Discussion Papers In Economics And Business

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August 2016

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Kohei Shiozawa*

Abstract

This paper presents some substantial relationships between the revealed preference test for a data set and the *shortest path problem* of a weighted graph. We give a unified perspective of several forms of rationalizability tests based on the shortest path problem and an additional graph theoretic structure, which we call the *shortest path problem with weight adjustment*. Furthermore, the proposed structure is used to extend the result of Quah (2014), which sharpened the classical Afriat's Theorem-type result.

Keywords: Revealed preference; rationalizability; Afriat's inequalities; generalized axiom of revealed preference; shortest path problem

JEL classification: C60; D11

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1 Introduction

Given n observations of consumer's choices $x^k \in \mathbb{R}^\ell_+$ (for $k = 1, \dots, n$) and prices $p^k \in \mathbb{R}^\ell_{++}$ (for $k = 1, \dots, n$) of ℓ goods, we say that the data set $\mathcal{O} = \{(p^k, x^k)\}_{k=1}^n$ is *rationalizable* if there exists a non-satiated utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that, for each $k = 1, \dots, n$, the choice x^k is generated from the utility maximization problem with the utility function U and the budget constraint created by the price p^k and the wealth level $p^k \cdot x^k$. Afriat (1967) provided the first characterization of the rationalizability of data set $\mathcal{O}$. The Afriat's Theorem says that a given data set $\mathcal{O}$ is rationalizable *if and only if* there is a solution $U_k, \lambda_k > 0\,(k = 1, \dots, n)$ to the following system of inequalities:

$$U_{k'} \leqq U_k + \lambda_k p^k \cdot (x^{k'} - x^k) \qquad \text{for all } \; k, k' = 1, \dots, n. \tag{1}$$

This system of inequalities is known as the *Afriat's inequalities.* Moreover, these conditions are also equivalent to the *generalized axiom of revealed preference* (GARP), which requires that there is no *revealed preference* cycle containing a *revealed strict preference* where revealed (strict) preference is defined on the set of observed consumption $\mathcal{X} = \{x^k\}_{k=1}^n$ as $x^k \succeq^* (\succ^*)\, x^{k'}$ if $p^k \cdot x^k \geqq (>)\, p^k \cdot x^{k'}$ (Afriat 1967; Varian 1982).

Subsequent research focused on the properties of the data set that is rationalizable by a particular form of utility function, e.g., homotheticity (Varian 1983) and quasi-linearity (Brown and Calsamiglia 2007), and extended Afriat's Theorem for the rationalizability with integer observations (Fujishige and Yang 2012; Polisson and Quah 2013; Forges and Iehlé 2014) and the rationalizability with more general budget constraints including non-linear budget constraints (Matzkin 1991; Chavas and Cox 1993; Forges and Minelli 2009; Quah 2014). These various rationalizabilities are also characterized by the similar threefold form: the rationalizability, the feasibility of a system of inequalities (like Afriat's inequalities), and the no-cycle condition (like the GARP). Therefore, it is intuitive that these rationalizability problems may share some common mathematical structure. The main objective of this paper is to answer this intuitive question, and we present a unified framework for those rationalizability problems.

As noted by Piaw and Vohra (2003), among others, the form (1) is related to a combinatorial optimization problem called the *shortest path problem* (SPP), which seeks shortest paths from a given start point to all the other points in a given network. It is known that the feasibility of the SPP is also characterized in a threefold way with the feasibility of a system of inequalities similar to (1) and the absence of negative cycles in the network. In this paper, we present a graph theoretic framework, which is a common structure of the various rationalizability problems, based on the SPP.[1] Our argument uses the standard SPP and a modified version of it that we define in this paper. We call the modified problem the *SPP with weight adjustment* (SPPWA), which asks whether there is a weight adjustment under which the adjusted network has shortest paths from a given start point to all the other points. We show that

[1] Kolesnikov et al. (2013) also demonstrated this relationship using the Monge–Kantorovich mass transportation problem, which is a general mathematical framework containing the SPP. Here, we demonstrate the relationship in a complete manner using elementary graph theoretic arguments that do not require linear programming techniques or knowledge of the Monge–Kantorovich problem.

the feasibility of the SPPWA is characterized by a graph theoretic counterpart of the GARP, which is easily checked in empirical studies. Moreover, our graph theoretic argument allows us to extend the result of Quah (2014), which concluded that one could always find a rationalization that is compatible with a given rationally extended preference $\succeq$ of the revealed preference relation $\succeq^*$, and hence provided a sharper conclusion than Afriat's Theorem.[2] In this paper, we present a graph theoretic counterpart of Quah's problem as well as the result using the SPPWA framework, and extend Quah's result so that we can treat both the integer observations and the real observations in a unified way.

The rest of the paper is organized as follows. In Section 2, first, we introduce various rationalizability problems and review the characterization results. Next, we introduce the SPP and discuss its relation to the rationalizability problems. In Section 3, we formalize the SPPWA, present our first main result (Theorem 4), and apply it to the rationalizaiblity problems. In Section 4, we formalize the graph theoretic counterpart of the problem in Quah (2014) and present our second main result (Theorem 6), which extends the result of Quah (2014) using the SPPWA framework. Section 5 concludes this paper. All proofs are consigned to the Appendix.

2 Rationalizability and Shortest Path Problem

2.1 Rationalizability Problem

Let X denote a *consumption set* (a subset of some finite dimensional Euclidean space). A *budget set* on X is defined as $B = \{x \in X \,|\, g(x) \leqq 0\}$ for some (real-valued or integer-valued) function g defined on X. A *data set* on X is $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$ where $B^k = \{x \in X \,|\, g^k(x) \leqq 0\}$ is a budget set on X for the k-th observation and x^k is interpreted as the k-th observed consumption bundle such that $x^k \in X$ and $g^k(x^k) = 0$ for all $k = 1, \ldots, n$.[3] Let $\mathcal{X}$ denote the set of observed consumption bundles, i.e., $\mathcal{X} = \{x^k\}_{k=1}^n$ for any data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$.

Let U be a real-valued or an integer-valued utility function defined on a consumption set X. Let $\mathcal{O}$ be a data set $\{(B^k, x^k)\}_{k=1}^n$ on X. We say that U *rationalizes* $\mathcal{O}$ (or, $\mathcal{O}$ is *rationalized* by U) if, for all $k = 1, \ldots, n,\ y \in B^k \Rightarrow U(y) \leqq U(x^k)$.

In the following, we define various rationalizability problems by restricting the data set $\mathcal{O}$ and the utility function U in various ways. For this reason, we introduce some notation for the data set $\mathcal{O}$. We say that a data set $\mathcal{O}$ is a *linear budget data set* (*LB*-data set) if $X = \mathbb{R}^\ell_+$ and $g^k(x) = p^k \cdot (x - x^k)$ where $p^k \in \mathbb{R}^\ell_{++}$ and $x^k \in \mathbb{R}^\ell_+$ for all $k = 1, \ldots, n$. A data set $\mathcal{O}$ is a *linear budget data set with integer observations* (*LBI*-data set) if $X = \mathbb{Z}^\ell_+$ and $g^k(x) = p^k \cdot (x - x^k)$ where $p^k \in \mathbb{Z}^\ell_{++}$ and $x^k \in \mathbb{Z}^\ell_+$ for all $k = 1, \ldots, n$. A data set $\mathcal{O}$ is a *general budget data set* (*GB*-data set) if $X = \mathbb{R}^\ell_+$ and $g^k : \mathbb{R}^\ell_+ \to \mathbb{R}$ is a monotone and continuous function for all $k = 1, \ldots, n$.[4]

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[2] Indeed, he extended the Forges–Minelli Theorem (Forges and Minelli 2009), and in particular, Afriat's Theorem.

[3] Because, in the following, we assume either $X = \mathbb{R}^\ell_+$ or $X = \mathbb{Z}^\ell_+$, we shall often refer to a data set $\mathcal{O}$ without mentioning its consumption set X explicitly.

[4] A function $f : \mathbb{R}^\ell_+ \to \mathbb{R}$ is *monotone* if it satisfies $x \gg y \Rightarrow f(x) > f(y)$. We say that f is *strictly monotone* if it satisfies $x \geqq y$ and $x \neq y \Rightarrow f(x) > f(y)$.

Linear budget rationalizability is the most classical presented in Afriat (1967).

Definition 1. *A LB-data set $\mathcal{O}$ is linear budget rationalizable (LB-rationalizable) if there is a non-satiated utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$, which rationalizes $\mathcal{O}$.*

Homothetic rationalizability (Varian 1983) requires that the rationalizing function to be homothetic.

Definition 2. *A LB-data set $\mathcal{O}$ is homothetic rationalizable (H-rationalizable) if there is a non-satiated homothetic utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$, which rationalizes $\mathcal{O}$.*[5]

Quasi-linear rationalizability (Brown and Calsamiglia 2007) requires the data set to be extended so that the extended data set is rationalized by a quasi-linear utility function.

Definition 3. *A LB-data set $\mathcal{O}$ is quasi-linear rationalizable (Q-rationalizable) if there are non-negative numbers $(s^k)_{k=1}^n$ and a wealth level $I > 0$ such that $p^k \cdot x^k + s^k = I$ for all $k = 1, \ldots, n$, and LB-data set $\bar{\mathcal{O}} = \{(\bar{B}^k, (x^k, s^k))\}_{k=1}^n$ defined as $X = \mathbb{R}^{\ell+1}_{+}$ and $\bar{B}^k = \{(x, s) \in X \,|\, g^k((x, s)) \leqq 0\}$ where $g^k((x, s)) = p^k \cdot (x - x^k) + (s - s^k)$ for all $k = 1, \ldots, n$, is rationalized by a quasi-linear utility function $U : \mathbb{R}^{\ell+1}_{+} \to \mathbb{R}$ such that $U(x, s) = V(x) + s$ where $V : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is a continuous, concave, strictly monotone function.*

Linear budget rationalizability with integer observations (Fujishige and Yang 2012) requires that the integer observations are utility maximizers in an indivisible goods setting and also cost minimizers among those utility maximizing choices.[6]

Definition 4. *A LBI-data set $\mathcal{O}$ is linear budget rationalizable with integer observations (LBI-rationalizable) if there is a discrete concave utility function $U : \mathbb{Z}^{\ell}_{+} \to \mathbb{Z}$, which rationalizes $\mathcal{O}$, and it holds that $y \in \operatorname{argmax}\{U(x) \,|\, x \in B^k\} \Rightarrow p^k \cdot y \geqq p^k \cdot x^k$ for all $k = 1, \ldots, n$.*[7]

General budget rationalizability (Forges and Minelli 2009) considers the data set with possibly non-linear budgets.[8]

Definition 5. *A GB-data set $\mathcal{O}$ is general budget rationalizable (GB-rationalizable) if there is a monotone and continuous utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$, which rationalizes $\mathcal{O}$.*

Now, we review the characterization results. In essence, these results are threefold: the rationalizability, the feasibility of a system of inequalities, and a no-cycle condition. First, the Q-rationalizability and the H-rationalizability are characterized as follows.

