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Ken Urai†

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Abstract

An overlapping generations model with the double infinity of commodities and agents is the most fundamental framework to introduce outside money into a static economic model. In this model, competitive equilibria may not necessarily be Pareto-optimal. Although Samuelson (1958) emphasized the role of fiat money as a certain kind of social contract, we cannot characterize it as a cooperative game-theoretic solution like a core. In this paper, we obtained a finite replica core characterization of Walrasian equilibrium allocations under non-negative wealth transfer and a core-limit characterization of Samuelson's social contrivance of money. Preferences are not necessarily assumed to be ordered.

Keywords: Monetary Equilibrium, Overlapping Generations Model, Core Equivalence, Replica Economy, Non-Ordered Preference

JEL Classification: C62, C71, D51, E00

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1 Introduction

In this paper, we generalize the Debreu-Scarf core limit theorem to a case with a *double infinity* economy that includes such typical examples as Samuelson consumption-loan models with money.

For standard finite general equilibrium settings, the Walrasian equilibrium is Pareto-optimal (the first fundamental theorem of welfare economics), and every Pareto-optimal allocation is an equilibrium allocation relative to a price-wealth system (the second fundamental theorem of welfare economics). If we strengthen the concept of Pareto-optimal allocation to a *replica core allocation* (an allocation whose replica is a core allocation for each replica economy), we obtain an equivalence result where every replica core allocation is a competitive allocation and every competitive allocation is a replica core allocation (Debreu and Scarf 1963, Theorems 1 and 3). This equivalence theorem between the replica core and the competitive equilibria is commonly known as the Debreu-Scarf core limit theorem.

In a double infinity economy, competitive equilibrium (with or without money) is not necessarily Pareto-optimal (Samuelson 1958), but it is known to be weakly Pareto-optimal (Esteban 1986 and Balasko and Shell 1980). It is also known that every weakly Pareto-optimal allocation is an equilibrium allocation relative to a price-wealth system (Balasko and Shell 1980). Chae (1987), Aliprantis and Burkinshaw (1990) and Chae and Esteban (1993) treat the core equivalence problem for Walrasian equilibrium allocations in overlapping generations models. Their approaches, however, fail to treat competitive equilibrium allocations with money.[1] Of course, an equilibrium with money (non-negative wealth transfers from the government) is one critical issue that the overlapping generations model tries to describe.

In this paper, we show that if we strengthen the concept of weak Pareto-optimal allocation to *replica finite core allocation* (an allocation whose replica is a certain kind of finite core allocation for each replica economy[2]), we obtain an equivalence result analogous to Aliprantis and Burkinshaw (1990) where a replica finite core allocation is a competitive allocation with *non-negative wealth transfers* and every competitive allocation with *non-negative wealth transfers* is a replica finite core allocation.[3]

Our replica core equivalence approach (as well as that of Aliprantis and Burkinshaw 1990) has three important advantages: by concentrating on the equivalence argument without using the equal treatment property, (i) we can show a weak core theoretic equivalence result merely based on weak optimality conditions, (ii) we obtain a limit theorem of the core instead of a theorem in the limit measure space like Chae and Esteban, and (iii) we can allow for an argument based on non-ordered preferences, and hence our result may also be considered a non-ordered extension of the Debreu-Scarf core equivalence theorem.[4]

[1] Their concepts, such as the *short-term core* (Aliprantis and Burkinshaw 1990) and the *short-run core* (Chae 1987), exclude equilibrium allocations with non-zero fiat money in two-period overlapping generations economies. For the short-term core argument, as a simple one-good per period economy example pointed out in Esteban (1986), such a monetary equilibrium is blocked by a coalition of all agents after a certain period without changing all but the first finite members' allocations. In the short-run core, we can also easily construct an example under which a typical Samuelson-type monetary equilibrium allocation is always blocked by the t-generation for each t-economy for all $t = 1, 2, \cdots$.

[2] More precisely, this is an allocation, x, whose replica is a finite core allocation for each replica economy even when the endowments of some members are replaced by the allocation, x, itself.

[3] In this paper, we use "wealth transfer" instead of "monetary transfer" because these two concepts are different unless we use the perfect-foresight assumption on the expectation for dynamics. In the sense of Esteban and Millán (1990), we concentrate on the set of all monetary equilibrium allocations and competitive equilibrium allocations without money.

[4] Aliprantis and Burkinshaw (1990), however, do not successfully treat non-ordered preference cases. In our model, a strong sense of the local non-satiation in (E.3) plays an essential role in the proof of Theorem 1.

2 The Model

Let $\boldsymbol{N}$ be the set of all positive integers and $\boldsymbol{R}$ be the set of real numbers. A *pure exchange overlapping generations economy*, or more simply, an *economy*, $\mathcal{E}$, is comprised of the following list:

(E.1) $\{I_t\}_{t=1}^{\infty}$: a countable family of mutually disjoint finite subsets of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} I_t \subset \boldsymbol{N}$, and if $I_t = \emptyset$, $I_{t+1} = \emptyset$ for each $t \in \boldsymbol{N}$. I_t is the index set of agents in *generation* t.

(E.2) $\{K_t\}_{t=1}^{\infty}$: a countable family of non-empty finite intervals, $K_t = \{k(t), k(t)+1, \cdots, k(t)+\ell(t)\}$ where $k(t)$ and $\ell(t)$ are elements of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} K_t = \boldsymbol{N}$, $k(t) < k(t+1) \leqq k(t) + \ell(t)$ for all $t \in \boldsymbol{N}$, and $\{t \mid n \in K_t\}$ is finite for each $n \in \boldsymbol{N}$. K_t is the index set of *commodities* available to generation t.

