag{2}$$

where τ_t denotes the income tax rate at time t. In contrast to Futagami et al. (2008) who assume a constant tax rate, we allow τ_t to vary over time, as we discuss later. In an open economy, the household can borrow from foreign countries, and hence, A_t can be negative. However, A_t must satisfy the no-Ponzi game (NPG) condition, $\lim_{T\to\infty} A_T e^{-\int_t^T (1-\tau_v) r dv} \geq 0$. The household maximizes (1) subject to (2), which yields

$$\dot{C}_t = \{(1 - \tau_t) r - \rho\} C_t, \tag{3}$$

and the transversality condition (TVC), $\lim_{T\to\infty} A_T e^{-\int_t^T (1-\tau_v) r dv} = 0$.

2.2 Firms

As in Futagami et al. (1993), the production function of the representative firm is given by $Y_t = F(K_t, K_{g,t}L_t)$, where Y_t, K_t, $K_{g,t}$, and L_t are output, private capital, stock of infrastructure, and labor input at time t, respectively.[6] The presence of $K_{g,t}$ in the production function reflects the external effects. Infrastructure stock, $K_{g,t}$, accumulates through government investments. The production function satisfies the standard neoclassical characteristics, especially the constant returns to scale with respect to K_t and $K_{g,t}L_t$. Accordingly, we can transform this into the following intensive form:

$$Y_t = F(x_t, 1)K_{g,t}L_t = f(x_t)K_{g,t}L_t, \tag{4}$$

where $x_t \equiv K_t/(K_{g,t}L_t)$ and $f(x_t) \equiv F(x_t, 1)$. Given perfect competition and profit maximization, we obtain

$$r = f'(x_t), \tag{5a}$$

$$w_t = \{f(x_t) - f'(x_t)x_t\}K_{g,t} \equiv \omega_t K_{g,t}. \tag{5b}$$

The world interest rate, r, is constant because of the assumption of a small country. Thus, x_t and ω_t become constant over time. The following discussion omits time index t from x_t and ω_t. In equilibrium, we have $L_t = 1$. Then, Y_t grows at the same rate as $K_{g,t}$ (see (4)). For later use, we define $K_{g,t}/Y_t = 1/f(x) \equiv k_g$.

2.3 Government

To construct infrastructure, the government makes public investments. The amount of public investments in time t is denoted by G_t.[7] Then, stock of infrastructure accumulates according to $\dot{K}_{g,t} = G_t$. For later use, we define $g_t \equiv G_t/Y_t$. The government finances its

[6] In contrast to us, Futagami et al. (2008) assume that the flow of public service enters the production function as an input.

[7] As in Futagami et al. (1993), we ignore the depreciation of public capital to simplify the analysis.

expenditure in two ways. One is by levying an income tax, and the other is by issuing bonds. Thus, the government's budget constraint is

$$rB_t + G_t = \dot{B}_t + \tau_t(rA_t + w_t), \tag{6}$$

where B_t stands for outstanding government debt, and $\dot{B}_t$ denotes newly-issued government bonds. Using $A_t = K_t + B_t + FA_t$, we rewrite (6) as $\dot{B}_t = (1-\tau_t)rB_t + G_t - \tau_t\{r(K_t + FA_t) + w_t\}$. If the government debt increases at a rate higher than $(1-\tau_t)r$, no agents are willing to hold government bonds. Therefore, the government must satisfy the NPG condition, $\lim_{T\to\infty} B_T e^{-\int_t^T (1-\tau_v)rdv} \leq 0$.

Similar to the reform of the SGP in 2011 for EU countries, we assume that the government must reduce its debt–GDP ratio at a steady pace if its level is beyond the criterion level. To simply formulate this rule, we follow Futagami et al. (2008) and assume that the government adjusts its debt–GDP ratio, $b_t \equiv B_t/Y_t$, according to the following rule:

$$\dot{b}_t = -\phi(b_t - \bar{b}), \quad \phi > 0, \tag{7}$$

where $\bar{b} > 0$ and $\phi > 0$ represent the criterion level of the government's debt–GDP ratio and the adjustment coefficient of the rule, respectively. If $b_t > \bar{b}$ holds, the government reduces its debt by 100ϕ percent of the difference between the current and target levels of b. Then, if $\phi = 0.05$ and $\bar{b} = 0.6$, the debt policy rule, (7), is well suited to the debt reduction benchmark introduced by the SGP (i.e., member states whose current debt–GDP ratio is above 60 percent must reduce their debt–GDP ratio distance to 60 percent by an average rate of one twentieth per year). We assume $b_0 \geq \bar{b}$, because the average debt–GDP ratio in the Euro area in 2011 reached 88 percent of GDP (which is higher than the criterion level of 60 percent). If ϕ takes a large (small) value, the government adjusts b_t to the criterion level at a fast (slow) pace.

Given this adjustment rule, the government chooses either τ_t or g_t to satisfy the budget constraint, (6).[8] The present study considers two types of budgetary regimes: (I) the government sets a constant τ and adjusts g_t to satisfy (6) and (7), or (II) the government chooses a constant g, and τ_t is then endogenously determined to satisfy (6) and (7).

3 Adjustments in Public Investments: Regime (I)

We first consider the economy under budgetary regime (I), where the government sets a constant $\tau \in (0,1)$ and adjusts public investments to satisfy (6) and (7). We begin with the derivation of the dynamic system under regime (I). We define $c_t \equiv C_t/Y_t$, $a_t \equiv A_t/Y_t$, and $\gamma_t \equiv \dot{Y}_t/Y_t$. Here, Y_t is given by (4), x remains constant, $K_{g,t}$ is a state variable, and $L_t = 1$ holds in equilibrium. We then have to treat Y_t as a predetermined variable. Then, a_t and b_t are state variables and c_t is a jump variable. From (4) and $\dot{K}_{g,t} = G_t$, we have $g_t = k_g\gamma_t$. From (2), (3), (5b), (6), and the definitions of I_t and γ_t, we obtain

$$\dot{b}_t = (r - \gamma_t)b_t + g_t - \tau_t(ra_t + \omega k_g), \tag{8a}$$

$$\dot{a}_t = \{(1-\tau_t)r - \gamma_t\}a_t - c_t + (1-\tau_t)\omega k_g, \tag{8b}$$

$$\dot{c}_t = \{(1-\tau_t)r - \rho - \gamma_t\}c_t. \tag{8c}$$

Eliminating $\dot{b}_t$ from (7) and (8a) and solving for γ_t using $g_t = k_g\gamma_t$, we obtain[9]

$$\gamma_t \left(= \frac{g_t}{k_g}\right) = \frac{\tau(ra_t + \omega k_g) - rb_t - \phi(b_t - \bar{b})}{k_g - b_t} \equiv \gamma(a_t, b_t). \tag{9}$$

This equation shows that if the government reduces $\bar{b}$, γ_t decreases (increases) in the short run if $b_t < (>) k_g$. The government can reduce its debt–GDP ratio in two ways. One is reducing its debt, B_t, and the other is enhancing the output growth through public investments. When b_t is sufficiently small to satisfy $b_t < k_g$, the government can reduce

[8] Dividing both sides of (6) by Y_t, we obtain $\dot{b}_t = (r - \gamma_t)b_t + g_t - \tau_t(ra_t + \omega k_g)$. The flow budget constraint of the government is represented by τ_t and g_t. This is Equation (8a) we derive later.

[9] Because $g_t = k_g\gamma_t$, the choice of g_t is equivalent to that of γ_t. When we discuss regime (I) in the following, we mainly focus on γ_t rather than g_t.

b_t easier by reducing B_t rather than by enhancing output growth. In contrast, when $b_t > k_g$ holds, enhancing output growth is the easier way to reduce b_t. Hence, when the government reduces $\bar{b}$, γ_t decreases (increases) in the short run if $b_t < (>) k_g$. Note that (i) the effects of reductions in $\bar{b}$ on $\gamma_t (= g_t/k_g)$ become stronger as ϕ becomes larger, because the government must reduce b_t at a faster pace and (ii) effects on $\gamma_t (= g_t/k_g)$ gradually disappear as b_t becomes close to $\bar{b}$.

Substituting (9) into (8b) and (8c) yields

$$\dot{a}_t = \{(1-\tau)r - \gamma(a_t, b_t)\}a_t - c_t + (1-\tau)\omega k_g, \tag{10a}$$

$$\dot{c}_t = \{(1-\tau)r - \rho - \gamma(a_t, b_t)\}c_t. \tag{10b}$$

The dynamic system is then given by (7), (10a), and (10b) together with the initial values, b_0 and a_0.

3.1 Steady State and Stability

We derive the steady-state equilibrium where $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ holds. We set $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ in (7), (10a), and (10b), and we solve for b_t, a_t, c_t, and γ_t using (9). The steady-state values of b_t, a_t, c_t, and γ_t are, respectively, given by $b_{\mathrm{I}}^* = \bar{b} (> 0)$,

$$a_{\mathrm{I}}^* = \frac{(\gamma_{\mathrm{I}}^* - \tau\omega)k_g + (\rho + \tau r)\bar{b}}{\tau r}, \tag{11a}$$

$$c_{\mathrm{I}}^* = \frac{\gamma_{\mathrm{I}}^*(\rho + \tau\omega)k_g + \rho(\rho + \tau r)\bar{b}}{\tau r} > 0, \tag{11b}$$

$$\gamma_{\mathrm{I}}^* = (1-\tau)r - \rho. \tag{11c}$$

To ensure positive growth, we assume that $\gamma_{\mathrm{I}}^* = (1-\tau)r - \rho > 0$. From (11a), household income at the steady state is given by $(ra_{\mathrm{I}}^* + \omega k_g)Y_t = \frac{(\tau r + \rho)\bar{b} + \gamma_{\mathrm{I}}^* k_g}{\tau} Y_t$ that is apparently positive. Because $\tau \in (0,1)$ and $\rho > 0$, we have $\gamma_{\mathrm{I}}^* < (1-\tau)r$; this ensures TVC, because $\dot{A}_t/A_t = \gamma_{\mathrm{I}}^*$ holds at the steady state. The NPG condition of the government is also satisfied because $\dot{B}_t/B_t = \gamma_I^* < (1-\tau)r$ in the steady state. The policy parameters, $\bar{b}$ and

ϕ, have no effects on the long-run growth rate, γ_I^*. We obtain the following proposition.