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[5] A utility function $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is homothetic if it is a positive monotonic transformation of a function that is homogeneous of degree 1; that is, if $U(x) = f(g(x))$ where $g(x) : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ is homogeneous of degree 1 and $f : \mathbb{R} \to \mathbb{R}$ is positive monotonic.

[6] Polisson and Quah (2013) and Forges and Iehlé (2014) also considered rationalizability problems in the indivisible goods settings.

[7] A utility function $U : \mathbb{Z}^{\ell}_{+} \to \mathbb{Z}$ is *discrete concave* if, for any $x^1, x^2, \ldots, x^m \in \mathbb{Z}^{\ell}_{+}$ where $m \leqq \ell + 1$ and any rational numbers $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \ldots, \lambda_m \geqq 0$ where $\sum_{t=1}^m \lambda_t = 1$ and $\sum_{t=1}^m \lambda_t x^t \in \mathbb{Z}^{\ell}_{+}$, we have $U(\sum_{t=1}^m \lambda_t x^t) \geqq \sum_{t=1}^m \lambda_t U(x^t)$.

[8] Matzkin (1991) and Chavas and Cox (1993) also considered non-linear budgets settings, which are special cases of the setting of Forges and Minelli (2009). See also Quah (2014).

Theorem 1 (Varian 1983; Brown and Calsamiglia 2007)**.** *Suppose that $\mathcal{O}$ is a LB-data set and $S \in \{Q, H\}$. For all $k, k' = 1, \ldots, n$ such that $k \neq k'$, let $\ell((x^k, x^{k'})) = p^k \cdot (x^{k'} - x^k)$ if $S = Q$ and $\ell((x^k, x^{k'})) = \log(p^k \cdot x^{k'})$ otherwise. Then, the following three conditions are equivalent:*

(i) *The data set $\mathcal{O}$ is S-rationalizable.*

(ii) *There is a feasible solution $(U_k)_{k=1}^n \in \mathbb{R}^n$ to the following system of inequalities: $U_{k'} \leqq U_k + \ell((x^k, x^{k'}))$ for all $k, k' = 1, \ldots, n$ such that $k \neq k'$.*

(iii) *For all $m \geqq 2$ and $k_1, \ldots, k_m \in \{1, \ldots, n\}$, we have $\ell((x^{k_1}, x^{k_2})) + \cdots + \ell((x^{k_m}, x^{k_1})) \geqq 0$.*

Note that, for the H-rationalizability, this result is an equivalent version of the original characterization result of Varian (1983). As mentioned in Varian (1983), we take the log values of the original characterization conditions. Indeed, Condition (ii) is the log version of the feasibility condition of the original system of inequalities with the form $U_{k'} \leqq U_k(p^k \cdot x^{k'})$ where $U_k > 0\,(k = 1, \ldots, n)$ and Condition (iii) is the log version of the *homothetic axiom of revealed preference* (HARP), which has the form $(p^{k_1} \cdot x^{k_2})(p^{k_2} \cdot x^{k_3}) \cdots (p^{k_m} \cdot x^{k_1}) \geqq 1$. It is clear that these conditions are equivalent to each other.

For the LB, GB, and LBI-rationalizability characterizations, we first define the revealed preference relation and the GARP. For a given data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$ where $B^k = \{x \in X \mid g^k(x) \leqq 0\}$ and $g^k(x^k) = 0$ for all $k = 1, \ldots, n$, we say that x^k is *directly revealed (strictly) preferred* to $x^{k'}$ if $g^k(x^{k'}) \leqq (<)\, 0$, and denote as $x^k \succeq^* (\succ^*)\, x^{k'}$, for any pair of observed consumptions $x^k, x^{k'} \in \mathcal{X}$. We say that x^k is *directly revealed indifferent* to $x^{k'}$ if both $x^k \succeq^* x^{k'}$ and $x^{k'} \succeq^* x^k$ hold, and denote as $x^k \sim^* x^{k'}$. We say that $\mathcal{O}$ satisfies the *generalized axiom of revealed preference* (GARP) if there is a subset of observations $\{(B^{k_i}, x^{k_i})\}_{i=1}^m \subset \mathcal{O}$ such that $x^{k_1} \succeq^* x^{k_2}, \ldots, x^{k_m} \succeq^* x^{k_1}$ then those direct-revealed preference relations do not hold with $\succ^*$.[9] Using the GARP condition, the LB, GB, and LBI-rationalizability are characterized as follows.

Theorem 2 (Afriat 1967; Forges and Minelli 2009; Fujishige and Yang 2012)**.** *Suppose $S \in \{LB, GB, LBI\}$ and $\mathcal{O}$ is a S-data set. Let $K = \mathbb{R}$ if $S \in \{LB, GB\}$ and $K = \mathbb{Z}$ otherwise. Then, the following three conditions are equivalent:*

(i) *The data set $\mathcal{O}$ is S-rationalizable.*

(ii) *There is a feasible solution, $(U_k)_{k=1}^n \in K^n$ and $(\lambda_k)_{k=1}^n \in K_{++}^n$, to the following system of inequalities: $U_{k'} \leqq U_k + \lambda_k g^k(x^{k'})$ for all $k, k' = 1, \ldots, n$, where g^k are the functions that define the budget sets of $\mathcal{O}$.*

(iii) *The data set $\mathcal{O}$ satisfies the GARP.*

Note that, for the LBI-rationalizability, the GARP condition is *not* equivalent to the simple utility maximization behavior. Instead the GARP is equivalent to the utility maximization and the cost effi-

[9] The original definition of the GARP was based on the *revealed preference relation* $\succeq^{**}$, which is the transitive closure of the direct-revealed preference relation $\succeq^*$ (Varian 1982). However, the original definition is equivalent to that defined here, which is also referred to as the *cyclical consistency* condition from Afriat (1967). Note that, for the rest of our analysis and results, we do not need the revealed preference relation $\succeq^{**}$ and hence we omit it. The treatment of the GARP and the revealed preference relation $\succeq^{**}$ in this manner is common practice in the current literature. (See, for example, Fostel et al. (2004) and Quah (2014).)

ciency in the sense of Definition 4.[10] In the following, we refer to both Condition (ii) in Theorem 1 and Condition (ii) in Theorem 2 as the *Afriat's inequalities* condition.

In the next subsection, we shall see that these rationalizability conditions are characterized also by the feasibility of particular instances of the SPP. Indeed, it is known that the feasibility of the SPP is characterized in a threefold way: the feasibility of the SPP, the feasibility of a system of inequalities, and a non-negative cycle condition. The latter two conditions are actually generalizations of Conditions (ii) and (iii) in Theorem 1 and hence, we can characterize the rationalizability also by the feasibility of the SPP.

2.2 Shortest Path Problem and Rationalizability

The SPP asks for the shortest directed paths from one vertex to each of the other vertices in a given directed graph with weighted edges. Consider the examples in Figure 1. Both graphs consist of the same vertex and edge sets, namely, five vertices $V = \{a, b, c, d, e\}$ and eight edges $E = \{(a, b), (a, d), (a, e), (b, d), (c, b), (c, e), (d, c), (e, d)\}$. However, the graph on the left has shortest paths from vertex a to all the other vertices, whereas the graph on the right does not have a shortest path from a to b, c, or d. This is because any path to those vertices can be shortened by extending it along the closed path $b \to d \to c \to b$, which has total length -1.

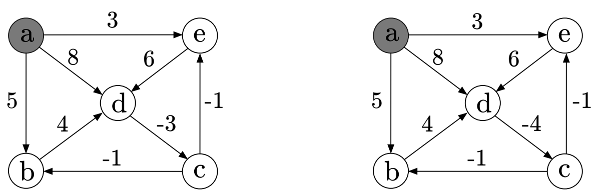

Figure 1

Formally, a *directed graph* is a pair $G = (V, E)$ where V is a finite set and E is a subset of the ordered pairs $V \times V$. We call an element $v \in V$ a *vertex* and an element $(v, u) \in E$ an *edge*.[11] A *path* (or a v_0-v_m *path*) $P : v_0 \to v_1 \to \cdots \to v_m$ is a sequence of vertices and edges $v_0(v_0, v_1)v_1(v_1, v_2)v_2 \cdots v_{m-1}(v_{m-1}, v_m)v_m$. A path is called a *cycle* if $v_0 = v_m$. A real-valued (or an integer-valued) function $\ell : E \to \mathbb{R}$ (or $\ell : E \to \mathbb{Z}$) is called a *weight function* and represents the *weight* (or *length*) *of the edge* $\ell((v, u))$ for each edge $(v, u) \in E$. The *weight* (or *length*) *of a path*

[10] Whereas Fujishige and Yang (2012) characterized the GARP in the integer observations setting by assuming the cost efficiency, Polisson and Quah (2013) characterized it by assuming the existence of an implicit divisible good in the sense that the data set can be extended in a similar way to the quasi-linear rationalizability of Brown and Calsamiglia (2007). In contrast, Forges and Iehlé (2014) characterized the simple utility maximization behavior in the integer observations setting by modifying the GARP to the *discrete axiom of revealed preference* (DARP), which is a relaxed version of the GARP in the integer observations setting.

[11] In this paper, we only consider *simple* graphs. That is, we assume that there are no *loops* or *parallel edges* in $G = (V, E)$.

$P : v_0 \to v_1 \to \cdots \to v_m$ is the sum of the lengths of its edges $\ell(P) := \sum_{k=0}^{m-1} \ell((v_k, v_{k+1}))$. A *shortest path* from vertex s to vertex v is a s-v path that has the minimum length over all s-v paths.[12]

The following theorem is a known characterization for the feasibility of the SPP. For completeness, we present a simple proof of this characterization in the Appendix. See Murota and Shioura (2013) and Korte and Vygen (2012) for more detailed discussions.

Theorem 3. *Let $K = \mathbb{R}$ or $\mathbb{Z}$. Let $G = (V, E)$ be a directed graph with n vertices and m edges. Let $\ell : E \to K$ be a weight function. Suppose that there is a path from vertex $s \in V$ to each of the other vertices $v \in V \setminus \{s\}$. Then, the following conditions are equivalent:*

- (i) *There are shortest paths from s to the other vertices v.*
- (ii) *There is a vector $\pi = (\pi(v))_{v \in V} \in K^n$ such that $\pi(u) \leqq \pi(v) + \ell((v, u))$ for all $(v, u) \in E$.*
- (iii) *There is no negative length cycle in the graph G with respect to weight ℓ.*

A vector π that satisfies Condition (ii) above is called a *feasible potential*. In the above examples, the graph on the left has a feasible potential $(\pi(a), \pi(b), \pi(c), \pi(d), \pi(d)) = (0, 4, 5, 8, 3)$. However, the graph on the right does not have a feasible potential because of the negative length cycle $b \to d \to c \to b$. If there was a feasible potential $\tilde{\pi}$, then $\tilde{\pi}(b) \leqq \tilde{\pi}(d) + 4 \leqq (\tilde{\pi}(c) - 4) + 4 \leqq (\tilde{\pi}(b) - 1) - 4 + 4$, which results in the contradiction $0 \leqq -1$.