(E.3) $\{(\precsim_i,\ \omega^i)\}_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$: countably many agents, where $\precsim_i$ is a reflexive binary relation on commodity space for each generation $\boldsymbol{R}_+^{K_t}$, representing a preference of $i \in I_t$. We write $x_i \sim_i y_i$ iff $x_i \precsim_i y_i$ and $y_i \precsim x_i$, and $x_i \succ_i y_i$ iff $x_i \precsim_i y_i$ and $x_i \not\sim y_i$. *Strict preference* $\succ_i$ is continuous (having an open graph in $\boldsymbol{R}_+^{K_t} \times \boldsymbol{R}_+^{K_t}$), strictly monotonic ($x_i \geqq y_i$ and $x_i \neq y_i$ implies $x_i \succ_i y_i$), and has a convex better set ($\{y_i | \, y_i \succ_i x_i\}$ is convex) at every x_i such that $\omega_i \not\succ_i x_i$. The closure of the graph of $\succ_i$ in $\boldsymbol{R}^{K_t} \times \boldsymbol{R}^{K_t}$ is the graph of $\precsim_i$ (a *strong sense of local non-satiation*). The *initial endowment* of i, ω^i, is an element of $\boldsymbol{R}_{++}^{K_t} = \{x \mid x : K_t \to \boldsymbol{R}_{++}\}$ for each $i \in I_t$.

It is convenient to identify the commodity space for each generation $\boldsymbol{R}_+^{K_t}$ with a subset of $\boldsymbol{R}^{\boldsymbol{N}}$, which is the set of all functions from $\boldsymbol{N}$ to $\boldsymbol{R}$, by considering $x \in \boldsymbol{R}_+^{K_t}$ a function that takes value 0 on $\boldsymbol{N} \setminus K_t$. Then we can define the total commodity space for economy $\oplus_{t=1}^{\infty} \boldsymbol{R}_+^{K_t}$ as the set of all finite sums among the points in the commodity spaces of the generations. Clearly, $\oplus_{t=1}^{\infty} \boldsymbol{R}_+^{K_t}$ can be identified with a subset of direct sum $\boldsymbol{R}_{\infty}$, the set of all finite real sequences, which is a subspace of the set of all real sequences, $\boldsymbol{R}^{\infty} \approx \boldsymbol{R}^{\boldsymbol{N}}$ with pointwise convergence topology.

Given an economy, $\mathcal{E} = (\{I_t\}_{t=1}^{\infty},\ \{K_t\}_{t=1}^{\infty},\ \{(\precsim_i,\ \omega^i)\}_{i\in\bigcup_{t\in\boldsymbol{N}} I_t})$, the *price space* for $\mathcal{E}$, $\mathcal{P}(\mathcal{E})$, is defined as the set of all p in $\boldsymbol{R}_+^{\boldsymbol{N}}$ such that under the duality between $\boldsymbol{R}_{\infty}$ (with relative topology) and $\boldsymbol{R}^{\infty}$ (with pointwise convergence topology), p positively evaluates all the agents' initial endowments:

(1) $$\mathcal{P}(\mathcal{E}) = \{p \in \boldsymbol{R}_+^{\boldsymbol{N}} \mid p \cdot \omega^i > 0 \text{ for all } i \in I_t, \text{ for all } t \in \boldsymbol{N}\}.$$

Since for all $i \in I_t$, ω^i belongs to $\boldsymbol{R}_{++}^{K_t}$ for all $t \in \boldsymbol{N}$, the price space of $\mathcal{E}$ always includes $\boldsymbol{R}_{++}^{\boldsymbol{N}}$ for all $\mathcal{E}$ in $\mathcal{E}con$, where $\mathcal{E}con$ denotes the set of all economies satisfying conditions (E.1), (E.2) and (E.3).

For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\precsim_i,\ \omega^i)\}) \in \mathcal{E}con$, sequence $(x^i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ is called an *allocation* for $\mathcal{E}$. Allocation $(x^i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ is said to be *feasible* if

(2) $$\sum_{t\in\boldsymbol{N}} \sum_{i\in I_t} x^i \leqq \sum_{t\in\boldsymbol{N}} \sum_{i\in I_t} \omega^i,$$

where the summability in $\boldsymbol{R}^{\boldsymbol{N}}$ of both sides of the inequality is assured by (E.2). The list of price vector $p^* \in \mathcal{P}(\mathcal{E})$, *non-negative wealth transfer function* $M_{\mathcal{E}}^* : \boldsymbol{N} = \bigcup_{t=1}^{\infty} I_t \to \boldsymbol{R}_+$, and feasible allocation $(x_*^i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$, is called a *non-negative wealth transfer Walrasian equilibrium* for $\mathcal{E}$, if for each $t \in \boldsymbol{N}$ and $i \in I_t$, x_*^i is a $\succ_i$-maximal element in set $\{x^i \in \boldsymbol{R}^{K_t} \mid p^* \cdot x^i \leqq p^* \cdot \omega^i + M_{\mathcal{E}}^*(i)\}$. Since the non-negative wealth transfer is an abstraction of the *money supply* in perfect-foresight overlapping generations settings, we denote the set of all non-negative wealth transfer Walrasian equilibrium allocations by $\mathcal{MWalras}(\mathcal{E})$.