Proposition 1 *Consider budgetary regime (I). Suppose that* $\gamma_{\mathrm{I}}^* = (1-\tau)r - \rho > 0$. *There exists a unique steady-state equilibrium where* $b_{\mathrm{I}}^* = \bar{b}$ *and* (11a)–(11c) *hold.* $\bar{b}$ *and* ϕ *have no effects on the long-run growth rate,* γ_I^*.

We next examine the stability of the steady state characterized by $b_{\mathrm{I}}^* = \bar{b}$ and (11a)–(11c). We linearize the dynamic system around the steady state characterized by $b_{\mathrm{I}}^* = \bar{b}$ and (11a)–(11c), and then we obtain

$$\begin{pmatrix} \dot{b}_t \\ \dot{a}_t \\ \dot{c}_t \end{pmatrix} = \begin{pmatrix} -\phi & 0 & 0 \\ -\frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} a_{\mathrm{I}}^* & \rho - \frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} a_{\mathrm{I}}^* & -1 \\ -\frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} c_{\mathrm{I}}^* & -\frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} c_{\mathrm{I}}^* & 0 \end{pmatrix} \begin{pmatrix} b_t - \bar{b} \\ a_t - a_{\mathrm{I}}^* \\ c_t - c_{\mathrm{I}}^* \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} \frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} &\equiv \left.\frac{\partial \gamma(a_t, b_t)}{\partial b_t}\right|_{(a_t, b_t) = (a_{\mathrm{I}}^*, \bar{b})} = -\frac{\phi + \rho + \tau r}{k_g - \bar{b}}, \\ \frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} &\equiv \left.\frac{\partial \gamma(a_t, b_t)}{\partial a_t}\right|_{(a_t, b_t) = (a_{\mathrm{I}}^*, \bar{b})} = \frac{\tau r}{k_g - \bar{b}}. \end{aligned} \tag{13}$$

We denote the Jacobian matrix of (12) by M. Let us denote the (i, j) element of M by m_{ij} $(i, j = 1, 2, 3)$. One of the eigenvalues of M is equal to $-\phi < 0$. The other two, λ_1 and λ_2, are the solutions of the characteristic equation $\Omega(z) \equiv z^2 - m_{22} z + m_{32} = 0$, thus satisfying

$$\lambda_1 + \lambda_2 = m_{22} = \rho - \frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} a_{\mathrm{I}}^* \quad \text{and} \quad \lambda_1 \cdot \lambda_2 = m_{32} = -\frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} c_{\mathrm{I}}^*. \tag{14}$$

The signs of m_{22} and m_{32} indicate the stability of the steady-state equilibrium and can prove the next proposition.

Proposition 2 *Consider budgetary regime (I). Suppose that* $\gamma_{\mathrm{I}}^* = (1-\tau)r - \rho > 0$.

1. *If* $\bar{b} < k_g$, *the steady state is locally saddle stable and exhibits local determinacy.*

2. *If $\bar{b} > k_g$, the following hold:*

   (a) *Let $\tilde{b} \equiv \frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r}k_g$. When $\tau > r/\omega$,*

      i. *if $k_g < \bar{b} < \tilde{b}$, the steady state is locally stable and exhibits local indeterminacy.*

      ii. *if $\bar{b} > \tilde{b}$, the steady state is locally unstable.*[10]

   (b) *When $\tau < r/\omega$, the steady state is locally unstable.*

(Proof) See Appendix A.

Proposition 2 shows that when $\bar{b}$ is large ($\bar{b} > k_g$), the steady state becomes unstable under some conditions. However, this does not mean that there is no equilibrium. Proposition 3 shows the possibility of an equilibrium that exhibits a limit cycle.

Proposition 3 *Consider budgetary regime (I). Suppose that $\gamma_{\mathrm{I}}^* = (1-\tau)r - \rho > 0$ and $\bar{b} > k_g$. In addition, suppose that $r/\omega < 1$ and $\hat{b}_- < \tilde{b} < \hat{b}_+$ and that $\bar{b}$ is sufficiently close to $\tilde{b}$, where $\hat{b}_-$ and $\hat{b}_+$ are as defined in Appendix B. Then, there exists at least a limit cycle around the steady state, and the steady state exhibits equilibrium indeterminacy.*

(Proof) See Appendix B.

The properties of the equilibrium path heavily depend on $\bar{b}$. When $\bar{b}$ is sufficiently small to satisfy $\bar{b} < k_g$, the equilibrium path is uniquely determined. However, when $\bar{b}$ takes a moderate value ($k_g < \bar{b} < \tilde{b}$) and $\tau > r/\omega$ holds, the steady state is locally stable and exhibits indeterminacy. In other cases, the steady state is unstable. However, as proven in Proposition 3, there exists a limit cycle around the steady state. In Figures 1 and 2, we illustrate a numerical example of a stable limit cycle.[11] In this example, indeterminacy arises because there are multiple equilibrium paths that converge to a stable limit cycle.

[10]Note that $\tilde{b}$ is larger than k_g if $\tau > r/\omega$ holds. See Appendix A.

[11]In this example, we assume the Cobb–Douglas production function: $Y_t = K_t{}^{\alpha}(K_{g,t}L_t)^{1-\alpha}$. We use the following parameters: $\alpha = 0.36$, $r = 0.03$, $\rho = 0.01$, $\phi = 0.05$, and $\tau = 0.3$. Under these parameters, we have $\omega \approx 2.59$, $k_g \approx 0.25$, $\hat{b}_- \approx 2.01$, $\hat{b}_+ \approx 238.61$, and $\tilde{b} \approx 6.612$. We then assume that $b_0 \approx 6.613$. These values satisfy $1 > \tau > r/\omega$, $\bar{b} > k_g$, and $\hat{b}_- < \tilde{b} < \hat{b}_+$.

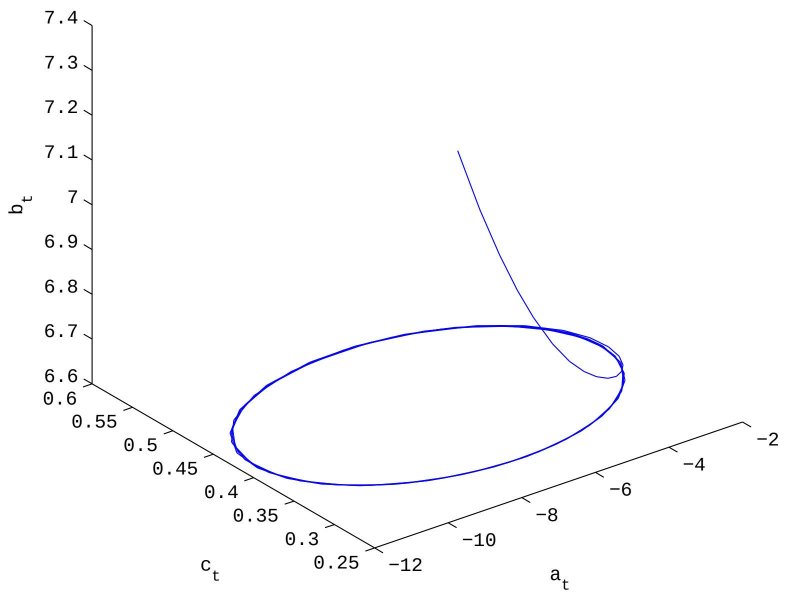

Figure 1: An Example of Limit Cycle

The intuition behind indeterminacy is as follows: Suppose that households expect future increases in G_t. Because increases in G_t have positive effects on output growth and labor wage (see (4) and (5b)), this expectation implies that households also expect future increases in output growth and labor income. Households then have a lesser incentive to save, and a_t thus decreases in the future. As a result, the government's tax revenue decreases, which tightens its budget constraint. As discussed just below (9), the government—which is constrained with tight budgets—has two ways to reduce $b_t \equiv B_t/Y_t$ according to (7): one is to reduce the debt, B_t, which requires reductions in public investments, G_t, and the other is to increase public investments, G_t, in order to stimulate output growth. When $b_t > k_g$ holds, enhancing output growth is an easier way to reduce b_t than paying back B_t. Then, if $\bar{b} > k_g$, households' expectations are self-fulfilling and equilibrium indeterminacy arises. In the numerical example of the limit cycle in Figure 2, government spending and private asset holdings move in opposite directions, which is

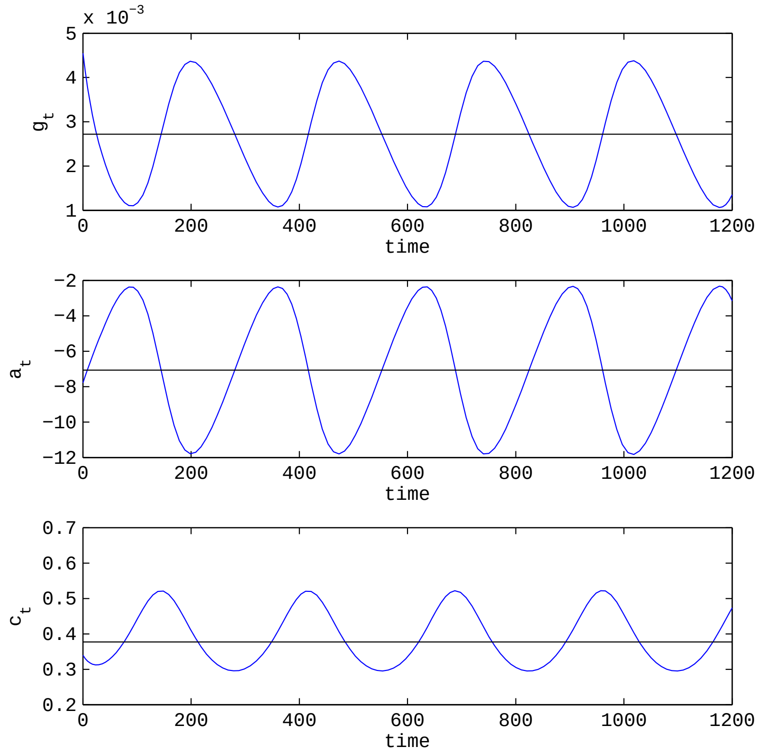

Figure 2: Fluctuations of (g_t, a_t, c_t)

consistent with this interpretation for indeterminacy. When $b_t < k_g$, the government can reduce b_t easier by paying back B_t rather than by enhancing output growth. Then, if $\bar{b} < k_g$, expectations of future increases in G_t are not self-fulfilling. Hence, the steady state exhibits equilibrium determinacy.