Now, observe that Condition (ii) in Theorem 3 is a generalization of Afriat's inequalities, (ii) in Theorem 1. Moreover, Condition (iii) in Theorem 1 is a particular case of the no-negative-cycle condition, (iii) in Theorem 3.[13] As mentioned in Varian (1983), Piaw and Vohra (2003), Fujishige and Yang (2012), and Nobibon et al. (2015), we can construct a directed graph and associate it with weights generated from the data set $\mathcal{O}$.[14] The directed graph $G = (V, E)$ is constructed by introducing a vertex for each consumption bundle $x^k \in \mathcal{X}$ and an edge for each pair of distinct indices $(x^k, x^{k'})$. That is,

$$V := \mathcal{X} = \{x^1, x^2, \ldots, x^n\} \quad \text{and} \quad E := \{(x^k, x^{k'}) \mid k, k' = 1, \ldots, n \text{ and } k \neq k'\}. \tag{2}$$

If we take the weights $\ell_Q : E \to \mathbb{R}$ and $\ell_H : E \to \mathbb{R}$ to be

$$\ell_Q((x^k, x^{k'})) := p^k \cdot (x^{k'} - x^k) \quad \text{and} \quad \ell_H((x^k, x^{k'})) := \log(p^k \cdot x^{k'}) \tag{3}$$

where p^k and x^k are the prices and quantities that define the budget sets of the *LB*-data set $\mathcal{O}$. Then, we have a SPP characterization of the rationalizability problems. (We omit the proof as it is obvious from Theorem 1 and Theorem 3.)

[12] Note that we use the word *weight* or *length* even if $\ell((v, u))$ or $\ell(P)$ is negative.

[13] It is interesting that almost the same form of the no-cycle condition appears in the mechanism design literature (Rochet 1987). Rochet's result concerns the *rationalizability of a public decision function* in a quasi-linear setting. The problem asks, for any arbitrary given decision function, whether there is a *monetary transfer function* with which the decision–transfer pair has the truth-telling property for a given agent equipped with a parameterized quasi-linear utility function. He showed that the rationalizability of a given decision function for a given agent is characterized by a *non-positive* cycle condition based on the agent's parameterized quasi-linear utility function and the decision function (Theorem 1, Rochet 1987).

[14] Koo (1971) also investigated the graph theoretic representations of revealed preferences.

Proposition 1. *Let $\mathcal{O}$ be a LB-data set and $S \in \{Q, H\}$. Let $G = (V, E)$ be defined by (2) and $\ell_S : E \to \mathbb{R}$ be defined as (3). Then, the following conditions are equivalent:*

(i) *The data set $\mathcal{O}$ is S-rationalizable.*
(ii) *There is a feasible solution $(U_k)_{k=1}^n \in \mathbb{R}^n$ to the following system of inequalities: $U_{k'} \leqq U_k + \ell_S((x^k, x^{k'}))$ for all $(x^k, x^{k'}) \in E$.*
(iii) *There is no negative length cycle in $G = (V, E)$ with $\ell_S : E \to \mathbb{R}$.*
(iv) *There are shortest paths from x^1 to every other vertices in $G = (V, E)$ with $\ell_S : E \to \mathbb{R}$.*

For the LB, GB, and LBI-rationalizabilities, the arguments are the same and straightforward. Indeed, the correct SPP instance is clear if we compare the feasible potential condition of the SPP, (ii) in Theorem 3, with Afriat's inequalities of the rationalizabilities, (ii) in Theorem 2. Formally, for any $S \in \{LB, GB, LBI\}$, we consider the weight function for each S-data set as follows.

$$\ell_S((x^k, x^{k'}); \lambda) := \lambda_k g^k(x^{k'}) \quad \text{for any} \quad \lambda = (\lambda_k)_{k=1}^n \in K_{++}^n \tag{4}$$

where $K = \mathbb{R}$ if $S \in \{LB, GB\}$ and $K = \mathbb{Z}$ otherwise, and g^k are functions that define the budget sets of the S-data set $\mathcal{O}$.

Proposition 2. *Suppose $S \in \{LB, GB, LBI\}$ and $\mathcal{O}$ is a S-data set. Let $K = \mathbb{R}$ if $S \in \{LB, GB\}$ and $K = \mathbb{Z}$ otherwise. Let $G = (V, E)$ be defined by (2) and $\ell_S(\lambda) : E \to K$ be defined as (4). Then, the following three conditions are equivalent:*

(i) *The data set $\mathcal{O}$ is S-rationalizable.*
(ii) *There is a feasible solution, $(U_k)_{k=1}^n \in K^n$ and $\lambda \in K_{++}^n$, to the following system of inequalities: $U_{k'} \leqq U_k + \ell_S((x^k, x^{k'}); \lambda)$ for all $(x^k, x^{k'}) \in E$.*
(iii) *The data set $\mathcal{O}$ satisfies the GARP.*
(iv) *There is some $\lambda \in K_{++}^n$ such that the directed graph G with the weight $\ell_S(\lambda) : E \to K$ has shortest paths from x^1 to each of the other vertices.*
(v) *There is some $\lambda \in K_{++}^n$ such that the directed graph G with the weight $\ell_S(\lambda) : E \to K$ has no negative length cycle.*

Again, we omit the proof for Proposition 2. However, note that for the LBI-rationalizability, the weight is integer-valued; $\ell_{LBI}((x^k, x^{k'}); \lambda) = \lambda_k g^k(x^{k'}) = \lambda_k p^k \cdot (x^{k'} - x^k)$, and hence, the Afriat's inequalities condition is equivalent to the *integer* feasible potential condition in Theorem 3.

As shown in Proposition 1, the Q-rationalizability and the H-rationalizability are actually characterized by particular instances of the SPP. However, as shown in Proposition 2, the LB, GB, and LBI-rationalizabilities are characterized by a modified problem. The modified problem asks if there is a *weight adjustment* $\lambda \in \mathbb{R}_{++}^n$ (or $\lambda \in \mathbb{Z}_{++}^n$) under which the graph with the adjusted weight $\ell_S((x^k, x^{k'}); \lambda) = \lambda_k g^k(x^{k'})$ has shortest paths from a given start point to the other points. In the next section, we formalize this modified problem with graph theoretic apparatus and characterize its feasibility. Through this argument, we can develop a graph theoretic counterpart of the GARP, which characterizes the feasibility of the modified problem.

3 Rationalizability and Shortest Path Problem with Weight Adjustment

Proposition 2 shows that we can characterize the LB, GB, and LBI-rationalizabilities using a modified SPP. The modified problem asks if there is a weight adjustment $\lambda = (\lambda_v)_{v\in V} \in \mathbb{R}^{|V|}_{++}$ (or $\lambda = (\lambda_v)_{v\in V} \in \mathbb{Z}^{|V|}_{++}$) under which the graph with adjusted weight $\lambda_v \ell((v,u))$ has shortest paths from a given start point to the other points. We call this modified problem the *SPP with weight adjustment* (SPPWA). Note that the adjustment λ never changes the signs of the weights $\ell((v,u))$ because $\lambda \gg 0$. Additionally, note that the adjustment $\lambda_v > 0$ is identical for all the edges from the same vertex v.

Consider the examples in Figure 2. Both graphs consist of the same vertex and edge sets, five vertices $V = \{a,b,c,d,e\}$ and eight edges $E = \{(a,b),(a,d),(a,e),(b,d),(c,b),(c,e),(d,c),(e,d)\}$. The graph on the left has a negative length cycle $b \to d \to c \to b$. However, it has an adjustment $(\lambda_a,\lambda_b,\lambda_c,\lambda_d,\lambda_e) := (1,2,1,1,1)$, so that adjusted weight never results in a negative length cycle (and hence, Theorem 3 implies that there are shortest paths from a to all other vertices in the adjusted network). In contrast, the graph on the right does not have such an adjustment, because the negative length cycle $b \to d \to c \to b$ is always a negative length cycle under any adjustment $\lambda \gg 0$, as it does not contain a positive edge. That is, $\lambda_b 0 + \lambda_d(-4) + \lambda_c(-1) \leqq \min\{\lambda_b,\lambda_d,\lambda_c\}(0-4-1) < 0$. Moreover, note that the graph on the left does not contain such a problematic cycle. An important point that is highlighted by these

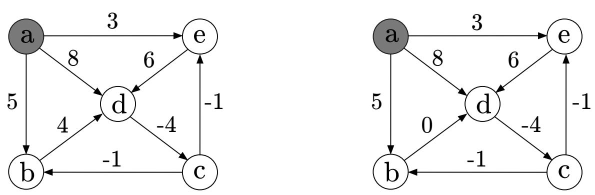

Figure 2

examples is *the existence of a negative length cycle that consists of only non-positive edges.* Indeed, we can characterize the feasibility of the SPPWA using this condition. The following theorem is our first main result.

Theorem 4. *Let $K = \mathbb{R}$ or $\mathbb{Z}$. Let $G = (V,E)$ be a directed graph with n vertices and m edges. Let $\ell : E \to K$ be a weight function. Suppose that there is a path from vertex $s \in V$ to each of the other vertices $v \in V \setminus \{s\}$. For any vector $\lambda = (\lambda_v)_{v\in V} \in K^n_{++}$, define an adjusted weight function $\ell(\lambda) : E \to K$ as*

$$\ell((v,u);\lambda) := \lambda_v \ell((v,u)) \quad \text{for all } (v,u) \in E.$$

Define a subset E_{np} of the edges E as

$$E_{np} := \{(v,u) \in E \,|\, \ell((v,u)) \leqq 0\}.$$

Then, the following conditions are equivalent:

(i) *There is a vector* $\lambda \in K^n_{++}$ *such that the graph* G *with the adjusted weight* $\ell(\lambda)$ *has shortest paths from* s *to each of the other vertices* v.

(ii) *There is a vector* $\lambda \in K^n_{++}$ *and a vector* $\pi \in K^n$ *such that* $\pi(u) \leqq \pi(v) + \ell((v,u);\lambda)$ *for all* $(v,u) \in E$.