A *coalition* in economy $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\precsim_i,\ \omega^i)\}) \in \mathcal{E}con$ is a set of consumers $S \subset \bigcup_{t=1}^{\infty} I_t$. Allocation x for economy $\mathcal{E}$ is said to be *blocked* by coalition S if it is possible to find commodity bundles

$\hat{x}^i$ for all $i \in S$ such that $\sum_{i\in S}(\hat{x}^i - \omega^i) = 0$ and $\hat{x}^i \succsim_i x^i$ for all $i \in S$, and $\hat{x}^i \succ_i x^i$ for at least one $i \in S$. For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega^i)\}) \in \mathcal{E}con$, the set of all feasible allocations that cannot be blocked by any coalition is said to be the *core* of economy $\mathcal{E}$ and is denoted by $\mathcal{C}ore(\mathcal{E})$. Element $x \in \mathcal{C}ore(\mathcal{E})$ is called a *core allocation*. The set of all feasible allocations that cannot be blocked by any finite coalition is called the *finite core* of economy $\mathcal{E}$ and is denoted by $\mathcal{F}core(\mathcal{E})$. Element $x \in \mathcal{F}core(\mathcal{E})$ is called a *finite core allocation* for $\mathcal{E}$.

3 Replica Core Equivalence Theorem

For each feasible allocation $x = (x^i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ for $\mathcal{E} = (\{I_t\}, \{K_t\}, \{(\succsim_i, \omega^i)\}) \in \mathcal{E}con$, we denote by $\mathcal{E}(x)$ an economy where initial endowment allocation $\omega = (\omega^i)$ is replaced by $x = (x^i)$.[5] Hence, we have $\mathcal{E} = \mathcal{E}(\omega)$.

Consider the following replica economy,

$$\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega), \tag{3}$$

which consists of all the members of the m-fold replica economy of $\mathcal{E}(x)$ and the n-fold replica economy of $\mathcal{E}(\omega)$ for each $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$. Let us denote by $\mathbb{C}^{m\oplus n}(\mathcal{E})$ the set of allocations x for $\mathcal{E}$ such that the $(m+n)$*-fold replica allocation of* x belongs to $\mathcal{F}core(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$.[6] Moreover, let us denote by $\mathbf{C}^n(\mathcal{E})$ the set of allocations x for $\mathcal{E}$ such that the n-fold replica allocation of x belongs to $\mathcal{C}ore(\mathcal{E}^n)$. It is easy to check that if x is a feasible allocation of $\mathcal{E} = \mathcal{E}(\omega)$ such that $(m+n)$-fold replica allocation of x does not belong to $\mathcal{F}core(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$, the replica allocation does not belong to $\mathcal{C}ore(\mathcal{E}^{m+n})$.[7] Therefore, we can write $\mathbb{C}^{m\oplus n}(\mathcal{E}) \supset \mathbf{C}^{m+n}(\mathcal{E})$ for each $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$. It is also easy to check that $\mathbb{C}^{m\oplus n}(\mathcal{E}) \supset \mathbb{C}^{m'\oplus n'}(\mathcal{E}) \supset \cdots$ where $m' \geqq m$, $n' \geqq n$.[8] For finite economy $\mathcal{E}$, the Debreu-Scarf limit theorem can be restated as $\bigcap_{m+n=2}^{\infty} \mathbf{C}^{m+n}(\mathcal{E}) = \mathcal{W}alras(\mathcal{E})$. We see below (Theorem 1), $\bigcap_{n=1}^{\infty}\bigcap_{m=1}^{\infty} \mathbb{C}^{m\oplus n}(\mathcal{E}) = \mathcal{MW}alras(\mathcal{E})$. Hence, the restriction of Theorem 1 to the case with finite economy $\mathcal{E}$ provides the following extension of the replica core version of the Debreu-Scarf limit theorem because there is no difference in our settings between $\mathcal{W}alras(\mathcal{E})$ and $\mathcal{MW}alras(\mathcal{E})$.

> For finite economy $\mathcal{E}$, feasible allocation x for $\mathcal{E}$ is a competitive equilibrium allocation iff its $(m+n)$-fold replica allocation belongs to $\mathcal{F}core(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ for every sufficiently large $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$. That is, $\mathcal{W}alras(\mathcal{E}) = \bigcap_{n=1}^{\infty}\bigcap_{m=1}^{\infty} \mathbb{C}^{m\oplus n}(\mathcal{E})$.

As above, concept $\mathcal{F}core$ gives a unified replica core equivalence characterization for all non-negative wealth transfer Walrasian equilibrium allocations. Note that allocation x such that $x \in \mathcal{F}core(\mathcal{E}(x))$ is the *weak Pareto-optimal* allocations in Balasko and Shell (1980). It is easy to check that the n-fold replica allocation of $x^* \in \mathcal{MW}alras(\mathcal{E})$ belongs to $\mathcal{F}core(\mathcal{E}^n(x^*))$ for all $n \in \boldsymbol{N}$ and $\mathcal{F}core(\mathcal{E}^n(\omega))$ for all

[5] In the following, we sometimes omit the subscript $i \in \bigcup_{t\in\boldsymbol{N}} I_t$ of an allocation for an economy as long as there is no risk of confusion.

[6] The $(m+n)$-fold replica allocation of x is the allocation for $\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega)$ such that for each replica agent i in $\mathcal{E}(x)$ or $\mathcal{E}(\omega)$, we assign the same allocation under x in economy $\mathcal{E}$.

[7] Clearly, the replica allocation, x^{m+n}, is feasible for $\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega)$ and $\mathcal{E}^{m+n} = \mathcal{E}^{m+n}(\omega)$. If $x^{m+n} \notin \mathcal{F}core(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$, then there exists a finite coalition S in $\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega)$ that blocks allocation x^{m+n}. We can write $S = S_1 \cup S_2$, where S_1 (resp. S_2) consists of members in $\mathcal{E}^m(x)$ (resp. $\mathcal{E}^n(\omega)$). Then, coalition S^* consisting of all members of $\mathcal{E}^m(x)$ and S_2 also blocks $x^{m+n} \in \mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega)$. Therefore $x^{m+n} \notin \mathcal{C}ore(\mathcal{E}^{m+n})$.