Proposition 2 also shows that the equilibrium determinacy condition is independent of ϕ. Intuitively, equilibrium determinacy depends only on the *directions* of the simultaneous movement of (b_t, a_t, c_t) as explained above. Because adjustment *speed* ϕ does not change the directions, ϕ does not affect equilibrium determinacy.

Maebayashi et al. (2013) show that in a closed economy where the government follows (7) and adopts regime (I), equilibrium indeterminacy does not arise. Consider a closed economy where public capital is productive. As in our open economy, increases in public investments have positive growth effects. Concurrently, increases in public investments

year	Italy	Netherlands	Norway	Portugal	Spain	Sweden	Switzerland	UK
1990	49.0	68.9	52.5	32.0	40.9	40.2	48.4	48.5
1991	49.7	67.8	52.4	32.2	43.1	41.5	50.2	49.6
1992	50.6	66.8	52.5	33.4	45.9	43.1	51.7	49.8
1993	52.1	66.9	52.7	36.4	48.8	45.0	53.4	49.1
1994	51.6	65.2	51.3	38.5	49.9	44.2	54.4	47.5
1995	50.4	64.3	50.6	38.9	50.7	44.1	55.5	46.8
1996	50.1	62.9	49.4	39.5	51.2	44.9	56.4	46.0
1997	49.4	61.2	48.4	40.5	50.3	45.0	56.3	44.4
1998	48.9	59.2	48.9	41.4	49.3	44.0	55.7	42.9
1999	48.6	57.6	50.1	42.3	48.7	42.8	55.6	41.8
2000	47.9	56.4	50.5	43.3	48.0	42.0	54.7	40.3
2001	47.5	56.7	50.9	44.4	47.8	42.0	54.7	39.3
average	49.7	62.8	50.9	38.6	47.9	43.2	53.9	45.5

Table 1: Real government net capital stocks as a percentage of real GDP (data source: Database on Capital Stocks in OECD Countries of Kiel Institute for the World Economy)

crowd out private investments, which counteracts the positive growth effects. Then, the growth effects of increases in public investments become weaker compared to those in our small open economy. The government—constrained with tight budgets—has only one choice: to cut public investments to pay back its debt. Households' expectations of increases in G_t are not self-fulfilling, and thus indeterminacy does not arise in a closed economy. In contrast, because an *open* economy can import capital from abroad and because economic activities in a *small* economy do not affect the world resource constraint, increases in public investments do not necessarily crowd out private investments in a *small open* economy and can thus have stronger growth effects. As such, in our small open economy, equilibrium indeterminacy can arise under some conditions.

Finally, we refer to an empirical implication of the determinacy condition: $\bar{b} < k_g$.

Table 1 shows the real government net capital stocks as a percentage of real GDP from 1990 to 2001 in several countries in the Euro area.[12] On average, except in the

[12]Constructing new data is not the scope of this paper. Therefore, we use existing data provided by the Database on Capital Stocks in OECD Countries of Kiel Institute for the World Economy. It is downloadable at http://www.ifw-kiel.de/forschung/Daten/netcap. Because only data up to 2001 are available for our purpose, we focus on data from 1990 to 2001. For details on data construction, see Kamps (2006).

Netherlands, the 60% criterion level of the debt–GDP ratio proposed by the SGP may not be sufficiently low to ensure equilibrium determinacy. Besides, this result holds stably in that term.

3.2 Welfare Analysis

To examine the welfare effects of reductions in $\bar{b}$ under budgetary regime (I), we consider the following scenario. The economy is initially in the steady-state equilibrium characterized by $b_{\mathrm{I}}^* = \bar{b}$ and (11a)–(11c). At time 0, the government unexpectedly reduces $\bar{b}$ marginally. After marginal reductions in $\bar{b}$, if the new steady state is stable, the economy begins to move toward the new steady state. If the steady state exhibits local indeterminacy, the transitional path is not unique and welfare effects of this policy change cannot be examined. Hence, here, we assume $\bar{b} < k_g$ to ensure determinacy. However, we analyze the case of indeterminacy in an extended model where the interest rate depends on the fiscal conditions of the economy. See Appendix G.

Our welfare measure is (1). Because $C_t = C_0 e^{\{(1-\tau)r-\rho\}t} = c_0 f(x) K_{g,0} e^{\{(1-\tau)r-\rho\}t}$, we can rewrite (1) as

$$U_0 = \frac{1}{\rho} \ln c_0 + \frac{1}{\rho} \left\{ \ln f(x) K_{g,0} + \frac{(1-\tau)r - \rho}{\rho} \right\}. \tag{15}$$

When the government unexpectedly reduces $\bar{b}$ at time 0, c_0 jumps to its new value just at time 0; however, other variables in (15) do not change. Then, marginal reductions in $\bar{b}$ affect household welfare only through c_0. Because $\bar{b} < k_g$ holds and the steady state is saddle-stable, we can consider $\lambda_1 < 0$ and $\lambda_2 > 0$ as solutions of the characteristic equation $\Omega(z) = 0$, as shown in Subsection 3.1. Appendix C shows that using $\lambda_2 > 0$, c_0 can be expressed as

$$c_0 = c_{\mathrm{I}}^* + (b_0 - \bar{b})v_3 + \{a_0 - a_{\mathrm{I}}^* - (b_0 - \bar{b})v_2\}\lambda_2, \tag{16}$$

where v_2 and v_3 are defined in (C.3a) and (C.3b).

We now explain our scenario in more detail. Initially, $\bar{b}$ is set equal to $\bar{b}^{init}$. We denote the initial steady-state value of a_t as a_{I}^{init}. Because the economy is initially in the steady-state equilibrium, we have $b_0 = \bar{b}^{init}$ and $a_0 = a_{\mathrm{I}}^{init}$ in (16). Denote the criterion level of b_t and the steady-state value of a_t after the policy change as $\bar{b}^{new}$ and a_{I}^{new}, respectively. Because we consider marginal changes, $\bar{b}^{new}$ and a_{I}^{new} are approximately equal to $\bar{b}^{init}$ and a_{I}^{init}, respectively. This generates $b_0 = \bar{b}^{init} \approx \bar{b}^{new}$ and $a_0 = a_{\mathrm{I}}^{init} \approx a_{\mathrm{I}}^{new}$. After the policy change, we have $\bar{b} = \bar{b}^{new}$ and $a_{\mathrm{I}}^{*} = a_{\mathrm{I}}^{new}$ in (16), which implies that $b_0 - \bar{b} = \bar{b}^{init} - \bar{b}^{new} \approx 0$ and $a_0 - a_{\mathrm{I}}^{*} = a_{\mathrm{I}}^{init} - a_{\mathrm{I}}^{new} \approx 0$. Then, the effect of marginal changes in $\bar{b}$ can be expressed as

$$\begin{aligned}
\left.\frac{\partial c_0}{\partial \bar{b}}\right|_{(b_0,a_0)\approx(\bar{b}^{new},a_{\mathrm{I}}^{new})} &= \frac{\partial c_{\mathrm{I}}^{*}}{\partial \bar{b}} - \frac{\partial a_{\mathrm{I}}^{*}}{\partial \bar{b}}\lambda_2 + v_2\lambda_2 - v_3, \\
&= \frac{\lambda_2 - \rho}{\tau r}\left\{\frac{\lambda_2(\phi+\rho+\tau r)}{\phi+\lambda_2} - (\rho+\tau r)\right\} \equiv \Psi(\phi). \qquad (17)
\end{aligned}$$

The calculation procedure from the first to the second lines in (17) is presented in Appendix D. We can easily show that $\lambda_2 - \rho > 0$ and $\Psi(0) = 0$.[13] Differentiating $\Psi(\phi)$, we obtain

$$\Psi'(\phi) = \frac{\lambda_2(\lambda_2-\rho)}{\tau r(\phi+\lambda_2)}\{\lambda_2 - (\rho+\tau r)\}.$$

We can show that if $r < (>)\omega$ holds, we have $\rho + \tau r < (>)\lambda_2$.[14] Then, if $r < (>)\omega$, $\Psi(\phi) > (<)0$ and $\Psi'(\phi) > (<)0$ for $\phi > 0$. From (15), we obtain $\partial U_0/\partial \bar{b}\big|_{(b_0,a_0)\approx(\bar{b}^{new},a_{\mathrm{I}}^{new})} = \Psi(\phi)/\rho c_{\mathrm{I}}^{*}$. Because ϕ has no effect on c_{I}^{*}, the discussion so far yields the following proposition.

Proposition 4 *Consider budgetary regime (I). Suppose that* $\gamma_{\mathrm{I}}^{*} = (1-\tau)r - \rho > 0$ *and* $\bar{b} < k_g$ *and that the economy is initially in the steady-state equilibrium.*

[13] λ_2 is a solution of $\Omega(z) = 0$. $0 < z < \lambda_2$ $(z > \lambda_2)$ if $\Omega(z) < (>)0$ holds for $z > 0$. Because $\Omega(\rho) = \rho^2 - m_{22}\rho + m_{32} = -(1-\tau)\omega k_g \tau r/(k_g - \bar{b}) < 0$, we have $\lambda_2 - \rho > 0$.

[14] As in footnote 13, $0 < z < \lambda_2$ $(z > \lambda_2)$ holds if $\Omega(z) < (>)0$ holds for $z > 0$. We have $\Omega(\rho+\tau r) = \tau r k_g(r-\omega)/(k_g - \bar{b})$. Because $\bar{b} < k_g$ is assumed, $\rho + \tau r < (>)\lambda_2$ if $r < (>)\omega$.

(a) If $r < \omega$ holds, marginal reductions in $\bar{b}$ reduce households' welfare. As $\phi > 0$ becomes smaller, reductions in household welfare also become smaller.

(b) If $r > \omega$ holds, marginal reductions in $\bar{b}$ raise households' welfare. As $\phi > 0$ increases, increases in household welfare also become larger.

The welfare effects of reductions in $\bar{b}$ depend on the relationship between r and ω. When $r > \omega$, reductions in $\bar{b}$ improve welfare. Furthermore, the value of ϕ provides an important implication. When $r > \omega$, the government can further improve welfare by reducing b_t at a faster pace. In contrast, when $r < \omega$, the government can mitigate welfare losses by reducing b_t at a slower pace.