(iii) *There is a vector* $\lambda \in K^n_{++}$ *such that the graph* G *with the adjusted weight* $\ell(\lambda)$ *has no negative length cycle.*

(iv) *The graph* $G_{np} := (V, E_{np})$ *has no negative length cycle with respect to the original weight* $\ell : E \to K$ *(a cycle containing a negative weighted edge with respect to the original weight* $\ell : E \to K$*).*

(v) *The graph* $G_{np} := (V, E_{np})$ *has no strongly connected component (SCC) that contains a negative weighted edge with respect to the original weight* $\ell : E \to K$.[15]

We call $(\pi, \lambda) \in K^n \times K^n_{++}$ in Condition (ii) a *feasible solution* of the SPPWA. Note that the equivalences of the first three conditions are obvious consequences of Theorem 3: the characterization conditions for the SPP. In the above examples in Figure 2, we observed that the non-existence of a negative length cycle that consists of only non-positive edges may characterize the feasibility of the SPPWA, and the equivalence (iii) $\Leftrightarrow$ (iv) says that the observation is correct. Indeed, the subgraph G_{np} can be constructed by dropping all the positive weighted edges of the originally given graph and weight, (G, ℓ). Therefore, a negative length cycle in G_{np} with respect to the original weight $\ell : E \to K$ is a negative length cycle in G with $\ell : E \to K$ that consists of only non-positive edges. Finally, (iv) $\Leftrightarrow$ (v) holds through a graph theoretic translation.[16]

We now return to the rationalizability problems. By Proposition 2, the *LB*, *GB*, and *LBI*-rationalizability problems are characterized by particular instances of the SPPWA. Formally, the directed graph is defined as the same as (2), i.e., $V := \{x^1, x^2, \ldots, x^n\}$ and $E := \{(x^k, x^{k'}) \,|\, k, k' = 1, \ldots, n \text{ and } k \neq k'\}$, and the weight and the adjusted weight functions are defined as follows: for all $S \in \{LB, GB, LBI\}$, and for any S-data set $\mathcal{O}$ and any $\lambda = (\lambda_k)_{k=1}^n \in K^n_{++}$,

$$\ell_S((x^k, x^{k'})) := g^k(x^{k'}) \quad \text{and} \quad \ell_S((x^k, x^{k'}); \lambda) = \lambda_k \ell_S((x^k, x^{k'})) \tag{5}$$

where $K = \mathbb{R}$ if $S \in \{LB, GB\}$ and $K = \mathbb{Z}$ otherwise, and g^k are functions that define the budget sets of the S-data set $\mathcal{O}$. Then, Proposition 2 and Theorem 4 imply the following proposition.

---

[15] A directed graph $G = (V, E)$ is *strongly connected* if each pair of vertices v and u are *strongly connected*; that is, there are v-u and u-v paths in G. A subgraph C of a graph $G = (V, E)$ is a SCC of G if there is no strongly connected subgraph C' of G where C is a subgraph of C' and $C \neq C'$, i.e., if C is a maximal strongly connected subgraph of G. Because the strong connectivity of a pair of vertices is an equivalence relation, we can decompose the set of vertices, V, into equivalence classes by the strong connectivity relation. This decomposition is called the SCC decomposition of the directed graph. Note that we use the term SCC of a directed graph to describe the vertex sets that decompose V into the SCC decomposition and the subgraphs that are induced by each of such vertex sets.

[16] Condition (v) can be checked easily using the SCC decomposition algorithm (e.g., STRONGLY CONNECTED COMPONENT ALGORITHM, Korte and Vygen 2012). Hence, as shown in Proposition 3 below, we can easily test the rationalizability of the economic data set. This procedure was also proposed by Fujishige and Yang (2012) and Nobibon et al. (2015) for the *LBI* and *LB*-rationalizabilities, respectively.

Proposition 3. *Suppose $S \in \{LB, GB, LBI\}$ and $\mathcal{O}$ is a S-data set. Let $G = (V, E)$ defined as (2) and g^k be the functions that define the budget sets of $\mathcal{O}$. Then, the following conditions are equivalent:*

(i) *The data set $\mathcal{O}$ is S-rationalizable.*

(ii) *The subgraph graph $G_{np} = (V, E_{np})$ of G, where $E_{np} = \{(x^k, x^{k'}) \in E \mid g^k(x^{k'}) \leqq 0\}$, has no cycle that contains an edge $(x^k, x^{k'})$ satisfying $g^k(x^{k'}) < 0$.*

(iii) *The graph G_{np} has no SCC that contains an edge $(x^k, x^{k'})$ satisfying $g^k(x^{k'}) < 0$.*

Remark 1: Under the language of preferences, the graph $G_{np} = (V, E_{np})$ has an edge $(x^k, x^{k'})$ if the consumption bundle x^k is directly revealed preferred to $x^{k'}$. Hence, the graph $G_{np} = (V, E_{np})$ is a graph theoretic representation of the direct-revealed preference relation $\succeq^*$ on the finite set $V = \mathcal{X} = \{x^k\}_{k=1}^n$. Formally, $\succeq^*$ is defined as $x^k \succeq^* x^{k'} \Leftrightarrow g^k(x^{k'}) \leqq 0$ and so, if we construct a directed graph $G_{\succeq^*} := (V, E_{\succeq^*})$ where $E_{\succeq^*} := \{(x^k, x^{k'}) \mid k, k' = 1, \ldots, n,\ k \neq k',$ and $x^k \succeq^* x^{k'}$ holds$\}$, then we have $G_{np} = G_{\succeq^*}$.

Remark 2: The GARP is actually equivalent to Condition (ii) in Proposition 3. In other words, Condition (ii) in Proposition 3 is a graph theoretic representation of the GARP. Indeed, the GARP requires that *if we have "$x^{k_1} \succeq^* x^{k_2}, \ldots, x^{k_m} \succeq^* x^{k_1}$ then, those revealed preference relations do not hold with $\succ^*$"*. If we restate it with the original data, then it becomes *if we have $g^{k_1}(x^{k_2}) \leqq 0, \ldots, g^{k_m}(x^{k_1}) \leqq 0$ then, those inequalities do not hold with strict inequality.* Finally, if we translate this condition into the graph theoretic language we developed above, then it becomes *each cycle in G_{np} contains zero-valued edges only*, and it is obviously equivalent to Condition (ii) in Proposition 3, which requires that *there is no cycle in G_{np} that contains a negative-valued edge; an edge $(x^k, x^{k'})$ satisfying $g^k(x^{k'}) < 0$.*

This section investigates a graph theoretic structure of the LB, GB, and LBI-rationalizability. In other words, we recover Afriat's Theorem-type results by a graph theoretic framework. However, Afriat's Theorem (for the LB-data set setting) and the Forges–Minelli Theorem (for the GB-data set setting) were extended by Quah (2014) for more sharper results. The result says *we can rationalize any rational completion of the revealed preference relation* and implies that no preference that is consistent with the data set $\mathcal{O}$ can be eliminated by rationality. In the next section, we further recover this sharper result based on the SPPWA framework and extend Quah's result for integer observations (for the LBI-data set setting).

4 Rationalizability of Consistent Preferences and SPPWA

4.1 The Quah's Theorem

Suppose that $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$ is a GB-data set. We call a binary relation $\succeq$ on a set of observed consumption bundles $\mathcal{X} = \{x^k\}_{k=1}^n$ a *preference relation* (or a *rational relation*) if it is reflexive, transitive, and complete. We say a relation between x^k and $x^{k'}$ is *strict* if $x^k \succeq x^{k'}$ and $x^{k'} \not\succeq x^k$, denoting it as $x^k \succ x^{k'}$, and a relation between x^k and $x^{k'}$ is *indifferent* if $x^k \succeq x^{k'}$ and $x^{k'} \succeq x^k$, denoting it $x^k \sim x^{k'}$. A preference relation $\succeq$ is said to be a *consistent preference* with the direct-revealed preference relation $\succeq^*$ (or, with the data set $\mathcal{O}$) if it satisfies the following two conditions: (i) $x^k \succeq^* x^{k'} \implies x^k \succeq x^{k'}$

and (ii) $x^k \succ^* x^{k'} \implies x^k \succ x^{k'}$.[17] In words, a consistent preference relation is a rational extension of the direct-revealed preference relation, which preserves the direct-revealed *strict* preferences as *strict* preferences. Note that condition (i) implies $x^k \sim^* x^{k'} \implies x^k \sim x^{k'}$ and a consistent preference also preserves any direct-revealed indifference relation. Finally, we say that the revealed preference relation $\succeq^*$ (or the data set $\mathcal{O}$) *admits a consistent preference* if there is a consistent preference with it.

As noted in Proposition 2 of Quah (2014), the data set $\mathcal{O}$ admits a consistent preference *if and only if* it satisfies the GARP, and in particular, any data set with a consistent preference can be rationalizable. The following result, which was shown in Quah (2014), further says that *any consistent preference on $\mathcal{X}$ can be extended to a rationalizing preference.*[18]

Theorem 5 (Quah 2014, Theorem 2)**.** *Suppose that a GB-data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$ admits a consistent preference $\succeq$. Then there is a monotone and continuous utility function $U : \mathbb{R}_+^\ell \to \mathbb{R}$ that satisfies the following two conditions:*

(i) *$U : \mathbb{R}_+^\ell \to \mathbb{R}$ rationalizes the data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$, and*
(ii) *$U : \mathbb{R}_+^\ell \to \mathbb{R}$ preserves the consistent preference $\succeq$, i.e., $U(x^k) > (=) U(x^{k'})$ if and only if $x^k \succ (\sim) x^{k'}$ for all $k, k' = 1, \ldots, n$.*

He shows this result by showing the following Lemma, which we focus on in the next subsection.

Lemma 1 (Quah 2014, Proof of Theorem 2)**.** *Suppose that a GB-data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$ admits a consistent preference $\succeq$. Then there is a solution of Afriat's inequalities that preserves $\succeq$, i.e., there are real numbers $U_k, \lambda_k > 0 \, (k = 1, \ldots, n)$ satisfying the following two conditions:*

(i) *$U_{k'} \leqq U_k + \lambda_k g^k(x^{k'})$ for all $k, k' = 1, \ldots, n$, and*
(ii) *$U_k > (=) U_{k'}$ if and only if $x^k \succ (\sim) x^{k'}$ for all $k, k' = 1, \ldots, n$,*

where g^k are the functions that define the budget sets of $\mathcal{O}$.

As noted in Remark 1 of Proposition 3, the graph G_{np} of the SPPWA constructed from the data set $\mathcal{O}$ is a graph theoretic representation of the direct-revealed preference relation $\succeq^*$. Hence, it is intuitive that a problem asking whether there is a feasible solution of the SPPWA that preserves a rational extension of G_{np} is a generalization of the above rationalizability problem of Quah (2014). In the next subsection, we present this problem with graph theoretic apparatus and extend the result of Quah (2014).

[17] As noted in Quah (2014), the consistency of a preference relation that we have defined here is equivalent to the one that is defined by the conditions (i) and (ii) with the revealed preference $\succeq^{**}$ and the revealed strict preference $\succ^{**}$, instead of the direct-revealed preference $\succeq^*$ and the direct-revealed strict preference $\succ^*$.

[18] For consistency in terminology, we state Quah's Theorem based on the Forges–Minelli-type general budget rationalizability setting. Hence, in the presented result, the rationalizing utility function is monotone and continuous, instead of *strongly* monotone and continuous as in the original Quah Theorem (Theorem 2, Quah 2014).

4.2 SPPWA with Consistent Rational Extension of G_{np}

Suppose we have an instance of the SPPWA, $G = (V, E)$ and $\ell : E \to K$, where $K = \mathbb{R}$ or $\mathbb{Z}$. We say that a graph with the same vertex set V, $G_r = (V, E_r)$, is *complete* if for any pair of vertices $v, u \in V$ there is at least one edge $(v, u) \in E_r$ or $(u, v) \in E_r$, and is *transitive* if it satisfies that for any pair of edges such that $(v, u), (u, w) \in E_r$ there is the edge $(v, w) \in E_r$. A graph $G_r = (V, E_r)$ is *rational* if it complete and transitive.[19] A rational graph $G_r = (V, E_r)$ is *consistent* with $(G, \ell : E \to K)$ if the following two conditions are satisfied: (i) G_{np} is a subgraph of G_r and (ii) if $(v, u) \in E_{np}$ and $\ell((v, u)) < 0$, then $(v, u) \in E_r$ and $(u, v) \notin E_r$, where $G_{np} = (V, E_{np})$ is defined as $E_{np} = \{(v, u) \in E \mid \ell((v, u)) \leqq 0\}$.