[8] Note that the equal treatment property is not necessary for ensuring the above inclusion relations. Here we are following the *replica core equivalence* approach in Aliprantis and Burkinshaw (1990).

$n \in \boldsymbol{N}$.[9] But we have an example where the n-fold replica of allocation x, which is not an element of $\mathcal{MWalras}(\mathcal{E})$, belongs to $\mathcal{Fcore}(\mathcal{E}^n(x)) \cap \mathcal{Fcore}(\mathcal{E}^n(\omega))$ for all $n \in \boldsymbol{N}$.[10]

Theorem 1: Feasible allocation x for $\mathcal{E}$ is a non-negative wealth transfer Walrasian equilibrium allocation iff its $(m+n)$-fold replica allocation belongs to $\mathcal{Fcore}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ for every $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$. That is, $\mathcal{MWalras}(\mathcal{E}) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathbb{C}^{m \oplus n}(\mathcal{E})$.

Proof: [**Sufficiency**] Let $x_* = (x_*^i)$ be an element of $\mathcal{MWalras}(\mathcal{E})$ under price p^* and non-negative wealth transfer function $M_{\mathcal{E}}^*$. Assume that $S = S_1 \cup S_2$ is a finite coalition of $\mathcal{E}^m(x_*) \oplus \mathcal{E}^n(\omega)$ for some m and n in $\boldsymbol{N}$ blocking the $(m+n)$-fold replica allocation of $x_* = (x_*^i)$, where S_1 is a coalition in $\mathcal{E}^m(x_*)$ and S_2 is a coalition in $\mathcal{E}^n(\omega)$. Then, under (E.3), especially by using the strong sense of local non-satiation, an allocation $(x^i)_{i \in S}$ for S exists such that $\sum_{i \in S} x^i = \sum_{i \in S_1} x_*^i + \sum_{i \in S_2} \omega^i$ and $x^i \succ_i x_*^i$ for all $i \in S$. Then we have $p^* \cdot x^i > p^* \cdot x_*^i$ for all $i \in S$. This implies, however, that $p^* \cdot (\sum_{i \in S_1} x^i + \sum_{i \in S_2} x^i) > p^* \cdot \sum_{i \in S_1} x_*^i + p^* \cdot \sum_{i \in S_2} x_*^i \geqq p^* \cdot (\sum_{i \in S_1} x_*^i + \sum_{i \in S_2} \omega^i)$, a contradiction to $\sum_{i \in S} x^i = \sum_{i \in S_1} x_*^i + \sum_{i \in S_2} \omega^i$.

[**Necessity**] Let $\bar{x} = (\bar{x}^i)$ be an allocation for $\mathcal{E} = (\{I_t\}_{t=1}^{\infty}, \{K_t\}_{t=1}^{\infty}, (\succsim_i, \omega^i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t})$ such that every $(m+n)$-fold replica allocation of $\bar{x}$ belongs to $\mathcal{Fcore}(\mathcal{E}^m(\bar{x}) \oplus \mathcal{E}^n(\omega))$ for all m and n in $\boldsymbol{N}$. In this proof, we denote by $I(t)$ the set of all agents in generations from 1 to t, $\bigcup_{s=1}^{t} I_s$, and by $K(t)$ the set of all commodities that are available for agents in $I(t)$, $\bigcup_{s=1}^{t} K_s$. Define for each $i \in I_t$, $t \in \boldsymbol{N}$, Γ_i as $\Gamma_i = \{\beta^i z_1^i + (1-\beta^i) z_2^i \mid \omega^i + z_1^i \succ_i \bar{x}^i,\ \bar{x}^i + z_2^i \succ_i \bar{x}^i,\ 0 \leqq \beta^i \leqq 1\} \subset \boldsymbol{R}^{K_t}$. Then, take the convex hull $\Gamma(t)$ of finite union $\bigcup_{i \in I(t)} \Gamma_i \subset \bigcup_{s=1}^{t} \boldsymbol{R}^{K_s} \approx \boldsymbol{R}^{K(t)}$ for each $t \in \boldsymbol{N}$. Since, for every i, Γ_i is non-empty and convex, non-empty convex set $\Gamma(t)$ consists of all vectors z that can be written as $\sum_{i \in I(t)} \alpha^i (\beta^i z_1^i + (1-\beta^i) z_2^i)$, with $\alpha^i \geqq 0$, $\sum_{i \in I(t)} \alpha^i = 1$, where $z_1^i + \omega^i \succ_i \bar{x}^i$ and $z_2^i + \bar{x}^i \succ_i \bar{x}^i$ for each i.

We will show in the similar way as in the proof of Debreu and Scarf (1963, Theorem 3) that $\Gamma(t)$ does not have 0 as its element for each $t \in \boldsymbol{N}$. Let us suppose that 0 belongs to $\Gamma(t)$. Then, one can write $\sum_{i \in I(t)} \alpha^i (\beta^i z_1^i + (1-\beta^i) z_2^i) = 0$, with $\alpha^i \geqq 0$, $\sum_{i \in I(t)} \alpha^i = 1$, and $z_1^i + \omega^i \succ_i \bar{x}^i$ and $z_2^i + \bar{x}^i \succ_i \bar{x}^i$ for each i. For each k sufficiently large, let a_{1k}^i and a_{2k}^i be the smallest integers greater than $k\alpha^i\beta^i$ and $k\alpha^i(1-\beta^i)$ respectively. Also, let I be the set of $i \in I(t)$ for which $\alpha^i > 0$. For each $i \in I$, we define z_{1k}^i as $\frac{k\alpha^i\beta^i}{a_{1k}^i} z_1^i$ and z_{2k}^i as $\frac{k\alpha^i(1-\beta^i)}{a_{2k}^i} z_2^i$. Observe that $z_{1k}^i + \omega^i$ belongs to the segment $[\omega^i,\ z_1^i + \omega^i]$, $z_{2k}^i + \bar{x}^i$ belongs to the segment $[\bar{x}^i,\ z_2^i + \bar{x}^i]$.