To interpret Proposition 4 intuitively, we rewrite households' income, $I_t = rA_t + w_t$, as $I_t = rA_t + \omega K_{g,t}$ by using (5b). Households' income depends on their assets, A_t, and public capital, $K_{g,t}$. A_t accumulates through savings of the private sector (households), whereas $K_{g,t}$ accumulates through public investments by the government. We can consider public investments as savings of the public sector. Consequently, r and ω can be considered the rates of return on private and public savings, respectively. If $r < \omega$ holds, accumulating assets through *public* savings is socially more efficient. Because reductions in $\bar{b}$ have negative effects on public investments (savings) when $\bar{b} < k_g$ holds as discussed below (9), reductions in $\bar{b}$ reduce households' welfare. When ϕ is small, the initial decline in γ_t is also small, which mitigates the negative welfare effects. When $r > \omega$ holds, accumulating assets through public savings is not socially efficient. Because reductions in $\bar{b}$ depress public investments (savings), welfare improves. When ϕ is large, the initial decline in γ_t is also large. Then, public investments are further depressed and households' welfare further improves.

4 Tax Adjustments: Regime(II)

We now move onto budgetary regime (II), where the government sets a constant $\gamma(= g/k_g)$ and adjusts τ_t to satisfy (7) and (8a). Under regime (II), it will be shown that there exists a unique unstable steady state. This is a new result by considering a small open economy because Maebayashi et al. (2013) have shown that the steady state is stable under regime (II) in the closed economy.

We first derive the dynamic system. Eliminating $\dot{b}_t$ from (7) and (8a) and solving for τ_t, we obtain

$$\tau_t = \frac{(r-\gamma)b_t + k_g\gamma + \phi(b_t - \bar{b})}{ra_t + \omega k_g} \equiv \tau(a_t, b_t). \tag{18}$$

When the government reduces $\bar{b}$, τ_t increases in the short run because the government must earn a larger primary surplus to reduce b_t. The effect on τ_t becomes stronger as ϕ increases and gradually disappears as b_t approaches $\bar{b}$. The following two points should be mentioned. First, when $a_t = -\omega k_g/r$, τ_t cannot be defined; thus, the optimization problem of the household is not well-defined. Then, the transitional paths along which a_t moves across $-\omega k_g/r$ must not be equilibrium. Second, when a_t is so small that household income $(ra_t + \omega k_g)Y_t$ is negative, τ_t tends to be negative; thus, household tax payments $\tau_t(ra_t + \omega k_g)Y_t$ tend to be positive despite that household income is negative. Then, the representative household must borrow from abroad to make tax payments beyond its income. In other words, the government forces the household to borrow from abroad in order to meet the budget constraint of the government. Although the case makes little sense in practice, it is theoretically possible and provides important implications as we later show. Substituting (18) into (8b) and (8c) yields

$$\dot{a}_t = (r-\gamma)(a_t - b_t) - \phi(b_t - \bar{b}) - c_t + (\omega - \gamma)k_g, \tag{19a}$$

$$\dot{c}_t = \{(1-\tau(a_t, b_t))r - \rho - \gamma\}c_t. \tag{19b}$$

The dynamic system is given by (7), (19a), and (19b) along with b_0 and a_0.

We next derive the steady-state equilibrium, where $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ holds. We set $\dot{b}_t = \dot{a}_t = \dot{c}_t = 0$ in (7), (18), (19a), and (19b), and we solve for b_t, a_t, c_t, and τ_t. The steady-state values of b_t, a_t, c_t, and τ_t are respectively given by $b^*_{\mathrm{II}} = \bar{b}(> 0)$,

$$a^*_{\mathrm{II}} = \frac{(r-\gamma)\bar{b} + \gamma k_g}{r-\rho-\gamma} - \frac{\omega k_g}{r}, \tag{20a}$$

$$c^*_{\mathrm{II}} = \frac{(r-\gamma)\bar{b} + \gamma k_g}{r-\rho-\gamma}\rho + \frac{\gamma\omega k_g}{r}, \tag{20b}$$

$$\tau^*_{\mathrm{II}} = \frac{r-(\rho+\gamma)}{r}. \tag{20c}$$

From (20a), households' income is given by $(ra^*_{\mathrm{II}} + \omega k_g)Y_t = \frac{(r-\gamma)\bar{b}+\gamma k_g}{r-\rho-\gamma} rY_t$ at the steady state. If $\gamma < r - \rho$ holds, the steady-state income level becomes strictly positive. This inequality ensures that $\tau^*_{\mathrm{II}} \in (0, 1)$ and $c^*_{\mathrm{II}} > 0$. Accordingly, the remainder of this section assumes that $\gamma < r - \rho$. Because $\dot{A}_t/A_t = \dot{B}_t/B_t = \gamma$ holds in the steady state and (20c) implies $(1 - \tau^*_{II})r - \gamma = \rho > 0$, the TVC of the household and the NPG condition of the government are both satisfied.

As shown in Appendix E, the Jacobian matrix of the linearized dynamic system has only one stable root, $-\phi$. Because there are two state variables, b_t and a_t, the economy cannot approach the steady-state equilibrium unless b_0 and a_0 are respectively equal to $\bar{b}$ and a^*_{II} by chance. We thus obtain the next proposition.

Proposition 5 *Consider budgetary regime (II). Suppose that $\gamma < r - \rho$. There exists a unique steady-state equilibrium where $b^*_{\mathrm{II}} = \bar{b}$ and* (20a)–(20c) *hold. Unless b_0 and a_0 are equal to $\bar{b}$ and a^*_{II}, respectively, the economy cannot approach the steady-state equilibrium.*

The intuition of Proposition 5 is as follows. We focus on the marginal rate of intertemporal substitution in the Euler equation of the household. Under regime (II), the income tax rate is inversely related to asset holding of the household. When a_t is lower (higher) than that of the steady state, the income tax rate is raised (lowered) to finance the constant government spending. This lowers (raises) the rate of return of savings. Then, the household decreases (increases) asset holdings and a_t gets away from the steady state. Therefore, the steady state is unstable in regime (II).

Because the economy cannot reach the unstable steady-state equilibrium, it makes little sense to continue studying the characteristics of the steady state. However, our model provides other important implications. Using (7), we solve (19a) and then rearrange the solution using inequality $c_t > 0$:[15]

$$a_t < b_t + \left\{ (a_0 - b_0) + \frac{\omega - \gamma}{r - \gamma} k_g \right\} e^{(r-\gamma)t} - \frac{\omega - \gamma}{r - \gamma} k_g. \tag{21}$$

Note that the debt rule, (7), ensures $\lim_{t \to +\infty} b_t = \bar{b}$ and $r - \gamma > 0$ is assumed. Suppose that $a_0 - b_0$ is sufficiently small so that $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$. Then, (21) implies $\lim_{t \to \infty} a_t = -\infty$. If $a_0 > -\omega k_g/r$ holds, a_t necessarily moves across $-\omega k_g/r$. As discussed below (18), such transitional paths cannot be an equilibrium, because τ_t is not defined at $a_t = -\omega k_g/r$ and the households' optimization problem is not well-defined.

It is also shown that even if $a_0 < -\omega k_g/r$ holds and a_t does not move across $-\omega k_g/r$ along the transition, there exists no equilibrium. As shown in (18), τ_t tends to zero as a_t tends to $-\infty$. Then, for a large t, the discounted sum of households' labor income can be written as

$$\int_t^{+\infty} (1 - \tau_v) w_v e^{-\int_t^v (1-\tau_s) r ds} dv = \int_t^{+\infty} \omega k_g Y_v e^{-r(v-t)} dv = \frac{\omega k_g}{r - \gamma} Y_t > 0. \tag{22}$$

In the first equality, we use $\tau_t = 0$ for a large t, (5b), and the definition of k_g. The second equality uses $\dot{Y}_t / Y_t = \gamma$ and $r > \gamma$. The assumption $\gamma < r - \rho (< r)$ ensures the last inequality. Because $a_t = -\infty$ for a large t, the inequality $-a_t > \omega k_g/(r - \gamma)$ holds for a large t. From (22) and the definition of a_t, the following inequality is obtained for a large t:

$$-A_t > \int_t^{+\infty} (1 - \tau_v) w_v e^{-\int_t^v (1-\tau_s) r ds} dv.$$

This inequality implies that for a large t, households' borrowing eventually exceeds the discounted sum of their labor income, and households would thus be unable to pay off

[15] Appendix G provides the derivation of (21).

their borrowing. Therefore, rational expectations and perfect foresight ensure that foreign agents do not lend to such households. The next remark summarizes the discussion so far.

Remark

Consider budgetary regime (II). Suppose that $\gamma < r - \rho$ *and* $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$. *Then, there exists no equilibrium such that the government can follow the debt-reduction rule,* (7).

Inequality $a_0 - b_0 < -(\omega - \gamma)k_g/(r - \gamma)$ indicates that households' initial asset holdings tend to be smaller than the government's initial outstanding debt, which means that private domestic savings cannot absorb the outstanding government debt; thus, the government borrows from foreign investors. Our result implies that in such a case, under regime (II), households eventually become overextended with foreign debt. Therefore, there exists no equilibrium where the government can follow the debt-reduction rule, (7).

Maebayashi et al. (2013) show that in a closed economy where the government follows (7) and adopts budgetary regime (II), the unique steady state is always saddle stable and the economy converges to the steady state where the debt–GDP ratio remains constant over time. Then, there always exists an equilibrium where the government can follow the debt-reduction rule, (7), if it is a closed economy. To reduce its debt–GDP ratio, the government imposes a high tax rate on the interest income under regime (II) (see (18)), which discourages household savings and negatively affects a_t. However, households in a closed economy cannot borrow from abroad, and hence, a_t cannot be smaller than b_t. Therefore, households in a closed economy do not lose the ability to pay back debt. In contrast, because households in an *open* economy can borrow from abroad and because households in a *small* economy do not affect the world asset market, households in a *small open* economy are at risk of becoming overextended with foreign debt (although this does not happen in equilibrium).

5 Concluding Remarks

We examine how the debt-reduction rule found in the SGP affects the dynamics of the economy and welfare in a small open economy model where the government can borrow from abroad to finance its debt. Public capital accumulated by public investments has positive externalities on goods production.