Note that a preference relation $\succeq$ on $\mathcal{X} = \{x^k\}_{k=1}^n$ in the previous subsection corresponds to a rational graph $G_r = (\mathcal{X}, E_r)$ where

$$E_r = \{(x^k, x^{k'}) \mid k', k = 1, \ldots, n,\ k' \neq k, \text{ and } x^k \succeq x^{k'} \text{ holds}\} \tag{6}$$

in the current framework. Hence, intuitively, an edge $(v, u) \in E_r$ indicates that v *is at least as good as* u. Note also that if we define $(G, \ell : E \to K)$ as in (2) and (5), and G_r as in (6) for any data set $\mathcal{O}$ and any preference $\succeq$, the consistency of $\succeq$ with $\mathcal{O}$ clearly is equivalent to the consistency of G_r with $(G, \ell : E \to K)$. Hence, our graph theoretic formalization actually generalizes Quah's problem.

In our framework, analogously to Quah's original formalization, we see that the existence of a consistent rational graph with an instance of the SPPWA, $(G, \ell : E \to K)$, is a necessary and sufficient condition for the feasibility of the SPPWA. (Hence, it is equivalent to all conditions in Theorem 4). Formally, the following proposition holds.

Proposition 4. *For any instance of the SPPWA, $G = (V, E)$ and $\ell : E \to K$, let G_{np} be the subgraph of G defined in Theorem 4. Then, the following two conditions are equivalent:*

(i) *The graph $G_{np} = (V, E_{np})$ has no negative length cycle with respect to the weight $\ell : E \to K$ (a cycle containing a negative weighted edge with respect to the weight $\ell : E \to K$).*
(ii) *There is a consistent rational graph $G_r = (V, E_r)$ with the instance $(G, \ell : E \to K)$.*

Finally, we state our second main result. It gives a graph theoretic generalization of Quah's Lemma.

Theorem 6. *For any instance of the SPPWA, $G = (V, E)$ and $\ell : E \to K$, with a consistent rational graph, $G_r = (V, E_r)$, there exists a feasible solution of the SPPWA, which preserves the consistent rational graph. That is, there is $(\pi, \lambda) \in K^n \times K^n_{++}$ that satisfies the following two conditions:*

(i) $\pi(u) \leqq \pi(v) + \ell((v, u); \lambda)$ *for all* $(v, u) \in E$, *and*
(ii) $\pi(v) > \pi(u)$ *if and only if* $(v, u) \in E_r$ *and* $(u, v) \notin E_r$ *and*
$\pi(v) = \pi(u)$ *if and only if* $(v, u) \in E_r$ *and* $(u, v) \in E_r$, *for all* $v, u \in V$.

[19] We say that a graph is rational without the reflexivity, i.e., we implicitly assume that there are loops $(v, v) \in E$ for all $v \in V$ whenever we say that a graph $G = (V, E)$ is rational. However, we do not need these loops for the results and/or analysis and hence, we do not mention them any further.

Remark 1: Because our result applies to any instance of the SPPWA with a real-valued or an integer-valued weight, it extends Quah's Lemma to the integer observations, and hence Quah's Theorem with the LBI-rationalizability. Indeed, as argued in the proof of Theorem 1 of Fujishige and Yang (2012), if we define a utility function $U : \mathbb{Z}^\ell_+ \to \mathbb{Z}$ as

$$U(x) := \min\{U_k + \lambda_k p^k \cdot (x - x^k) \,|\, k = 1, \dots, n\}$$

where $((U_k)_{k=1}^n, (\lambda_k)_{k=1}^n)$ is a solution of Afriat's inequalities obtained by Theorem 6 then, it LBI-rationalizes the data set $\mathcal{O}$. Moreover, it is known that this utility function satisfies that $U(x^k) = U_k$ for all $k = 1, \dots, n$. Hence, from Condition (ii) in Theorem 6, the utility function U clearly preserves the consistent preference $\succeq$ represented by G_r, which is defined as in (6). Therefore, Quah's Theorem is extended for the integer observations setting.

Remark 2: The implication "(iv) ⇒ (ii)" of Theorem 4 follows from Proposition 4 and Theorem 6, and it follows that, in particular, the existence of a feasible solution of the SPPWA preserving a consistent rational graph G_r of $(G, \ell : E \to K)$ is also equivalent to all conditions in Theorem 4.

Remark 3: The proof of Theorem 6 is based on an algorithm to compute a feasible solution of the SPPWA preserving the consistent rational graph. This algorithm is a modified version of the Varian–Quah algorithm (Varian 1982; Quah 2014) based on the SCC decomposition of G_r. The idea to use the SCC decomposition structure is similar to the one in Fujishige and Yang (2012). Moreover, this algorithm can easily be modified to compute a feasible solution of the SPPWA based only on the instance $G = (V, E)$ with $\ell : E \to K$. (See the proof of Theorem 6 in the Appendix.)

5 Concluding Remark

Apart from the revealed preference tests, many researchers investigated *goodness-of-fit measures* for the GARP (Afriat 1973; Houtman and Maks 1985; Varian 1990; Echenique et al. 2011; Smeulders et al. 2013; Dean and Martin 2015). They proposed some indices as goodness-of-fit measures for the GARP, but some of them have been shown to be computationally difficult. More precisely, the computations of some indices are NP-hard problems (Smeulders et al. 2013; Smeulders et al. 2014; Shiozawa 2015).[20] These indices were defined according to the weights of the GARP violations (or the cyclical consistency violations). Hence, their validities are based on Condition (ii) in Proposition 3. However, the GARP has another equivalent combinatorial form: Condition (iii) in Proposition 3. Hence, we have another candidate for an index of a goodness-of-fit measure for the GARP:

$$\text{SCCI} := \frac{\sum_{\kappa=1}^{d} \sum_{(x^k, x^{k'}) \in H_\kappa} g^k(x^{k'})}{\sum_{(x^k, x^{k'}) \in E_{np}} g^k(x^{k'})} \tag{7}$$

[20] Specifically, Smeulders et al. (2013) and Smeulders et al. (2014) showed that it is NP-hard to compute the indices defined in Houtman and Maks (1985), Varian (1990), and Echenique et al. (2011). Shiozawa (2015) showed that computing the minimum cost index of Dean and Martin (2015) is also NP-hard. Note, in contrast, that Smeulders et al. (2013) proposed two computationally feasible indices and Smeulders et al. (2014) proposed a feasible (polynomial-time exact) algorithm for computing the index of Afriat (1973).

where g^k are the functions that define the budget sets of the LB, GB, or LBI-data set $\mathcal{O} = \{(B^k, x^k)\}_{k=1}^n$, $H_\kappa(\kappa = 1, \dots, d)$ is the SCC decomposition of $G_{np} = (V, E_{np})$, which is constructed from the data set $\mathcal{O}$ as in Proposition 3, and "$(x^k, x^{k'}) \in H_\kappa$" expresses that the edge $(x^k, x^{k'})$ is contained in the set of edges of H_κ.[21] Note that SCCI $\in [0, 1]$ for any data set. This index has a natural interpretation: the ratio of the weight of irrational parts (negative weight SCCs) of the data set over the entire weight of the revealed preference relation. Moreover, it has an $O(n^2)$ time algorithm: compute SCCs of G_{np} using the SCC decomposition algorithm and compute SCCI defined by (7). Hence, based on Proposition 3, SCCI may be a new valid and computationally feasible goodness-of-fit measure for the GARP.

Acknowledgments

The author would like to thank Ken Urai for helpful discussions and comments. The author thanks Jun Iritani, Shigehiro Serizawa, Tomoyuki Kamo, and Kazuhiko Hashimoto for their helpful comments. The author also thanks the anonymous referee for the insightful comments and suggestions.

Appendix

1. A proof of Theorem 3

This argument is based on Murota and Shioura (2013).

(i) $\Rightarrow$ (ii): Let $d(v)$ be the length of the shortest path from the vertex s to each of the other vertices $v \in V$. We define $d(s) := 0$. (Note that if the weight function is integer-valued ($\ell : E \to \mathbb{Z}$), the shortest path length $d(v)$ must be an integer for all $v \in V$.) Take an arbitrary edge $(v, u) \in E$. Because $d(u)$ is the length of the shortest path from s to u, we have

$$d(u) \leqq d(v) + \ell((v, u)),$$

where the right-hand side is the length of a path from s to u that follows the shortest path from s to v and the edge (v, u). Thus, $d(u) \leqq d(v) + \ell((v, u))$ for all $(v, u) \in E$.

(ii) $\Rightarrow$ (iii): Take an arbitrary cycle $K : v_0 \to v_1 \to \cdots \to v_{m-1} \to v_m$, where $v_m = v_0$. Then,

$$\ell(K) = \sum_{h=0}^{m-1} \ell((v_h, v_{h+1})) \geqq \sum_{h=0}^{m-1} (\pi(v_{h+1}) - \pi(v_h)) = \pi(v_m) - \pi(v_0) = 0.$$

Hence, there is no negative length cycle.

(iii) $\Rightarrow$ (i): Take an s-v path P^* that has shortest length over all simple paths from s to v. (We say a path is *simple* if it contains no cycle.) Such P^* exists because the set of all simple paths from s to v is not empty by assumption, and this set is finite; a simple path must consist of at most $n - 1$ edges out of m edges in E. Now, take an arbitrary s-v path P that is not simple. Because P is not simple, there is a vertex v' through which P passes at least twice. Thus, P contains a cycle K. Because there is no

[21] Note that if the denominator $\sum_{(x^k, x^{k'}) \in E_{np}} g^k(x^{k'})$ is zero, then the data set is rationalizable by Proposition 3. In such cases, we define SCCI := 0.

negative length cycle, the length of a s-v path P' generated by removing K from P is no more than the length of P. That is, $\ell(P') \leqq \ell(P)$. Continuing this procedure until P is reduced to a simple s-v path $\bar{P}$, we still have $\ell(\bar{P}) \leqq \ell(P)$. However, because P^* is the shortest length simple path from s to v, we must have $\ell(P^*) \leqq \ell(\bar{P})$. Therefore, $\ell(P^*) \leqq \ell(P)$ holds. Hence, P^* is a shortest s-v path. □

2. Proof of Theorem 4

As previously mentioned, the equivalences (i) ⇔ (ii) ⇔ (iii) are consequences of Theorem 3. Hence, we prove the remaining equivalences in the following.