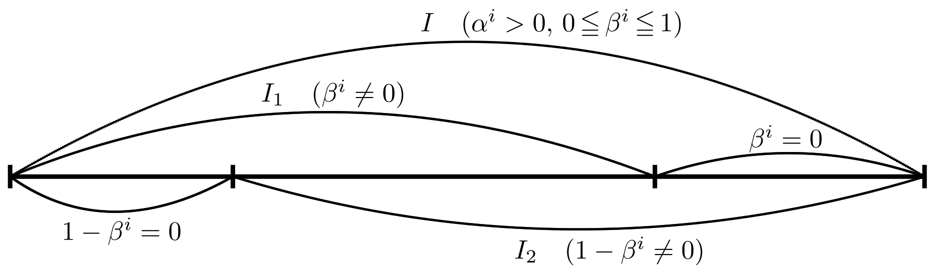

Figure 1: The union of I_1 and I_2 is equals to I.

Let I_1 be the set of $i \in I$ such that $\beta^i \neq 0$, and I_2 be the set of $i \in I$ such that $1 - \beta^i \neq 0$. Note that $I_1 \cup I_2 = I$ (see Figure 1). For $i \in I_1$, $z_{1k}^i + \omega^i$ tends to $z_1^i + \omega^i$, and for $i \in I_2$, $z_{2k}^i + \bar{x}^i$ tends to $z_2^i + \bar{x}^i$

[9] To see this, in the proof of Theorem 1, [Sufficiency], let S_1 or S_2 be an empty set.
[10] See Appendix.

as k tends to infinity. The continuity assumption on preferences implies that $z^i_{1k} + \omega^i \succ_i \bar{x}^i$ for all $i \in I_1$ and $z^i_{2k} + \bar{x}^i \succ_i \bar{x}^i$ for all $i \in I_2$ for all k sufficiently large. Select one of such k. Then we have

$$
\begin{aligned}
(4) \qquad 0 &= k \sum_{i \in I} \alpha^i (\beta^i z^i_1 + (1-\beta^i) z^i_2) = \sum_{i \in I} (a^i_{1k} z^i_{1k} + a^i_{2k} z^i_{2k}) = \sum_{i \in I} k\alpha^i (\beta^i z^i_1 + (1-\beta^i) z^i_2) \\
&= \sum_{i \in I_1 \setminus I_2} k\alpha^i \beta^i z^i_1 + \sum_{i \in I_1 \cap I_2} k\alpha^i (\beta^i z^i_1 + (1-\beta^i) z^i_2) + \sum_{i \in I_2 \setminus I_1} k\alpha^i (1-\beta^i) z^i_2 \\
&= \sum_{i \in I_1 \setminus I_2} a^i_{1k} z^i_{1k} + \sum_{i \in I_1 \cap I_2} (a^i_{1k} z^i_{1k} + a^i_{2k} z^i_{2k}) + \sum_{i \in I_2 \setminus I_1} a^i_{2k} z^i_{2k}
\end{aligned}
$$

Let us consider the $\left((\max_{i \in I} \ a^i_{1k}) + (\max_{i \in I} \ a^i_{2k})\right)$-fold replica economy of $\boldsymbol{\mathcal{E}}$. Take the coalition composed of a^i_{1k} replica members of i for each $i \in I_1$ to each one of whom we assign $\omega^i + z^i_{1k}$, and a^i_{2k} replica members of i for each $i \in I_2$ to each one of whom we assign $\bar{x}^{i_2} + z^{i_2}_{2k}$. This coalition blocks the allocation $(\bar{x}^i)$ as equation (4) and the fact that $z^i_{1k} + \omega^i \succ_i \bar{x}^i$ for all $i \in I_1$ and $z^i_{2k} + \bar{x}^i \succ_i \bar{x}^i$ for all $i \in I_2$ show. This is a contradiction to the definition of $\boldsymbol{\mathcal{F}core}(\boldsymbol{\mathcal{E}}^m(\bar{x}) \oplus \boldsymbol{\mathcal{E}}^n(\omega))$. Hence, we have established that 0 does not belong to the convex set $\Gamma(t)$ for each $t \in \boldsymbol{N}$.

Let $\pi(t) \subset \boldsymbol{R}^\infty$ be the set of prices such that $p \cdot z \geqq 0$ for all $z \in \Gamma(t) \subset \boldsymbol{R}^{K(t)} \subset \boldsymbol{R}_\infty$, which is non-empty by the separating hyperplane theorem. $\pi(t)$ is closed in $\boldsymbol{R}^{K(t)} \times \boldsymbol{R} \times \cdots = \boldsymbol{R}^\infty$. Moreover, under the resource-related structure assured by (E.2) and $\omega^i \gg 0$ for all i, $\pi(t)$ is a subset of $\boldsymbol{R}^{K(t)}_{++} \times \boldsymbol{R} \times \cdots \subset \boldsymbol{R}^\infty$.[11] Next, we will obtain $p^* \in \bigcap_{t \in \boldsymbol{N}} \pi(t)$. From the definition of each $\Gamma(t) \subset \boldsymbol{R}^{K(t)}$, we have $\Gamma(1) \subset \Gamma(2) \subset \Gamma(3) \subset \cdots$ in $\boldsymbol{R}_\infty$. Hence, we have $\pi(1) \supset \pi(2) \supset \pi(3) \supset \cdots$ in $\boldsymbol{R}^\infty$. Thus we see $\bigcap_{s=1}^t \pi(s) = \pi(t)$. For finite economy, we have $\pi(t) = \pi(t+1) = \cdots$ for all sufficiently large t, hence the result is obvious. If the number of agents is infinite, for each $t \in \boldsymbol{N}$, choose price $p(t) = (p_1(t), p_2(t), \cdots)$ in $\pi(t) \subset \boldsymbol{R}^{K(t)}_{++} \times \boldsymbol{R} \times \cdots \subset \boldsymbol{R}^\infty$ (see Figure 2).