In budgetary regime (I), where the government controls public spending to follow a debt policy rule, there exists a unique steady state equilibrium. When the long-run debt–GDP ratio is sufficiently low (high), the steady state exhibits (in)determinacy. Focusing on the case where the steady state exhibits determinacy, we obtain the following welfare implications. If the rate of return from private savings is larger (smaller) than that from public savings, reductions in government debt improve (reduce) welfare and the government can realize (may suffer) larger welfare gains (losses) by reducing its debt at a faster pace. In budgetary regime (II), where the government controls the income tax rate to follow a debt policy rule, the steady state is always unstable. If the initial asset holdings of households are smaller than the initial outstanding government debt, there exists no equilibrium where the government can follow the debt-reduction rule, (7).

Appendix

A Proof of Proposition 2

1. If $\bar{b} < k_g$ holds, we have $m_{32} < 0$ because $c_{\mathrm{I}}^* > 0$ and $\partial\gamma(a_I^*, \bar{b})/\partial a_t = \tau r/(k_g - \bar{b}) > 0$ (see (13)). From the second equation of (14), one of the eigenvalues of the Jacobian matrix in (12) is negative while the other is positive. Then, the steady state is locally saddle stable and exhibits local determinacy.

2. If $\bar{b} > k_g$ holds, $m_{32} > 0$ (see (14) and (13)). Using (11a) and (13), we rewrite m_{22} as

$$m_{22} = \frac{\{(1-\tau)r - 2\rho - \tau\omega\}k_g + (2\rho + \tau r)\bar{b}}{\bar{b} - k_g}. \tag{A.1}$$

(a) If $\tau > r/\omega$ holds, $(1-\tau)r - 2\rho - \tau\omega$ takes a negative value. Let $\tilde{b} \equiv \frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r}k_g$. Then, if $k_g < \bar{b} < \tilde{b}$ holds, $m_{22} < 0$. Inequality $\tau > r/\omega$ ensures that $\frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r} = 1 + \frac{\tau\omega - r}{2\rho+\tau r} > 1$ and thus ensures the existence of a $\bar{b}$ that satisfies $k_g < \bar{b} < \tilde{b}$. From $m_{22} < 0$, $m_{32} > 0$, and (14), we know that the real parts of both λ_1 and λ_2 are negative. The steady state is locally stable and exhibits local indeterminacy.

If $\bar{b} > \tilde{b}$ holds, $\bar{b} > k_g$ because inequality $\tau > r/\omega$ ensures that $\frac{2\rho+\tau\omega-(1-\tau)r}{2\rho+\tau r} = 1 + \frac{\tau\omega - r}{2\rho+\tau r} > 1$. Then, $m_{22} > 0$ and $m_{32} > 0$, which implies that the real parts of the two eigenvalues are positive.

(b) We now examine the signs of m_{22} and m_{32} in the case of $\tau < r/\omega$ and $\bar{b} > k_g$. $m_{32} > 0$ holds. Because $\bar{b} > k_g$, we have

$$\begin{aligned} m_{22} &> \frac{\{(1-\tau)r - 2\rho - \tau\omega\}k_g + (2\rho + \tau r)k_g}{\bar{b} - k_g} = \frac{r - \tau\omega}{\bar{b} - k_g}k_g \\ &> 0. \end{aligned}$$

Then, $m_{22} > 0$ and $m_{32} > 0$, which implies that the real parts of the two eigenvalues are positive.

B Proof of Proposition 3

We show the existence of limit cycles by applying the Hopf bifurcation theorem.[16] Because Jacobian matrix M (defined in (12)) has one negative eigenvalue, $-\phi$, the dynamic system can exhibit Hopf bifurcation if its submatrix

$$M' \equiv \begin{pmatrix} \rho - \frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t} a_{\mathrm{I}}^* & -1 \\ -\frac{\partial\gamma(a_I^*, \bar{b})}{\partial a_t} c_{\mathrm{I}}^* & 0 \end{pmatrix}$$

[16] To see the Hopf bifurcation theorem in detail, refer to Theorem 3.4.2 (pp. 151) of Guckenheimer and Holmes (1983).

has a couple of complex eigenvalues. A necessary and sufficient condition for complex eigenvalues is $(trM')^2 - 4detM' < 0$, that is,[17]

$$\left(\rho - \frac{\tau r}{k_g - \bar{b}} a_{\mathrm{I}}^*\right)^2 < -4 \frac{\tau r}{k_g - \bar{b}} c_{\mathrm{I}}^*. \tag{B.1}$$

Substituting (11a) and (11b) into (B.1), we obtain

$$\left[\rho - \frac{(\gamma_{\mathrm{I}}^* - \tau\omega)k_g + (\rho + \tau r)\bar{b}}{k_g - \bar{b}}\right]^2 < -4 \frac{\gamma_{\mathrm{I}}^*(\rho + \tau\omega)k_g + \rho(\rho + \tau r)\bar{b}}{k_g - \bar{b}}.$$

Multiplying both sides by $(k_g - \bar{b})^2$ and arranging it as a polynomial of $\bar{b}$, we have

$$(\tau r)^2 \bar{b}^2 - 2\tau r\big[(1-\tau)(\omega - r) + \omega\big]k_g \bar{b} + \big[(1-\tau)r + \tau\omega\big]^2 k_g^2 < 0. \tag{B.2}$$

The critical values of quadratic inequality (B.2) are

$$\hat{b}_- \equiv \frac{(1-\tau)(\omega - r) + \omega - 2\sqrt{(1-\tau)(\omega - r)\omega}}{\tau r},$$
$$\hat{b}_+ \equiv \frac{(1-\tau)(\omega - r) + \omega + 2\sqrt{(1-\tau)(\omega - r)\omega}}{\tau r},$$

where these real numbers are well-defined under the assumption that $r/\omega < 1$. Thus, a necessary and sufficient condition for complex eigenvalues is $\hat{b}_- < \bar{b} < \hat{b}_+$. M' has complex eigenvalues because $\hat{b}_- < \tilde{b} < \hat{b}_+$. The real part of these eigenvalues, $\frac{detM'}{2} = \frac{m_{22}}{2}$, equals zero only when $\bar{b} = \tilde{b}$; it is also differentiable and increasing in $\bar{b}$. Thus, $\tilde{b}$ is the unique bifurcation value of parameter $\bar{b}$, and submatrix M' has a couple of complex eigenvalues in any sufficiently small neighborhood of $\tilde{b}$. Therefore, by the Hopf bifurcation theorem, there exists a sufficiently small $\varepsilon > 0$ such that either of the following cases is true: (i) there exists a stable limit cycle for every $\bar{b}$ such that $\tilde{b} < \bar{b} < \tilde{b} + \varepsilon$, or (ii) there exists an unstable limit cycle for every $\bar{b}$ such that $\tilde{b} - \varepsilon < \bar{b} < \tilde{b}$. Because any solution emerging

[17] This inequality requires $\bar{b} > k_g$, and holds even in case 2.(a).ii of Proposition 2 where $\tau > \frac{r}{\omega}$ and $\bar{b} > \tilde{b}$, as mentioned previously.

from the neighborhoods of the cycle winds around it in case (i) and remains in the cycle (asymptotically on the (a_t, c_t) plane) in case (ii), there are innumerable initial values of c_t consistent with TVC in each case. This implies the multiplicity of equilibria. The proof of Proposition 3 is thus complete.

C Derivation of c_0

We derive the saddle path that converges to the steady state we consider. Because $\bar{b} < k_g$ holds, we can assume $\lambda_1 < 0$ and $\lambda_2 > 0$. Because $m_{32} < 0$ holds in the case of $\bar{b} < k_g$ (see Appendix A), $\sqrt{{m_{22}}^2 - 4m_{32}} > |m_{22}|$. Then, λ_1 and λ_2 are given by

$$\lambda_1 = \frac{m_{22} - \sqrt{{m_{22}}^2 - 4m_{32}}}{2} < 0 \quad \text{and} \quad \lambda_2 = \frac{m_{22} + \sqrt{{m_{22}}^2 - 4m_{32}}}{2} > 0. \tag{C.1}$$

Given b_0 and a_0, we solve (12) and obtain

$$\begin{pmatrix} b_t \\ a_t \\ c_t \end{pmatrix} = \begin{pmatrix} b_{\mathrm{I}}^* \\ a_{\mathrm{I}}^* \\ c_{\mathrm{I}}^* \end{pmatrix} + \theta_1 \begin{pmatrix} 1 \\ v_2 \\ v_3 \end{pmatrix} e^{-\phi t} + \theta_2 \begin{pmatrix} 0 \\ 1 \\ \lambda_2 \end{pmatrix} e^{\lambda_1 t}. \tag{C.2}$$

In (C.2), $\theta_1 \equiv b_0 - \bar{b}$ and $\theta_2 \equiv a_0 - a_{\mathrm{I}}^* - (b_0 - \bar{b})v_2$ are determined by the initial condition. Vectors $(1 \ v_2 \ v_3)^T$ and $(0 \ 1 \ \lambda_2)^T$ are the eigenvectors associated with eigenvalues $-\phi$ and λ_1, respectively.[18] v_2 and v_3 are expressed as

$$v_2 = \frac{m_{11}m_{21} - m_{31}}{m_{11}(m_{11} - m_{22}) + m_{32}} = \frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} \frac{\phi a_{\mathrm{I}}^* + c_{\mathrm{I}}^*}{\phi^2 + m_{22}\phi + m_{32}} = \frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} \frac{\phi a_{\mathrm{I}}^* + c_{\mathrm{I}}^*}{(\phi + \lambda_1)(\phi + \lambda_2)}, \tag{C.3a}$$

$$v_3 = \frac{(m_{11} - m_{22})m_{31} + m_{21}m_{32}}{m_{11}(m_{11} - m_{22}) + m_{32}} = \frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} \frac{(\rho + \phi)c_{\mathrm{I}}^*}{\phi^2 + m_{22}\phi + m_{32}} = \frac{\partial \gamma(a_I^*, \bar{b})}{\partial b_t} \frac{(\rho + \phi)c_{\mathrm{I}}^*}{(\phi + \lambda_1)(\phi + \lambda_2)}. \tag{C.3b}$$

[18] To derive the eigenvector associated with λ_1, we use (14). Let us denote the eigenvector as $\hat{v}^T \equiv (\hat{v}_1 \ 1 \ \hat{v}_3)^T$. From $M\hat{v} = \lambda\hat{v}$, we obtain $\hat{v}_1 = 0$ and $\hat{v}_3 = m_{22} - \lambda_1 = m_{32}/\lambda_1$. Then, the two equations of (14) imply that $\hat{v}_3 = \lambda_2$.