(iii) ⇒ (iv): We show the contraposition. If (iv) does not hold, then there is a cycle in G_{np} that contains a negative weighted edge. Let $v_0 \to v_1 \to \cdots \to v_{m-1} \to v_0$ denote this cycle. Note that this cycle $v_0 \to v_1 \to \cdots \to v_{m-1} \to v_0$ is also contained in the original graph G, because G_{np} is a subgraph of G. Take an arbitrary adjustment $\lambda \gg 0$. Then, the weight of the cycle with respect to the adjusted weight $\ell(\lambda)$ is

$$\begin{aligned}\ell((v_0, v_1); \lambda) + \cdots + \ell((v_{m-1}, v_0); \lambda) &= \lambda_0 \ell((v_0, v_1)) + \cdots + \lambda_{m-1} \ell((v_{m-1}, v_0)) \\ &\leqq \min\{\lambda_h | h = 0, 1, \ldots, m-1\}(\ell((v_0, v_1)) + \cdots + \ell((v_{m-1}, v_0))) < 0.\end{aligned}$$

Thus, the graph G has a negative cycle with respect to the adjusted weight $\ell(\lambda)$. Because the adjustment $\lambda \gg 0$ is arbitrary, there is no $\lambda \gg 0$ such that the graph G with adjusted edge weight $\ell(\lambda)$ has no negative length cycle (i.e., Condition (iii) does not hold).

(iv) ⇔ (v): We show the contraposition. If (iv) does not hold, then there is a cycle in G_{np} that contains a negative weighted edge $(v^*, u^*) \in E_{np}$. For all pairs of vertices $v, u \in V$ that are in this cycle, there are v-u and u-v paths. Hence, vertices v^* and u^* are in the same SCC. Therefore, Condition (v) does not hold. Conversely, if (v) does not hold, there is a SCC of G_{np} that contains a negative weighted edge $(v^*, u^*) \in E_{np}$. Because the vertices v^* and u^* are in the same SCC, there is a u^*-v^* path

$$u^*(u^*, w_1)w_1 \cdots w_{m-1}(w_{m-1}, v^*)v^*$$

in G_{np}. Therefore, a cycle $v^*(v^*, u^*)u^*(u^*, w_1)w_1 \cdots w_{m-1}(w_{m-1}, v^*)v^*$ is in G_{np} and contains the negative weighted edge (v^*, u^*) (i.e., Condition (iv) does not hold).

(v) ⇒ (ii): As we noted in Remark 2 of Theorem 6, "(iv) ⇒ (ii)" in Theorem 4 is a consequence of Proposition 4 and Theorem 6. Hence, as we have shown "(iv) ⇔ (v)" above, the implication "(v) ⇒ (ii)" is also a consequence of those. However, as shown in the proof of Theorem 6 below, an algorithm given in the proof of Theorem 6 can be used to compute a solution (π, λ) of any feasible instances of the SPPWA. Hence, the proof of Theorem 6 can be applied to show "(v) ⇒ (ii)" in Theorem 4 directly. □

3. Proof of Proposition 4

(i) ⇒ (ii): Suppose that the graph $G_{np} = (V, E_{np})$ has no negative length cycle with respect to the weight $\ell : E \to K$. Then, form Theorem 4, we have a feasible solution of the SPPWA, $(\pi, \lambda) \in K^n_+ \times K^n_{++}$.

Define a graph $G_r = (V, E_r)$ in the following way

$$E_r := \{(v, u) \mid v, u \in V,\ v \neq u,\ \text{and } \pi(v) \geqq \pi(u)\}. \tag{8}$$

Then, since $K = \mathbb{Z}$ or $\mathbb{R}$ are totally ordered, it is clear that G_r is a rational graph, i.e., complete and transitive. Moreover, G_r is consistent with the instance (G, ℓ). Indeed, since (π, λ) is a feasible solution of the SPPWA, we have

$$\pi(u) \leqq \pi(v) + \lambda_v \ell((v, u)) \ \text{ for all } \ (v, u) \in E.$$

If we have $(v, u) \in E_{np}(\subset E)$ then $\ell((v, u)) \leqq 0$, and hence

$$\pi(u) \leqq \pi(v) + \lambda_v \ell((v, u)) \leqq \pi(v),$$

i.e., $(v, u) \in E_r$. Moreover, if $\ell((v, u)) < 0$, we have

$$\pi(u) \leqq \pi(v) + \lambda_v \ell((v, u)) < \pi(v),$$

and hence $(v, u) \in E_r$ and $(u, v) \notin E_r$ from the definition (8).

(ii) $\Rightarrow$ (i): Suppose that, there is a consistent rational graph $G_r = (V, E_r)$ with the instance (G, ℓ). Suppose also that, to the contrary, graph $G_{np} = (V, E_{np})$ has a negative length cycle with respect to the weight $\ell : E \to K$, i.e., a cycle in G_{np} that contains a negative weighted edge $\ell((v^*, u^*)) < 0$. Let $w_0(w_0, w_1)w_1 \cdots w_{m-1}(w_{m-1}, w_m)w_m$ be that cycle where $w_0 = w_m = v^*$ and $w_1 = u^*$. Since, $(w_k, w_{k+1}) \in E_{np}$ for all $k = 0, \ldots, m-1$ and G_r is consistent, we have $(w_k, w_{k+1}) \in E_r$ for all $k = 0, \ldots, m-1$. This implies, in particular, that there is edge $(w_1, w_0) \in E_r$, since G_r is transitive. However, since $\ell((w_0, w_1)) = \ell((v^*, u^*)) < 0$ and G_r is consistent, we must have $(w_1, w_0) \notin E_r$ by definition of the consistency. That is a contradiction. □

4. Proof of Theorem 6 (and Proof of "(v) $\Rightarrow$ (ii)" in Theorem 4).

In the following, we show that proofs for Theorem 6 and for the implication "(v) $\Rightarrow$ (ii)" in Theorem 4 can be done by an almost the same argument based on an algorithm we define below. For this reason, we denote $G_{sub} = G_r$ or G_{np}. Here, we assume that G_{sub} is a consistent rational graph with the SPPWA instance $(G = (V, E), \ell : E \to K)$ or that G_{np} satisfies Condition (v) of Theorem 4, respectively. First, we show a combinatorial property between $(G = (V, E), \ell : E \to K)$ and G_{sub}.

Claim 1. *Suppose that $(H_1, \ldots, H_d)$ is the SCC decomposition of the graph G_{sub} where $G_{sub} = G_r$ or G_{np}. Here, H_κ are assigned a topological order by their subscripts $\kappa = 1, \ldots, d$.*[22] *Then, we have the*

[22] That is, $H_1, \ldots, H_d$ are assigned their subscripts so that the following statement holds: *if there is a v-u path in G_{sub} where $v \in H_\kappa, u \in H_{\kappa'}$ and $\kappa \neq \kappa'$ then, $\kappa > \kappa'$*. It is known that we can have such subscripts for any directed graph as follows. Consider each SCC as a single vertex and re-define a graph consisting of the re-defined vertices (SCCs) and edges from SCC H' to another SCC H'' where there is a path from a vertex of H' to a vertex of H''. Then, the new graph is clearly acyclic and hence, has a topological order (Proposition 2.9, Korte and Vygen 2012). When we assign subscripts $\kappa = 1, \ldots, d$ to the SCCs that correspond to the topological order of the new graph, we get the stated structure. It is also known that the SCC decomposition algorithm (STRONGLY CONNECTED COMPONENT ALGORITHM, Korte and Vygen 2012) simultaneously provides such a topological order of SCCs in an inverse manner we define here (Theorem 2.20, Korte and Vygen 2012).

following two properties:

(i) $(v, u) \in E$ *such that* $v, u \in H_\kappa$ *for some* $\kappa = 1, \ldots, d \Rightarrow \ell((v, u)) \geqq 0$.

(ii) $(v, u) \in E$ *such that* $v \in H_\kappa$ *and* $u \in H_{\kappa'}$ *such that* $\kappa < \kappa' \Rightarrow \ell((v, u)) > 0$.

Case 1: $G_{sub} = G_r$. First, we show that (i) holds. Suppose, to the contrary, $v, u \in H_\kappa$ and $\ell((v, u)) < 0$. Then we have $(v, u) \in E_{np}$ and hence, from the consistency of G_r, we also have $(v, u) \in E_r$ and $(u, v) \notin E_r$. However, because v and u are in the same SCC, H_κ, there is a path $u \to \cdots \to v$ in G_r. Therefore, by transitivity of G_r, there must be an edge $(u, v) \in E_r$, which is absurd. Next, we show (ii). Suppose $v \in H_\kappa$ and $u \in H_{\kappa'}$ such that $\kappa < \kappa'$. Suppose also $\ell((v, u)) \leqq 0$. Then, $(v, u) \in E_{np}$ and hence by the consistency of G_r, we have $(v, u) \in E_r$, i.e., there is a path in G_r which connects a vertex in H_κ to a vertex in $H_{\kappa'}$. Because we take the index $\kappa = 1, \ldots, d$ as the topological order, we have $\kappa > \kappa'$ and this is a contradiction.

Case 2: $G_{sub} = G_{np}$. This case is more straightforward. Because we assume that G_{np} has no SCCs with a negative weighted edge in it, (i) follows. Furthermore, if we have $v \in H_\kappa$ and $u \in H_{\kappa'}$ such that $\kappa < \kappa'$ and $\ell((v, u)) \leqq 0$, it contradicts the topological ordering of SCCs represented by $\kappa = 1, \ldots, d$, because $\ell((v, u)) \leqq 0$ means $(v, u) \in E_{np}$ and, in particular, $\kappa > \kappa'$. □

For the time being, our goal is to show that, in either of the cases $K = \mathbb{R}$ or $K = \mathbb{Z}$, there are *real* numbers $(\pi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n_{++}$ with Condition (i) in Theorem 6 and, in addition, Condition (ii) in Theorem 6 if $G_{sub} = G_r$. For this purpose, let E_p and E_n denote the positive edges and negative edges of (G, ℓ), respectively:

$$E_p := \{(v, u) \in E \mid \ell((v, u)) > 0\} \text{ and}$$
$$E_n := \{(v, u) \in E \mid \ell((v, u)) < 0\}.$$

We can compute numbers $(\pi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n_{++}$, which satisfies Conditions (i) and (ii) in Theorem 6 by the following four-step algorithm:

— ALGORITHM-1 ————————————————

STEP 1: Compute the SCC decomposition, $H_1, \ldots, H_d$ of G_{sub}. Here, H_κ are assigned a topological order by their subscripts $\kappa = 1, \ldots, d$.

STEP 2: Let P and N be numbers such that

$$P := \begin{cases} \min\{\ell((v, u)) \mid \ell((v, u)) > 0\} & \text{if } E_p \neq \emptyset \\ 1 & \text{otherwise} \end{cases} \tag{9}$$

and

$$N := \begin{cases} \min\{\ell((v, u)) \mid \ell((v, u)) < 0\} & \text{if } E_n \neq \emptyset \\ -1 & \text{otherwise.} \end{cases} \tag{10}$$

STEP 3: Define numbers $(\phi_1, \ldots, \phi_d)$ and $(\mu_1, \ldots, \mu_d)$ as follows:

(1): $\phi_1 = 1$ and $\mu_1 = 1$.