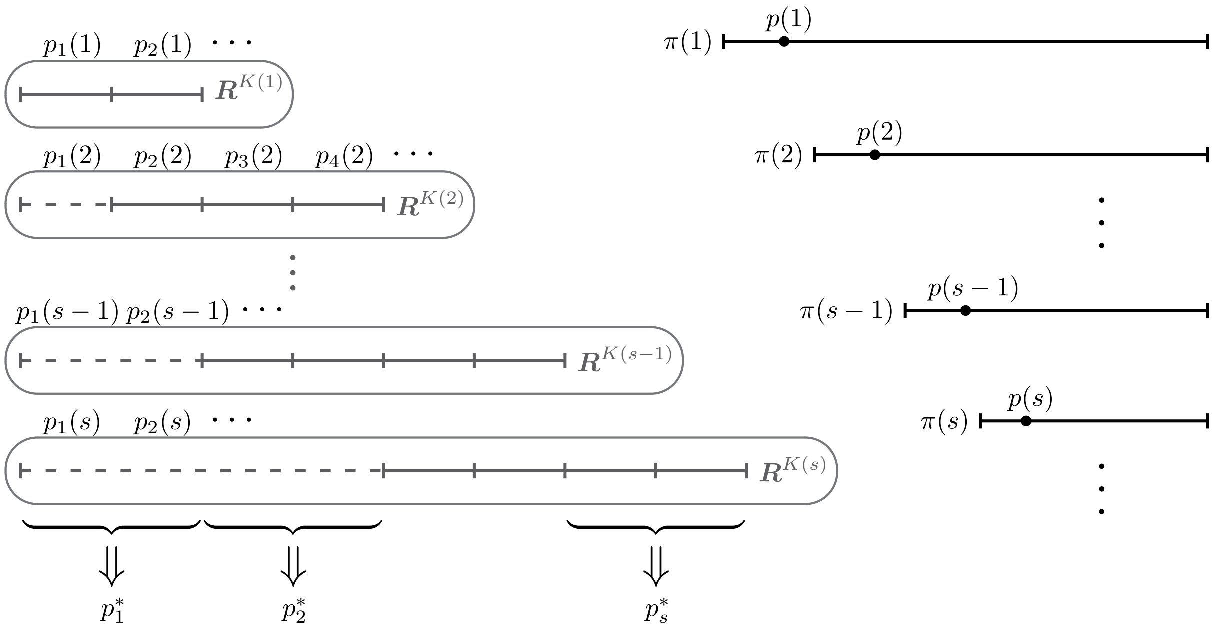

Figure 2: How to construct the limit price p^*.

[11] Indeed, $\pi(t)$ is the set of supporting price vectors for the better set at $\bar{x}^i$ under the strictly monotonic preference for each $i \in I(t)$, where every $\bar{x}^i$ is necessarily evaluated positively (at least as great as the value of ω^i).

Let us define for each $t \in \boldsymbol{N}$, compact set $\Delta^{K(t)} = \{q_t \mid q_t \in \boldsymbol{R}^{K(t)},\ \|q_t\| = 1\}$. Moreover, for each $s, t \in \boldsymbol{N}$, $s \leqq t$, define mapping $h_{st} : \pi(t) \to \Delta^{K(s)}$ as $h_{st}(p) = \frac{\mathrm{pr}_{K(s)}\, p}{\|\,\mathrm{pr}_{K(s)}\, p\|}$ for each $p \in \pi(t)$, where pr denotes the projection.[12] Since $\Delta^{K(1)}$ is compact, if $t \to \infty$, $h_{1t}(p(t))$ converges to a limit, $\hat{p}_1^* \in \Delta^{K(1)} \cap h_{11}(\pi(1))$. Then we take a subsequence, $\{p(t)\}_{t \in \boldsymbol{N}(1)}$, of $\{p(t)\}_{t \in \boldsymbol{N}}$, where $\boldsymbol{N}(1)$ is a cofinal subset of $\boldsymbol{N}$, such that $h_{1t}(p(t))$ converges to $\hat{p}_1^*$. Define $p_1^* \in \boldsymbol{R}^{K(1)}$ as $p_1^* = \hat{p}_1^*$. Next, since $\Delta^{K(2)}$ is compact, when $t \to \infty$, by taking a subsequence, $\{p(t)\}_{t \in \boldsymbol{N}(2)}$, of $\{p(t)\}_{t \in \boldsymbol{N}(1)}$, where $\boldsymbol{N}(2)$ is a cofinal subset of $\boldsymbol{N}(1)$ constructed by elements greater than or equal to 2, $h_{2t}(p(t))$ also has a limit $\hat{p}_2^* \in \Delta^{K(2)} \cap\ h_{22}(\pi(2))$. We define $p_2^* \in \boldsymbol{R}^{K(2) \setminus K(1)}$ as $\frac{1}{\|\,\mathrm{pr}_{K(1)}\, \hat{p}_2^*\|}\, \mathrm{pr}_{K(2) \setminus K(1)}\, \hat{p}_2^*$. Generally, from the compactness of $\Delta^{K(s)}$ for each $s \in \boldsymbol{N}$, if $t \to \infty$, by taking a subsequence, $\{p(t)\}_{t \in \boldsymbol{N}(s)}$, of $\{p(t)\}_{t \in \boldsymbol{N}(s-1)}$, where $\boldsymbol{N}(s)$ is a cofinal subset of $\boldsymbol{N}(s-1)$ constructed by elements greater than equal to s, $h_{st}(p(t))$ has a limit $\hat{p}_s^* \in \Delta^{K(s)} \cap h_{ss}(\pi(s))$. Hence we can define $p_s^* \in \boldsymbol{R}^{K(s) \setminus K(s-1)}$ as $p_s^* = \frac{1}{\|\,\mathrm{pr}_{K(1)}\, \hat{p}_s^*\|}\, \mathrm{pr}_{K(s) \setminus K(s-1)}\, \hat{p}_s^*$. By repeating the above procedure, we obtain $p^* = (p_1^*, p_2^*, \cdots)$. Since for each $s \in \boldsymbol{N}$, $\frac{(p_1^*, \cdots, p_s^*)}{\|(p_1^*, \cdots, p_s^*)\|} = \hat{p}_s^*$ is an element of $h_{ss}(\pi(s))$, p^* belongs to $h_{ss}^{-1}(\hat{p}_s^*) = \pi(s)$, so we have $p^* \in \bigcap_{t \in \boldsymbol{N}} \pi(t)$.