In (C.3a) and (C.3b), we use (14), which implies that $\phi^2 + m_{22}\phi + m_{32} = (\phi + \lambda_1)(\phi + \lambda_2)$. From (C.2), c_0 is given by (16).

D Derivation of (17)

From (11a) and (11b), we have $\partial a_{\mathrm{I}}^* / \partial \bar{b} = (\rho + \tau r)/(\tau r)$ and $\partial c_{\mathrm{I}}^* / \partial \bar{b} = \rho \partial a_{\mathrm{I}}^* / \partial \bar{b}$. Using the definitions of v_2 and v_3, we rearrange $v_2 \lambda - v_3$ as

$$v_2 \lambda_2 - v_3 = \frac{\phi + \rho + \tau r}{(k_g - \bar{b})(\phi + \lambda_1)(\phi + \lambda_2)} \{(\rho - \lambda_2) c_{\mathrm{I}}^* + \phi(c_{\mathrm{I}}^* - \lambda_2 a_{\mathrm{I}}^*)\}. \tag{D.1}$$

Because λ_2 is a solution of $\Omega(z) = 0$, $\Omega(\lambda_2) = {\lambda_2}^2 - m_{22}\lambda_2 + m_{32} = 0$, which implies that

$$c_{\mathrm{I}}^* - \lambda_2 a_{\mathrm{I}}^* = \left(\frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} \right)^{-1} \lambda_2 (\lambda_2 - \rho),$$

where $\partial \gamma(a_I^*, \bar{b}) / \partial a_t$ is given by (13). In deriving this equation, we use the definitions of m_{22} and m_{32}. Substituting this equation into (D.1) yields

$$\begin{aligned} v_2 \lambda_2 - v_3 &= \frac{(\lambda_2 - \rho)(\phi + \rho + \tau r)}{\tau r (\phi + \lambda_1)(\phi + \lambda_2)} \left(\phi \lambda_2 - \frac{\partial \gamma(a_I^*, \bar{b})}{\partial a_t} c_{\mathrm{I}}^* \right), \\ &= \frac{(\lambda_2 - \rho)(\phi + \rho + \tau r)}{\tau r (\phi + \lambda_1)(\phi + \lambda_2)} \left(\phi \lambda_2 + m_{32} \right), \\ &= \frac{(\lambda_2 - \rho)\lambda_2(\phi + \rho + \tau r)}{\tau r (\phi + \lambda_2)}. \end{aligned}$$

On the second line, we use the definition of m_{32}. The second equation of (14) is used on the last line. Substituting the above equation, $\partial a_{\mathrm{I}}^* / \partial \bar{b} = (\rho + \tau r)/(\tau r)$ and $\partial c_{\mathrm{I}}^* / \partial \bar{b} = \rho \partial a_{\mathrm{I}}^* / \partial \bar{b}$ into the first line of (17) yields the second line of (17).

E Stability of the Steady State: Regime (II)

To examine the stability of the steady-state equilibrium, we linearize the dynamic system around the steady state characterized by $b^*_{\mathrm{II}} = \bar{b}$ and (20a)–(20c):

$$\begin{pmatrix} \dot{b}_t \\ \dot{a}_t \\ \dot{c}_t \end{pmatrix} = \begin{pmatrix} -\phi & 0 & 0 \\ -(r-\gamma+\phi) & r-\gamma & -1 \\ -\frac{r-\gamma+\phi}{ra^*_{\mathrm{II}}+\omega k_g} rc^*_{\mathrm{II}} & \frac{r-\rho-\gamma}{ra^*_{\mathrm{II}}+\omega k_g} rc^*_{\mathrm{II}} & 0 \end{pmatrix} \begin{pmatrix} b_t - \bar{b} \\ a_t - a^*_{\mathrm{II}} \\ c_t - c^*_{\mathrm{II}} \end{pmatrix}.$$

One of the eigenvalues of the Jacobian matrix is given by $-\phi < 0$. The other two, μ_1 and μ_2, satisfy $\mu_1 + \mu_2 = r - \gamma > 0$ and $\mu_1 \cdot \mu_2 = \frac{r-\rho-\gamma}{ra^*_{\mathrm{II}}+\omega k_g} rc^*_{\mathrm{II}} > 0$. These inequalities hold because $\gamma < r - \rho$ is assumed. The real parts of both μ_1 and μ_2 are positive. There are two state variables, b_t and a_t. Unless b_0 and a_0 are respectively equal to $\bar{b}$ and a^*_{II} by chance, the economy cannot achieve the steady-state equilibrium.

F Derivation of (21)

We define $z_t \equiv a_t - b_t$. Using (7) and the definition of z_t, we rewrite (19a) as $\dot{z}_t = (r-\gamma)z_t - c_t + (\omega-\gamma)k_g$. Given $z_0 \equiv a_0 - b_0$, the solution for this is

$$z_t = \left(z_0 + \frac{\omega-\gamma}{r-\gamma} k_g \right) e^{(r-\gamma)t} - \frac{\omega-\gamma}{r-\gamma} k_g - \int_0^t c_v e^{(r-\gamma)(t-v)} dv.$$

Because $c_t > 0$ holds for all $t \geq 0$, this equation implies the following inequality:

$$z_t < \left(z_0 + \frac{\omega-\gamma}{r-\gamma} k_g \right) e^{(r-\gamma)t} - \frac{\omega-\gamma}{r-\gamma} k_g.$$

Substituting $z_t = a_t - b_t$ into this equation yields (21).

G An Extention: Variable Interest Rate

We extend the model in the main text to include an interest rate that is affected by an endogenous variable representative of the fiscal conditions of the economy. Intuitively, even in a small open economy, an interest rate for the agents in the economy can vary according to fiscal conditions because the risk premium for debt depends on them.

Some works such as Chatterjee et al. (2003) incorporate it in the following way. Let $n_t = \frac{B_t - A_t}{Y_t}$. This is net foreign debt per GDP. We think of this as an indicator of fiscal conditions and interest rate r_t depends on n_t:

$$r_t = r(n_t) > 0,$$

where $r(\cdot)$ is a nondecreasing function. Following Chatterjee et al. (2003), we adopt the following specification of $r(\cdot)$ when we conduct numerical analyses:

$$r_t = r^* + \exp(\sigma n_t) - 1 \equiv r(n_t), \tag{G.1}$$

where r^* is a constant and parameter $\sigma > 0$ is response strength of the interest rate for the fiscal conditions. When $n_t > 0$, the economy borrows from abroad and then the interest rate for the economy rises from natural level r^*, which reflects a risk premium.

G.1 Model

We incorporate a positive rate of depreciation of private capital, denoted by δ_k, and adopt a Cobb–Douglas type production function. The profit of the representative firm is

$$\theta K_t^\alpha (K_{g,t} L_t)^{1-\alpha} - (r_t + \delta_k) K_t - w_t L_t. \tag{G.2}$$

By (G.2), the first-order conditions of the firm's problem are

$$r_t = \alpha \theta x_t^{\alpha - 1} - \delta_k, \tag{G.3}$$

$$w_t = (1 - \alpha) \theta x_t^\alpha K_{g,t}, \tag{G.4}$$

where $x_t = \frac{K_t}{K_{g,t}L_t}$. From (G.3) and (G.4), we have

$$x_t = \left[\frac{\alpha\theta}{r(n_t)+\delta_k}\right]^{\frac{1}{1-\alpha}} \equiv x(n_t).$$

Besides, public capital-to-GDP ratio is

$$k_{g,t} = \frac{K_{g,t}}{Y_t} = \frac{1}{\theta x(n_t)^{\alpha}} \equiv k_g(n_t).$$

Note that by the specification of $r(\cdot)$, the price system and other key variables depend on n_t and vary with time.

The government adopts the same debt policy rule:

$$\dot{b}_t = -\phi(b_t - \bar{b}),$$

and faces the flow-budget constraint

$$r_t B_t + G_t = \dot{B}_t + \tau(r_t A_t + w_t).$$

For numerical analyses, we assume a positive rate of depreciation for public capital, denoted by δ_g. The dynamic equation of public capital accumulation is

$$\dot{K}_{g,t} = G_t - \delta_g K_{g,t}.$$

G.2 Regime (I)

We first consider regime (I) where the government sets a constant $\tau \in (0,1)$ and adjusts public investment. As in the basic model, using the budget constraint of the representative household, we can obtain government expenditure per GDP as a function of (b_t, n_t, c_t):

$$
\begin{aligned}
g_t = g(b_t, n_t, c_t) = & \frac{k_g(n_t)}{k_g(n_t) - b_t + \frac{\alpha\varepsilon(n_t)}{1-\alpha}\frac{(b_t - n_t)k_g(n_t)}{n_t}} \Bigg[\Big[\tau(\omega - r(n_t)n_t) - \big((1-\tau)r(n_t) + \delta_g\big)b_t \\
& - \phi(b_t - \bar{b}) \Big] - \frac{\alpha\varepsilon(n_t)}{1-\alpha} \Big[\tau(\omega + r(n_t)(b_t - n_t) + \frac{c_t - \omega}{n_t} b_t - \phi(b_t - \bar{b}) \Big] \Bigg],
\end{aligned}
$$

where $\varepsilon(n_t) = \frac{r'(n_t)n_t}{r(n_t)+\delta_g}$.

Together with the Euler equation, the model is reduced to a dynamic system of b_t, n_t, and c_t. Through some long manipulations, we obtain the following differential equations:

$$
\begin{aligned}
\dot{b}_t &= -\phi(b_t - \bar{b}), \\
\dot{n}_t &= \Delta(n_t)\Gamma(b_t, n_t, c_t)n_t, \\
\dot{c}_t &= \big[(1-\tau)r(n_t) - \rho + (\Delta(n_t) - 1)\Gamma(b_t, n_t, c_t) - g(b_t, n_t, c_t)k_g(n_t)^{-1} + \delta_g\big]c_t,
\end{aligned}
$$

where $\Delta(n_t) = \big[1 - \frac{\alpha}{1-\alpha}\varepsilon(n_t)\big]^{-1}$ and $\Gamma(b_t, n_t, c_t) = r(n_t) + \frac{g(b_t,n_t,c_t)}{n_t} - \frac{g(b_t,n_t,c_t)}{k_g(n_t)} + \frac{c_t - \omega}{n_t} + \delta_g$.