(2): For all $\kappa = 2, \ldots, d$, define $(\phi_\kappa, \mu_\kappa)$ recursively as

$$\phi_\kappa := (\min_{s<\kappa}\{\phi_s + \mu_s P\} + \phi_{\kappa-1})2^{-1}, \text{ and} \tag{11}$$

$$\mu_\kappa := (\phi_{\kappa-1} - \phi_\kappa)N^{-1}. \tag{12}$$

STEP 4: For any $v \in V$ such that $v \in H_\kappa$, let $\pi(v)$ and λ_v be

$$\pi(v) := \phi_\kappa \text{ and } \lambda_v := \mu_\kappa. \tag{13}$$

In the following, we show that the numbers $(\pi, \lambda) = ((\pi(v))_{v\in V}, (\lambda_v)_{v\in V})$ computed in ALGORITHM-1 actually satisfy the inequality condition (i) in Theorem 6. Moreover, we also show that if $G_{sub} = G_r$ then (π, λ) also satisfies Condition (ii) in Theorem 6. First, we show some properties of $(\phi_1, \ldots, \phi_d)$ and $(\mu_1, \ldots, \mu_d)$ computed in ALGORITHM-1.

Claim 2. *By* ALGORITHM-1*, the computed numbers $\phi = (\phi_1, \ldots, \phi_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ has the following conditions: for all $\kappa = 2, \ldots, d$,*

$$\phi_\kappa > \phi_{\kappa-1} \textit{ and} \tag{14}$$

$$\phi_\kappa < \min_{s<\kappa}\{\phi_s + \mu_s P\}. \tag{15}$$

In particular, $\mu_\kappa > 0$ for all $\kappa = 1, \ldots, d$.

We show this claim by mathematical induction argument on κ. Because $P > 0$, the base case $\kappa = 2$ is clear. Indeed, from (11) and $\mu_1 = 1$, it follows

$$\phi_2 = \phi_1 + \frac{1}{2}P > \phi_1$$

and

$$\phi_2 = \phi_1 + \frac{1}{2}P < \phi_1 + P = \phi_1 + \mu_1 P.$$

Moreover, because $N < 0$ and $\phi_1 < \phi_2$, it follows

$$\mu_2 = \frac{(\phi_1 - \phi_2)}{N} > 0.$$

Next, assume that we have (14) and (15) for κ (where $\kappa \geqq 2$), and $\mu_\kappa > 0$. Then, by assumption, we have $\min_{s<\kappa}\{\phi_s + \mu_s P\} > \phi_\kappa$ and, because $\mu_\kappa P > 0$, we have $\phi_\kappa + \mu_\kappa P > \phi_\kappa$. Hence, from (11), we have

$$\phi_{\kappa+1} = (\min_{s<\kappa+1}\{\phi_s + \mu_s P\} + \phi_\kappa)\frac{1}{2} = (\min\{\min_{s<\kappa}\{\phi_s + \mu_s P\}, \phi_\kappa + \mu_\kappa P\} + \phi_\kappa)\frac{1}{2} > (\phi_\kappa + \phi_\kappa)\frac{1}{2} = \phi_\kappa.$$

Moreover, because $\min_{s<\kappa}\{\phi_s + \mu_s P\} > \phi_\kappa$ and $\phi_\kappa + \mu_\kappa P > \phi_\kappa$ imply

$$\min_{s<\kappa+1}\{\phi_s + \mu_s P\} - \phi_\kappa > 0,$$

we also have

$$\phi_{\kappa+1} = (\min_{s<\kappa+1}\{\phi_s + \mu_s P\} + \phi_\kappa)\frac{1}{2} = (\min_{s<\kappa+1}\{\phi_s + \mu_s P\} - \phi_\kappa)\frac{1}{2} + \phi_\kappa$$
$$< (\min_{s<\kappa+1}\{\phi_s + \mu_s P\} - \phi_\kappa) + \phi_\kappa = \min_{s<\kappa+1}\{\phi_s + \mu_s P\}.$$

Finally, because $N < 0$ and $\phi_\kappa < \phi_{\kappa+1}$, it follows

$$\mu_{\kappa+1} = \frac{(\phi_\kappa - \phi_{\kappa+1})}{N} > 0.$$

Hence, Claim 2 is valid. □

Next, we show that $\pi = (\pi(v))_{v\in V}$ and $\lambda = (\lambda_v)_{v\in V}$ satisfy Condition (i) of Theorem 6.

Claim 3. ALGORITHM-1 *computes real numbers $(\pi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n_{++}$, which satisfies the following condition:*

$$\pi(u) \leqq \pi(v) + \lambda_v \ell((v,u)) \;\; \textit{for all} \;\; (v,u) \in E.$$

Take any edge $(v,u) \in E$. We have the following three cases.

Case 1: $(v,u) \in E$ such that $v, u \in H_\kappa$ for some $\kappa = 1, \ldots, d$.

By definition (13), we have $\pi(v) = \pi(u) = \phi_\kappa$ and $\lambda_v = \mu_\kappa > 0$. Moreover, by Claim 1, we have $\ell((v,u)) \geqq 0$. Hence, it follows

$$\pi(u) = \phi_\kappa \leqq \phi_\kappa + \mu_\kappa \ell((v,u)) = \pi(v) + \lambda_v \ell((v,u)).$$

Case 2: $(v,u) \in E$ such that $v \in H_\kappa$ and $u \in H_{\kappa'}$ such that $\kappa > \kappa'$.

From the definition (13), we have $\pi(v) = \phi_\kappa, \lambda_v = \mu_\kappa > 0$, and $\pi(u) = \phi_{\kappa'}$. From Condition (14) of Claim 2 and $\kappa > \kappa'$, we have

$$\pi(u) = \phi_{\kappa'} < \phi_{\kappa'+1} < \cdots < \phi_{\kappa-1}.$$

Moreover, by definitions (12) and (10), we have

$$\phi_{\kappa-1} = \phi_\kappa + \mu_\kappa N \leqq \phi_\kappa + \mu_\kappa \ell((v,u)) = \pi(v) + \lambda_v \ell((v,u)).$$

Hence, $\pi(u) \leqq \pi(v) + \lambda_v \ell((v,u))$ holds.

Case 3: $(v,u) \in E$ such that $v \in H_\kappa$ and $u \in H_{\kappa'}$ such that $\kappa < \kappa'$.

From the definition (13), we have $\pi(v) = \phi_\kappa, \lambda_v = \mu_\kappa > 0$, and $\pi(u) = \phi_{\kappa'}$. From Condition (15) of Claim 2 and $\kappa < \kappa'$, we have

$$\pi(u) = \phi_{\kappa'} < \phi_\kappa + \mu_\kappa P.$$

Moreover, by the Condition (ii) of Claim 1, we have $\ell((v,u)) > 0$ and hence, $E_p \neq \emptyset$. Therefore, by the definition (9), we have

$$\phi_\kappa + \mu_\kappa P \leqq \phi_\kappa + \mu_\kappa \ell((v,u)) = \pi(v) + \lambda_v \ell((v,u)).$$

Therefore, we have $\pi(u) \leqq \pi(v) + \lambda_v \ell((v,u))$.

Hence, Claim 3 is valid. □

From Claim 3, we have *real* numbers (π, λ) satisfying Condition (i) in Theorem 6. In addition to this, we show that if $G_{sub} = G_r$ then (π, λ) preserves G_r.

Claim 4. *If* $G_{sub} = G_r$*,* ALGORITHM-1 *computes numbers* $(\pi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n_{++}$*, which satisfy the following condition:*

$$\pi(v) > \pi(u) \text{ if and only if } (v, u) \in E_r \text{ and } (u, v) \notin E_r, \text{ for all } v, u \in V, \text{ and}$$
$$\pi(v) = \pi(u) \text{ if and only if } (v, u) \in E_r \text{ and } (u, v) \in E_r, \text{ for all } v, u \in V.$$

First, we show the former part of the condition. From the definition (13) and the condition (14) of Claim 2, we have

$$\pi(v) > \pi(u) \Leftrightarrow v \in H_\kappa \text{ and } u \in H_{\kappa'} \text{ where } \kappa > \kappa'.$$

From the definition of the SCC decomposition $(H_1, \ldots, H_d)$ of G_r with the topological order in its subscripts and from the rationality of G_r, we have

$$v \in H_\kappa \text{ and } u \in H_{\kappa'} \text{ where } \kappa > \kappa' \Leftrightarrow \text{there is } v\text{-}u \text{ path but no } u\text{-}v \text{ path in } G_r.$$

Finally, from the rationality of G_r, we have

$$\text{there is } v\text{-}u \text{ path but no } u\text{-}v \text{ path in } G_r \Leftrightarrow (v, u) \in E_r \text{ and } (u, v) \notin E_r.$$

Therefore, the the former part of the condition is valid. For the the latter part, we have the following equivalences from almost the same argument:

$$\begin{aligned} \pi(v) = \pi(u) &\Leftrightarrow v, u \in H_\kappa \\ &\Leftrightarrow \text{there is } v\text{-}u \text{ path and } u\text{-}v \text{ path in } G_r \Leftrightarrow (v, u) \in E_r \text{ and } (u, v) \in E_r. \end{aligned}$$

Therefore, Claim 4 is valid. □

Finally, we complete the proof. Note that we have shown that if the weight is a real-valued function, $\ell : E \to \mathbb{R}$, there is a feasible solution of the SPPWA computed by ALGORITHM-1 (Claim 3). Moreover, if $G_{sub} = G_r$ then, this feasible solution preserves the consistent rational graph G_r (Claim 4).

If the weight is an integer-valued function, $\ell : E \to \mathbb{Z}$, then the numbers (π, λ) computed by ALGORITHM-1 are, in general, *rational* numbers. Hence, we modify ALGORITHM-1 slightly, so that we can have integers $(\tilde{\pi}, \tilde{\lambda})$ with the same properties. In essence, the following algorithm calculates the numerators and denominators of the rational numbers (π, λ), and uniformly and positively normalizes these rational numbers to be *integers*.

— ALGORITHM-2 ———

STEP 1: Decompose G_{sub} into SCCs $(H_1, \ldots, H_d)$ as same as the STEP 1 of ALGORITHM-1.

STEP 2: Define P and N as same as the STEP 2 of ALGORITHM-1.

STEP 3: Define numbers $(\alpha_1, \ldots, \alpha_d)$ and $(\beta_1, \ldots, \beta_d)$ as follows:

(1): $\alpha_1 := 1$ and $\beta_1 := 1$.

(2): $\alpha_2 := 2\alpha_1 + \beta_1 P$ and $\beta_2 := 2\alpha_1 - \alpha_2$.