Since $x^i \succ_i \bar{x}^i$ means that both $x^i - \omega^i$ and $x^i - \bar{x}^i$ belong to Γ_i, we have $p^* \cdot x^i \geqq p^* \cdot \omega^i$ and $p^* \cdot x^i \geqq p^* \cdot \bar{x}^i$. By taking x^i arbitrarily near to $\bar{x}^i$ (from the local non-satiation property), we can see that $p^* \cdot \bar{x}^i \geqq p^* \cdot \omega^i$. Define $M_{\boldsymbol{\mathcal{E}}}^*(i) \geqq 0$ as $M_{\boldsymbol{\mathcal{E}}}^*(i) = p^* \cdot \bar{x}^i - p^* \cdot \omega^i$ for all i. Then, we have $p^* \cdot \bar{x}^i = p^* \cdot \omega^i + M_{\boldsymbol{\mathcal{E}}}^*$. In addition, the condition of initial endowments $\omega^i \gg 0$ for all i, implies that $p^* \cdot \omega^i > 0$. Since $x^i \succ_i \bar{x}^i$ means that $p^* \cdot x^i \geqq p^* \cdot \bar{x}^i$, the continuity of preference together with $p^* \cdot \omega^i + M_{\boldsymbol{\mathcal{E}}}^*(i) > 0$ implies that for every i, $\bar{x}^i$ is an individual maxima under p^* and $M_{\boldsymbol{\mathcal{E}}}^*$. ■

As we mentioned before, $\mathbb{C}^{m \oplus n}(\boldsymbol{\mathcal{E}}) \supset \mathbb{C}^{m' \oplus n'}(\boldsymbol{\mathcal{E}}) \supset \cdots$, where $m' \geqq m$, $n' \geqq n$, and $\boldsymbol{\mathcal{MWalras}}(\boldsymbol{\mathcal{E}}) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathbb{C}^{m \oplus n}(\boldsymbol{\mathcal{E}})$. Thus we obtained a replica finite-core limit equivalence theorem to the non-negative wealth transfer Walrasian equilibrium allocations, especially, the perfect-foresight monetary Walrasian equilibrium allocations for overlapping generation economies. Note that in the above proof, we do not assume preferences $\succ_i$s to be ordered. Our equivalence theorem can also be utilized to axiomatically characterize the price-wealth message mechanisms as Sonnenschein (1974), where the Debreu-Scarf limit theorem plays an essential role in showing the category theoretic main result (see, Urai and Murakami 2015).

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[12] Let $\Delta^{K(t)} \cap h_{tt}(\pi(t))$ be X_t for each $t \in \boldsymbol{N}$. Then the family $(X_t)_{t \in \boldsymbol{N}}$ and the family of mappings (h_{st}), $s \leqq t$, $s, t \in \boldsymbol{N}$ form an inverse system of compact spaces. By Bourbaki (1966, Chapter I, §9, no.6, Proposition 8), the inverse limit $X = \varprojlim X_t$ is non-empty. And price p^* that we seek in the following can be identified as an element of X.

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Samuelson, P. A. (1958): "An exact consumption loans model of interest with or without social contrivance of money," *Journal of Political Economy 66*(6), 467–482.

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Appendix

Let us consider overlapping generations economy $\mathcal{E}$ such that every agent lives for young and old two periods, where there is one consumption good for each period, and every t generation consists of two agents, i_t and i'_t, for $t = 1, 2, \cdots$. Each agent has initial endowment $(2 + \frac{\epsilon}{2}, \frac{\epsilon}{2})$, where $\epsilon > 0$ will be defined in the following as sufficiently small.