G.3 Local Determinacy

Let (b^*, n^*, c^*) be the stationary point. It satisfies

$$
\begin{aligned}
& b^* = \bar{b}, \\
& \Gamma(b^*, n^*, c^*) = 0, \\
& (1-\tau)r(n^*) - \rho - g(b^*, n^*, c^*)k_g(n^*)^{-1} + \delta_g = 0.
\end{aligned}
$$

Because this nonlinear system has no closed-form solution, we hereafter conduct numerical analyses. We linearize the dynamical system around the stationary point and check the signs of the eigenvalues of the Jacobian matrix. If the number of eigenvalues with a negative real part is three, the stationary point is a sink. Equilibrium indeterminacy arises because the system includes just one jumpable variable, c_t.

We adopt a benchmark parameter value as follows. On the household's preference and the production function, we choose a popular value: $\rho = 0.05$, $\theta = 1$, $\alpha = 0.36$. As a standard income tax rate and interest rate, we set $\tau = 0.15$ and $r^* = 0.05$. Following Chatterjee et al. (2003), we set $\delta_k = 0.05$, $\delta_g = 0.04$. We analyze the model under various values of $(\phi, \bar{b}, \sigma)$ because these are key parameters. As a benchmark value, we adopt $(\phi, \bar{b}, \sigma) = (0.05, 1, 0.1)$. We follow Chatterjee et al. (2003) and choose $\sigma = 0.1$ in (G.1). According to the SGP, we set $\phi = 0.05$. Although the long-run target of b_t is 0.6 in the SGP, we adopt $\bar{b} = 1$. This is because the actual debt–GDP ratios in the EU countries are distributed around 1 (with slightly large variance) and such a value ensures some goodness of fit for this model. With this parameter set, consumption–output ratio is 0.584, capital–output ratio is 2.915, and growth rate is 0.020 in the steady state.

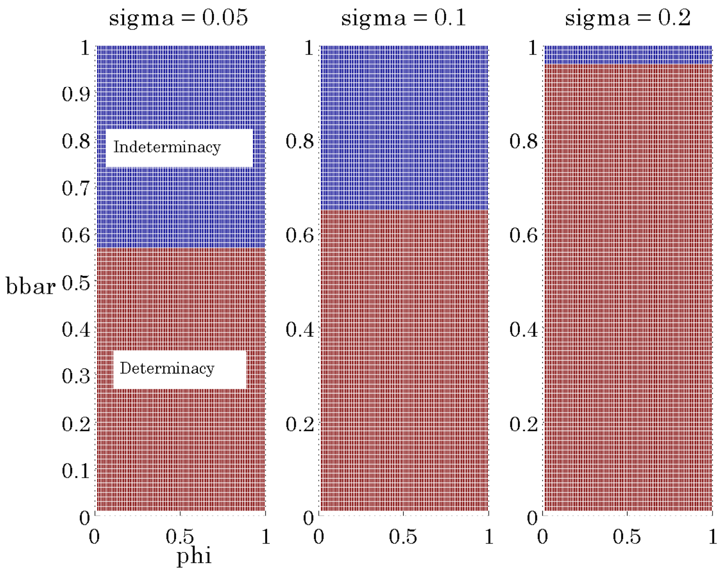

Figure 3: ϕ and equilibrium determinacy

First, we explore whether adjustment speed ϕ affects equilibrium determinacy under an endogenously varying interest rate. Figure 3 provides the answer: ϕ does not affect equilibrium determinacy. As in the original model of the main text, only the *directions* of the simultaneous movement of (a_t, b_t, g_t) are relevant to equilibrium determinacy. Therefore, adjustment *speed* ϕ does not affect equilibrium determinacy.

Second, we investigate the relationship between equilibrium determinacy and the re-

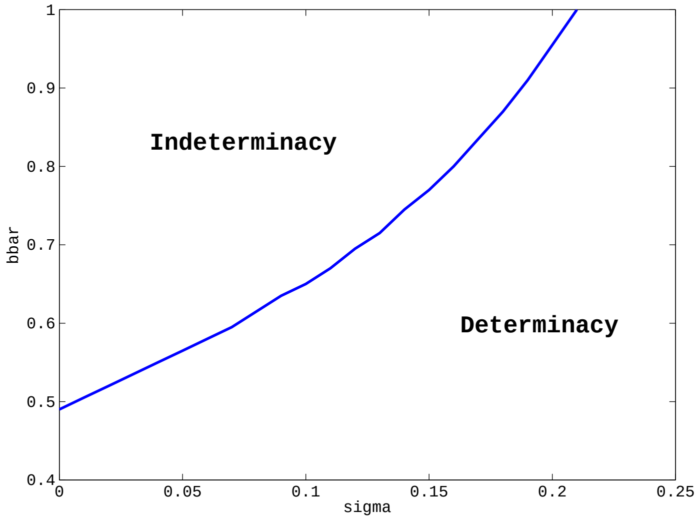

Figure 4: σ and equilibrium determinacy

sponse strength of the interest rate to fiscal conditions, σ. Figure 4 illustrates the same. As σ increases, the unique equilibrium area becomes larger. This result is very intuitive. Since one of the sources of indeterminacy in a small open economy is fixed interest rates, the existence of a variable component in the interest rate weakens this indeterminacy. However, more importantly, we should note that a variable interest rate does not eliminate equilibrium indeterminacy. Besides, for plausible values of σ, indeterminacy can arises under a not so high $\bar{b}$ in our model. The mechanism underlying indeterminacy is given in the main text, and is important when creating debt-policy rules.

G.4 Welfare

The lifetime utility of the representative household is

$$U = \int_0^\infty e^{-\rho t} \log C_t dt. \tag{G.5}$$

From the Euler equation, we have $C_t = C_0 e^{\int_0^t \gamma_{c,u} du}$, where $\gamma_{c,u} = (1-\tau)r(n_u) - \rho$. By $C_0 = c_0 Y_0$ and $Y_0 = \theta x_0^\alpha K_{g,0}$, we have $C_t = c_0 \theta x(n_0)^\alpha K_{g,0} e^{\int_0^t \gamma_{c,u} du}$. Substituting this into

(G.5) and differentiating it with respcet to $\bar{b}$, we obtain[19]

$$\frac{\partial U}{\partial \bar{b}} = \frac{1}{\rho c_0}\frac{\partial c_0}{\partial \bar{b}} + \int_0^{\infty} e^{-\rho t}\int_0^t (1-\tau)r'(n_u)\frac{\partial n_u}{\partial \bar{b}}dudt. \tag{G.6}$$

We analyze $E(\frac{\partial U}{\partial \bar{b}})$ for both cases of unique equilibrium and multiple equilibria.[20]

Based on the welfare analysis in the main text, we consider the following scenario. At the initial time, the economy stays at the stationary point: $b_0 = \bar{b}^{init}$, $n_0 = n^{init}$. Then, the government unexpectedly marginally reduces $\bar{b}$ to $\bar{b}^{new}$. Since this is a marginal change, we may take the following approximation:

$$b_0 = \bar{b}^{init} \approx \bar{b}^{new} = \bar{b}, \tag{G.7}$$

$$n_0 = \bar{n}^{init} \approx \bar{n}^{new} = n^*. \tag{G.8}$$

In the case of a unique equilibrium, the economy jumps into a new saddle path and monotonically converges to the new stationary point. In the case of equilibrium indeterminacy, for simplicity, we assume that a sunspot shock hits the economy only at the same time the government reduces $\bar{b}$. Wherever it jumps, the economy converges as in the determinate case because the stationary point of the local dynamical system is a sink.

Case of a Unique Equilibrium

In the case of a unique equilibrium, using (G.7) and (G.8), we find the equilibrium path of the linearized model and substitute it into (G.6). Table 2 provides the results.[21] See the rows labeled as "D". The main findings are as follows.

- For plausible parameter values, a marginal reduction in $\bar{b}$ worsens social welfare. This is consistent with the result in the original model (see Proposition 4). In the

[19]Note that $\frac{\partial n_0}{\partial \bar{b}} = 0$ because n_t is a predetermined variable.

[20]In the case of multiple equilibria, we focus on solutions with sunspot shocks. Therefore, the partial derivative should be evaluated by the expected value. In the case of a unique equilibrium, the model does not contain any stochastic component.

[21]Because we should treat both cases of unique and multiple equilibria and the threshold (with respect to $\bar{b}$) depends strongly on σ, we properly change the sets of alternative values of $\bar{b}$ according to the values of σ as described in Table 1.

σ	$\bar{b}$	D or I	ϕ 0.01	0.025	0.05	0.075	0.1	0.2
0	0.45	D	-212.15	-525.36	-1034.43	-1527.95	-2006.63	-3785.51
	0.5	I/N	510.21	1279.55	2575.53	3894.83	5225.15	10750.92
		I/U	509.86	1279.57	2576.57	3893.95	5228.69	10755.31
	0.55	I/N	96.67	247.11	514.38	802.92	1118.72	2709.08
		I/U	96.88	247.04	513.95	804.05	1119.22	2709.03
0.075	0.55	D	-1.51	-3.62	-6.95	-10.17	-13.33	-25.44
	0.6	D	-1.86	-4.50	-8.67	-12.96	-17.14	-33.77
	0.65	I/N	-17.49	-45.24	-93.99	-149.50	-210.91	-546.70
		I/U	-17.92	-45.32	-94.59	-149.87	-211.67	-546.47
0.1	0.55	D	-1.02	-2.38	-4.47	-6.46	-8.39	-15.65
	0.6	D	-1.17	-2.77	-5.25	-7.65	-9.99	-19.05
	0.65	I/N	-37.13	-94.53	-200.37	-320.06	-455.25	-1256.82
		I/U	-37.47	-94.82	-200.73	-319.97	-455.54	-1257.18
0.125	0.65	D	-0.95	-2.21	-4.13	-5.97	-7.77	-14.71
	0.7	D	-1.11	-2.61	-4.94	-7.21	-9.46	-18.38
	0.75	I/N	-4.10	-9.75	-20.06	-31.30	-42.95	-104.79
		I/U	-3.98	-9.96	-20.30	-31.20	-43.31	-104.97
1.5	0.75	D	-0.90	-2.07	-3.85	-5.55	-7.23	-13.82
	0.8	I/N	-3.17	-7.86	-16.00	-25.06	-34.39	-83.65
		I/U	-3.26	-7.69	-15.89	-24.59	-34.40	-83.65
	0.85	I/N	-2.05	-5.83	-12.02	-18.34	-24.76	-56.51
		I/U	-2.61	-5.91	-11.42	-18.32	-24.86	-57.10

Table 2: $E(\frac{\partial U}{\partial \bar{b}})$ given $(\sigma, \bar{b}, \phi)$. D (I) means equilibrium is determinate (indeterminate) under $(\sigma, \bar{b})$. N (U) means sunspot shocks follow a normal distribution (uniform distribution). Note that ϕ does not affect equilibrium determinacy.

original model, a reduction in $\bar{b}$ has a detrimental effect, if $r < \omega(= 1 - \alpha)$, which widely holds in usual cases. Inequality $r < \omega(= 1 - \alpha)$ also holds in this numerical analysis, and accordingly, a marginal reduction in $\bar{b}$ worsens social welfare.