(3): For all $\kappa = 3, \ldots, d$, define $(\alpha_\kappa, \beta_\kappa)$ recursively as

$$\alpha_\kappa := \min\{(2N)^{\kappa-2}(\alpha_1 + \beta_1 P), \min_{1<s<\kappa}\{(2N)^{\kappa-1-s}(N\alpha_s + \beta_s P)\}\} + N\alpha_{\kappa-1} \quad \text{and} \tag{16}$$

$$\beta_\kappa := 2N\alpha_{\kappa-1} - \alpha_\kappa. \tag{17}$$

STEP 4: Define numbers $(\tilde{\phi}_1, \ldots, \tilde{\phi}_d)$ and $(\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$ as follows:

(1): $\tilde{\phi}_1 := (2\,|N|)^{d-1}\alpha_1$ and $\tilde{\mu}_1 := (2\,|N|)^{d-1}\beta_1$.

(2): For all $\kappa = 2, \ldots, d$, define $(\tilde{\phi}_\kappa, \tilde{\mu}_\kappa)$ as

$$\tilde{\phi}_\kappa := (-1)^{\kappa-2}\, 2^{d-\kappa}\, |N|^{d-\kappa+1}\alpha_\kappa, \text{ and} \tag{18}$$

$$\tilde{\mu}_\kappa := (-1)^{\kappa-1}\, 2^{d-\kappa}\, |N|^{d-\kappa}\beta_\kappa. \tag{19}$$

STEP 5: For any $v \in V$ such that $v \in H_\kappa$, let $\tilde{\pi}(v)$ and $\tilde{\lambda}_v$ be

$$\tilde{\pi}(v) := \tilde{\phi}_\kappa \quad \text{and} \quad \tilde{\lambda}_v := \tilde{\mu}_\kappa. \tag{20}$$

Clearly, if the weight is an integer-valued function, $\ell : E \to \mathbb{Z}$, the numbers N, P, $\alpha = (\alpha_1, \ldots, \alpha_d)$, and $\beta = (\beta_1, \ldots, \beta_d)$ are integers, and hence, $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_d)$, $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_d)$, $\tilde{\pi} = (\tilde{\pi}(v))_{v\in V}$, and $\tilde{\lambda} = (\tilde{\lambda}_v)_{v\in V}$ are also integers. Moreover, we shall show in the Claim 5 below, that

$$(\tilde{\pi}, \tilde{\lambda}) = M(\pi, \lambda) \tag{21}$$

where $M := (2\,|N|)^{d-1} > 0$ and (π, λ) are numbers computed in ALGORITHM-1.[23] Note that if we show Equation (21) then it is clear from Claims 3 and 4 that $(\tilde{\pi}, \tilde{\lambda})$ is an *integer* feasible solution of the SPPWA $(G, \ell : E \to \mathbb{Z})$, and that if $G_{sub} = G_r$ then it preserves the consistent rational graph G_r. Hence, the following claim completes the proof.

Claim 5. *Let (ϕ, μ) and (π, λ) be numbers defined in* ALGORITHM-1*, and (α, β), $(\tilde{\phi}, \tilde{\mu})$, and $(\tilde{\pi}, \tilde{\lambda})$ be numbers defined in* ALGORITHM-2*. Then we have*

$$\phi_1 = \alpha_1 \quad \text{and} \quad \mu_1 = \beta_1, \tag{22}$$

$$\phi_\kappa = \frac{1}{2^{\kappa-1}N^{\kappa-2}}\alpha_\kappa \quad \text{and} \quad \mu_\kappa = \frac{1}{(2N)^{\kappa-1}}\beta_\kappa, \quad \text{for all } \kappa = 2, \ldots, d. \tag{23}$$

In particular, we have $(\tilde{\phi}, \tilde{\mu}) = M(\phi, \mu)$ where $M := (2\,|N|)^{d-1} > 0$, and hence $(\tilde{\pi}, \tilde{\lambda}) = M(\pi, \lambda)$.

Equation (22) is obvious since $\phi_1 = 1 = \alpha_1$ and $\mu_1 = 1 = \beta_1$. We show Equation (23) by induction. The case $\kappa = 2$ follows, from the definitions $\alpha_2 := 2\alpha_1 + \beta_1 P$, $\beta_2 := 2\alpha_1 - \alpha_2$, (11), and (12), as

$$\frac{1}{2}\alpha_2 = \frac{1}{2}(2\alpha_1 + \beta_1 P) = \frac{1}{2}(2\phi_1 + \mu_1 P) = \frac{1}{2}((\phi_1 + \mu_1 P) + \phi_1) = \phi_2$$

and

$$\frac{1}{2N}\beta_2 = \frac{1}{2N}(2\alpha_1 - \alpha_2) = \frac{1}{N}(\alpha_1 - \frac{1}{2}\alpha_2) = \frac{1}{N}(\phi_1 - \phi_2) = \mu_2.$$

[23] We mean $M(\pi, \lambda) := ((M\pi(v))_{v\in V}, (M\lambda_v)_{v\in V})$.

Now, suppose we have (23) for all s such that $2 \leqq s < \kappa$ (where $\kappa > 2$). Then, we have

$$\begin{aligned}
\frac{1}{2^{\kappa-1}N^{\kappa-2}}\alpha_\kappa &= \frac{1}{2^{\kappa-1}N^{\kappa-2}}(\min\{(2N)^{\kappa-2}(\alpha_1+\beta_1P), \min_{1<s<\kappa}\{(2N)^{\kappa-1-s}(N\alpha_s+\beta_sP)\}\}+N\alpha_{\kappa-1}) \\
&= \frac{1}{2}\frac{1}{(2N)^{\kappa-2}}(\min\{(2N)^{\kappa-2}(\alpha_1+\beta_1P), \min_{1<s<\kappa}\{(2N)^{\kappa-1-s}(N\alpha_s+\beta_sP)\}\}+N\alpha_{\kappa-1}) \\
&= \frac{1}{2}(\min\{\alpha_1+\beta_1P, \min_{1<s<\kappa}\{(2N)^{1-s}(N\alpha_s+\beta_sP)\}\}+\frac{1}{2^{\kappa-2}N^{\kappa-3}}\alpha_{\kappa-1}) \\
&= \frac{1}{2}(\min\{\alpha_1+\beta_1P, \min_{1<s<\kappa}\{\frac{1}{2^{s-1}N^{s-2}}\alpha_s+\frac{1}{(2N)^{s-1}}\beta_sP\}\}+\frac{1}{2^{\kappa-2}N^{\kappa-3}}\alpha_{\kappa-1}) \\
&= \frac{1}{2}(\min\{\phi_1+\mu_1P, \min_{1<s<\kappa}\{\phi_s+\mu_sP)\}\}+\phi_{\kappa-1}) \\
&= (\min_{s<\kappa}\{\phi_s+\mu_sP)\}+\phi_{\kappa-1})2^{-1} = \phi_\kappa,
\end{aligned}$$

where the first equation follows from (16), the last equation follows from (11), and the third equation from the last follows from the equations in (23) for the cases s such that $2 \leqq s < \kappa$.

Moreover, we have

$$\frac{1}{(2N)^{\kappa-1}}\beta_\kappa = \frac{1}{(2N)^{\kappa-1}}(2N\alpha_{\kappa-1}-\alpha_\kappa) = \frac{1}{N}(\frac{1}{2^{\kappa-2}N^{\kappa-3}}\alpha_{\kappa-1}-\frac{1}{2^{\kappa-1}N^{\kappa-2}}\alpha_\kappa) = \frac{1}{N}(\phi_{\kappa-1}-\phi_\kappa) = \mu_\kappa,$$

where the first equation follows from the definition (17), the last equation follows from the definition (12), and the second equation from the last follows from the former part of equations in (23) for the cases $\kappa-1$ and κ. Therefore, all the equations in (23) hold. Hence, if we define $M := (2\,|N|)^{d-1} > 0$, then we have $M(\phi,\mu) = (\tilde{\phi},\tilde{\mu})$ from equations (22) and (23). Indeed, from (22) and the definitions of $\tilde{\phi}_1$ and $\tilde{\mu}_1$, we have

$$M\phi_1 = (2\,|N|)^{d-1}\alpha_1 = \tilde{\phi}_1 \quad \text{and} \quad M\mu_1 = (2\,|N|)^{d-1}\beta_1 = \tilde{\mu}_1.$$

Moreover, from (23) and the definitions (18) and (19), we have, for any $\kappa = 2,\ldots,d$,

$$\begin{aligned}
M\phi_\kappa &= \frac{(2\,|N|)^{d-1}}{2^{\kappa-1}N^{\kappa-2}}\alpha_\kappa = \frac{2^{d-1}\,|N|^{d-1}}{2^{\kappa-1}(-1)^{\kappa-2}|N|^{\kappa-2}}\alpha_\kappa = (-1)^{\kappa-2}\,2^{d-\kappa}\,|N|^{d-\kappa+1}\alpha_\kappa = \tilde{\phi}_\kappa \quad \text{and} \\
M\mu_\kappa &= \frac{(2\,|N|)^{d-1}}{(2N)^{\kappa-1}}\beta_\kappa = \frac{2^{d-1}|N|^{d-1}}{2^{\kappa-1}(-1)^{\kappa-1}|N|^{\kappa-1}}\beta_\kappa = (-1)^{\kappa-1}\,2^{d-\kappa}\,|N|^{d-\kappa}\beta_\kappa = \tilde{\mu}_\kappa.
\end{aligned}$$

Finally, from the definitions (13) and (20), and from the equation $(\tilde{\phi},\tilde{\mu}) = M(\phi,\mu)$, we have $(\tilde{\pi},\tilde{\lambda}) = M(\pi,\lambda)$. Hence, Claim 5 is valid. □

Therefore, the proof of Theorem 6 is completed. (The proof of the implication "(v) ⇒ (ii)" in Theorem 4 is also completed.) □

Remark 1: As we see in the proof above, if we set $G_{sub} = G_{np}$, we can compute a feasible solution of the SPPWA, (π,λ), for a real-valued feasible instance, $(G, \ell : E \to \mathbb{R})$, by ALGORITHM-1. If we need an integer solution for an integer-valued feasible instance, $(G, \ell : E \to \mathbb{Z})$, we can compute one by ALGORITHM-2. Moreover, if we set $G_{sub} = G_r$ we can compute a solution, which preserves the consistent rational graph G_r.

Remark 2: The complexity of ALGORITHM-1 is $O(n^2)$, where n is the number of vertices of $G = (V,E)$, given that we can find a maximum (or a minimum) value of given κ numbers in $O(\kappa)$ time.

Indeed, Step 1 requires $O(n + m_{sub})$ time, where m_{sub} is the number of edges of G_{sub} (Theorem 2.19, Korte and Vygen 2012).[24] Step 2 requires $O(m)$ time where m is the number of edges of G. Step 3 requires $O(\kappa)$ time for each $\kappa = 1, \ldots, d$ where $d \leqq n$. Finally, Step 4 requires $O(n)$ time. Hence, because $m_{sub}, m \leqq n(n-1)$, all steps can be bounded by $O(n^2)$ time for any instances. (By a similar analysis, we can see that ALGORITHM-2 is also $O(n^2)$ complexity.) To the best of our knowledge, this is one of the most efficient algorithm for computing a feasible solution of the SPPWA, because other algorithms for computing a solution of Afriat's inequalities are $O(n^3)$ time complexity, where n is the number of observations. (Piaw and Vohra 2003; Fostel et al. 2004; Fujishige and Yang 2012).

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