Consider allocation $x = (x^{i_1}, x^{i'_1}, x^{i_2}, x^{i'_2}, \cdots)$ such that $x^{i_t} = (0.1 + \epsilon, 0.1 + \epsilon)$ for all $t = 1, 2, \cdots$, $x^{i'_1} = (3.9, 0.9)$ and $x^{i'_t} = (2.9, 0.9)$ for all $t = 2, 3, \cdots$. Clearly, x is feasible. Assuming that all the agents' marginal rate of substitution at x is 1 (see Figure 3 for generations $t \geqq 2$), we have $p = (1, 1, \cdots)$ as the supporting price for allocation x, which means that the replica allocation of x is weakly Pareto-optimal for all replica economies: the n-fold replica allocation of x is an element of $\mathcal{F}core(\mathcal{E}^n(x))$ for all $n \in \boldsymbol{N}$. Assume further that all agents' preferences are of the Cobb-Douglas type; their utility becomes arbitrarily a small level when their consumption at one of their life time periods is near to 0. Then we can check that the replica allocation of x is a finite core allocation for all replica economies: the n-fold replica allocation of x is an element of $\mathcal{F}core(\mathcal{E}^n(\omega))$ for all $n \in \boldsymbol{N}$.

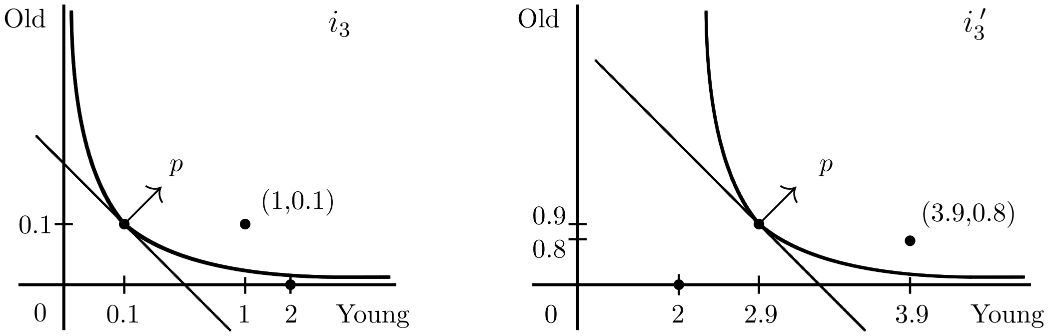

Figure 3: MRS for each agent is 1. Parameter $\epsilon > 0$ for each allocation is neglected.

For example, let $u^{i_t}(y_1, y_2) = y_1^{0.5} y_2^{0.5}$ for each $t = 1, 2, \cdots$, $u^{i'_1}(y_1, y_2) = (y_1 - 3)^{0.5} y_2^{0.5}$ when $y_1 \geqq 3$, $u^{i'_1}(y_1, y_2) = y_1 - 3$ when $0 \leqq y_1 \leqq 3$, $u^{i'_t}(y_1, y_2) = (y_1 - 2)^{0.5} y_2^{0.5}$ when $y_1 \geqq 2$ and $u^{i'_t}(y_1, y_2) = y_1 - 2$ when $0 \leqq y_1 \leqq 2$ for each $t = 2, 3, \cdots$. Then, no finite coalition in $\boldsymbol{\mathcal{E}}^n(\omega)$ can improve upon the n-fold replica allocation of x. Any finite coalition among members in the first generation fails to improve upon $(0.1 + \epsilon, 0.1 + \epsilon)$ as long as $\epsilon < 10^{-3}$. Indeed, utility level of at least one of such coalition members, i^*, should be less than or equal to $u^{i^*}(2 + \frac{\epsilon}{2}, \frac{\epsilon}{2}) = (2 + \frac{\epsilon}{2})^{0.5} \frac{\epsilon}{2}^{0.5}$ under the maximality for utility allocation with Cobb-Douglas type utility functions of homogeneity of degree 1. When $\epsilon < 10^{-3}$, such coalition fails to block any utility allocations greater than or equals to those under $(0.1 + \epsilon, 0.1 + \epsilon)$, hence never improve upon those under the n-times replica allocation of x. Suppose that the n-times replica allocation of x cannot be improved upon by any finite coalition among members of generations from 1 to $k - 1$. We show in the following that any finite coalition, S, among members from 1 to k also fails to block the n-times replica allocation of x. Let us denote S by $S_1 \cup S_2$, where S_1 is the set of members in generations from 1 to $k - 1$, and S_2 is the set of members in generation k. Note that between S_1 and S_2, we have only to consider two cases that there is a non-negative transfer of endowment commodity in period k from S_2 to S_1 or that there is a positive transfer of it from S_1 to S_2. For the first case, under the same discussion in the previous paragraph, it is impossible to make utility levels of members of S_2 greater than or equal to those under $(0.1 + \epsilon, 0.1 + \epsilon)$. For the second case, it would be possible to keep all utility levels of members of S_1 as good as those under x, but if so, by not doing such a positive endowment transfer, S_1 can improve upon the replica allocation of x, which contradicts the assumption. It follows that, by mathematical induction, the n-fold replica allocation of x is an element of $\boldsymbol{\mathcal{F}core}(\boldsymbol{\mathcal{E}}^n(\omega))$.

Allocation x is not a non-negative wealth transfer Walrasian equilibrium under p. (The wealth transfer for type i_t agents should be negative.) The two-fold replica allocation of x does not belong to $\boldsymbol{\mathcal{F}core}(\boldsymbol{\mathcal{E}}^1(x) \oplus \boldsymbol{\mathcal{E}}^1(\omega))$ since, for example, i_3 in $\boldsymbol{\mathcal{E}}^1(\omega)$ and i'_3 in $\boldsymbol{\mathcal{E}}^1(x)$ block the replica allocation of x with $(1 + \frac{\epsilon}{2}, 0.1 + \frac{\epsilon}{2})$ for i_3 and $(3.9, 0.8)$ for i'_3.