- The role of adjustment speed too is similar to that in the case of fixed interest rates. We can conclude that at least with regard to the qualitative aspect, the welfare effect of a reduction in $\bar{b}$ and the mechanism underlying it do not change basically even under an endogenous interest rate.

- The welfare implication of a change in $(\bar{b}, \phi)$ is similar to the basic model in which the interest rate is perfectly fixed.

- The existence of a variable component in the interest rate (i.e., $\sigma > 0$) yields a quantitatively crucial effect on social welfare.[22] This is because the mechanism underlying the welfare implication in the basic model depends on the assumption that the interest rate is perfectly fixed. Thus, it is natural that the absolute value of the welfare effect is decreasing in the degree of variability of the interest rate.

Case of Multiple Equilibria

In the case of multiple equilibria, we consider sunspot solutions.[23] Denote the three eigenvalues of the Jacobian matrix by λ_1, λ_2, and λ_3 and the associated eigenvectors by v_1, v_2, and v_3, respectively. The general solution of the linearized model is

$$z_t = z^* + \kappa_1 \exp(\lambda_1 t) v_1 + \kappa_2 \exp(\lambda_2 t) v_2 + \kappa_3 \exp(\lambda_3 t) v_3, \tag{G.9}$$

where $z_t = (b_t, n_t, c_t)'$, $z^* = (b^*, n^*, c^*)'$, and κ_1, κ_2, and κ_3 are coefficients. Because one of the eigenvalues is $-\phi < 0$, we may set $\lambda_1 = -\phi$. In the case of indeterminacy, $Re(\lambda_2) < 0$ and $Re(\lambda_3) < 0$, where $Re(\lambda_i)$ is the real part of λ_i. Suppose that coefficient κ_2 in (G.9) is a random variable that generates a sunspot shock that hits the economy only at the initial

[22] Note that this model coincides with the original model in the main text if $\sigma = 0$.
[23] We use the method of Lubik and Schorfheide (2003) to compose sunspot equilibria.

time at which the government sets a new long-run target for the debt–GDP ratio.[24] We consider the cases where κ_2 is (i) normally distributed such a way that c_0 jumps into the range $[0.95c^*, 1.05c^*]$ with probability greater than 0.999 and (ii) is uniformly distributed over $[0.97c^*, 1.03c^*]$. Since c_0 is now a random variable, we should consider an expected welfare effect for a marginal reduction in $\bar{b}$: $E(\frac{\partial U}{\partial \bar{b}})$. In Table 2, the rows labeled "I" present the results for this analysis. Labels, "N" and "U" indicate that sunspot shocks follow a normal and uniform distribution, respectively. The main findings are as follows.

- Both the distributions yield qualitatively and quantitatively similar results. Although the welfare effect in general varies for each realization of κ_2, the expected value is stable due to the symmetry of the distributions. We conjecture that this is from sunspot shocks following any symmetric distributions with sufficiently small variances.

- Except for the case of $\sigma = 0$, the relationships between adjustment speed ϕ and welfare effect $E(\frac{\partial U}{\partial \bar{b}})$ are similar to that in the case of equilibrium determinacy and hence the original model. In the case of $\sigma = 0$, the sign of $E(\frac{\partial U}{\partial \bar{b}})$ reverses. Its absolute value increases in ϕ even in this case.[25]

G.5 Regime (II)

We now consider regime (II) where τ is endogenous and $g = G/Y$ is exogenous. The dynamics system is given by

$$
\begin{aligned}
\dot{b}_t &= -\phi(b_t - \bar{b}), \\
\dot{n}_t &= \Delta(n_t)\Psi(b_t, n_t, c_t)n_t, \\
\dot{c}_t &= \left[(1 - \tau(b_t, n_t, c_t))r(n_t) - \rho + (\Delta(n_t) - 1)\Psi(b_t, n_t, c_t) - gk_g(n_t)^{-1} + \delta_g\right]c_t,
\end{aligned}
$$

[24] According to the dynamical equations, $\kappa_1 = b_0 - \bar{b}$. Given κ_2, the value of κ_3 is determined so that the initial value satisfies (G.9) .

[25] We obtain similar results for sufficiently small values of $\sigma > 0$, for example, $\sigma = 10^{-5}$. Thus, the welfare effect seems to have continuity at $\sigma = 0$ with respect to σ. Besides, similar results hold when there are no sunspot shocks. The phase diagram seems to exhibit some complicated changes between the cases of a saddle and sink nearby $\sigma = 0$.

where $\Delta(n_t) = \left[1 - \frac{\alpha}{1-\alpha}\varepsilon(n_t)\right]^{-1}$ and $\Psi(b_t, n_t, c_t) = r(n_t) + \frac{g}{n_t} - \frac{g}{k_g(n_t)} + \frac{c_t - \omega}{n_t} + \delta_g$. The tax rate is a function of b_t, n_t, and c_t:

$$\tau(b_t, n_t, c_t) = \frac{(r(n_t) + \frac{\alpha}{1-\alpha}\varepsilon(n_t)\frac{\dot{n}_t}{n_t} - gk_g(n_t)^{-1} + \delta_g)b_t + g - \phi(b_t - \bar{b})}{r(n_t)(b_t - n_t) + \omega}.$$

We derive the steady state where $\dot{b}_t = \dot{n}_t = \dot{c}_t = 0$. From $\dot{n}_t = \Psi(b_t, n_t, c_t) = 0$ and $\dot{c}_t = 0$, we obtain

$$\tau = 1 - \frac{1}{r(n)}\left\{\rho + \frac{g}{k_g(n)} - \delta_g\right\}. \tag{G.10}$$

If r is fixed, the right-hand side (RHS) of (G.10) is constant. If $r'(n) > 0$, the RHS is an increasing function of n as long as the long-run growth rate, $\gamma = \frac{g}{k_g(n)} - \delta_g$, is strictly positive. From the definition of $n_t = b_t - a_t$, we have $\dot{a}_t = 0$ in the steady state, which implies $c = \rho(\bar{b} - n) + (1 - \tau)\omega$. Using this equation and $\dot{n}_t = \Psi(b_t, n_t, c_t) = 0$, we obtain

$$\tau = \frac{\rho\bar{b} + \rho}{\omega - r(n)n}. \tag{G.11}$$

Irrespective of whether $r'(n) > 0$ or $r'(n) = 0$, there exists a unique $\bar{n}$ such that $\omega = r(\bar{n})\bar{n}$ and the denominator of the RHS is strictly positive if $n < \bar{n}$. The RHS of (G.11) increases with n. Equations (G.10) and (G.11) determine the steady-state value of n.

As shown in Proposition 5 of the main text, if r is fixed, there exists a unique steady state. Point A in panel (a) of Figure 5 corresponds to this steady state. The curve of (G.11) intersects that of (G.10) from below, at the steady state. The main text shows that this steady state is unstable. Even when r increases with n, there exists a steady state where the curve of (G.11) intersects that of (G.10) from below (see Point B in panel (b) of Figure 5).

To examine the stability of this steady state, we use the specification of (G.1). As in Subsection G.2, we assume $\rho = 0.05$, $\alpha = 0.36$, $\delta_k = 0.05$, $\delta_g = 0.04$, $\phi = 0.05$, and $r^* = 0.05$. To ensure positive growth, we use $\theta = 1.3$. We set $g = 0.038$. The values of $\bar{b}$

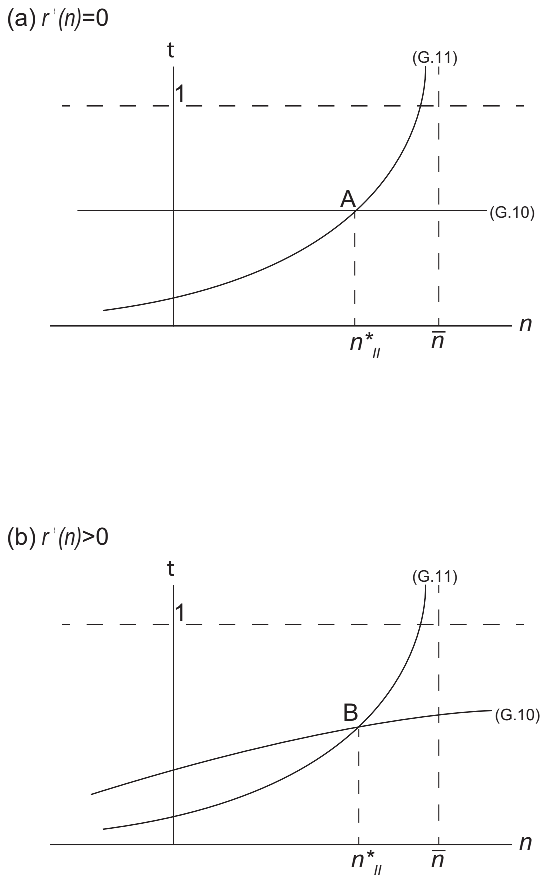

Figure 5: Steady-state equilibrium under regime (II)

range from 0.1 to 3. We linearly approximate the dynamic system around this steady state and then calculate the eigenvalues for $\sigma = 0.1$ and $\sigma = 0.2$. Our calculation shows that for all values of $\bar{b}$, this steady state is unstable. For example, when $\sigma = 0.1$ and $\bar{b} = 0.6$, the Jacobian of the linearized system has the following eigenvalues: 0.866, 0.0757, -0.05. This result shows that even when r is endogenously determined, the steady state may be unstable for a wide range of parameters.

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