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Abstract

The axiomatic characterization of price or market equilibrium is one of the most important problems in the general equilibrium theory. There does not seem to exist, however, so many papers on the axiomatic characterization problem of monetary equilibrium. The overlapping-generations model with a double infinity of commodities and agents is one of the most fundamental frameworks for introducing money into an economic model, although a simple game-theoretic or welfare characterization on the role of money under competitive mechanism is widely known to be difficult. In this paper, we show that the *informational efficiency* axiomatic characterization as in Hurwicz (1960), Mount and Reiter (1974) and Sonnenschein (1974) is possible for the price-money competitive mechanism for overlapping-generations economies among the class of all *allocation mechanisms with messages*. In particular, the category theoretic universal mapping characterization in Sonnenschein (1974) is generalized and applied to the overlapping-generations framework through our monetary version of Debreu-Scarf's core limit theorem of Urai and Murakami (2015). Our argument is also closely related to the *replica characterization* approaches of Walrasian social choice mechanism like Thomson (1988) and Nagahisa (1994), and provides a comprehensive perspective on them.

Keywords: Axiomatic Characterization, Resource Allocation Mechanism, Informational Efficiency, Monetary Equilibrium, Overlapping-Generations Model, Replica Core Equivalence, Universal Mapping Property

JEL Classification: C60, C71, D51, D82, E00

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1 Introduction

The axiomatic characterization of price or market equilibrium is one of the most important problems in the general equilibrium theory. There does not seem to exist, however, so many papers on the axiomatic characterization problem of monetary equilibrium (an equilibrium with fiat money or the non-negative wealth transfer).

In order to describe money in the static economic model, it is widely known that an ordinary finite economic setting is insufficient and the double infinity of agent-commodity structure, e.g., the overlapping-generations framework, is necessary. The overlapping-generations model was firstly proposed by Samuelson in 1958 (Samuelson 1958), and has generated much discussion and many papers. An outstanding characteristic of this model is that *competitive equilibria may not necessarily be Pareto-optimal* and that the outside money has the potential to improve on the welfare of the economy. Although Samuelson's argument was not entirely accurate, it is known that the characterization of money in an overlapping-generations economy is not so straightforward from the welfare and game theoretic viewpoint (see Hayashi 1976, Okuno and Zilcha 1980 and Esteban 1986) except for the elementary welfare relation between the monetary equilibrium and the weakly Pareto-optimal allocation (Balasko and Shell 1980 and Esteban 1986).

In this paper, to obtain an alternative explanation for the importance of money, we provide the axiomatic characterization of the price-money competitive mechanism in overlapping-generations economies among the class of allocation mechanisms with messages together with several axioms and category theoretic treatments in Sonnenschein (1974). In his seminal paper (1974 Propositions 1, 2 and 7), Sonnenschein characterizes the *price mechanism* as the unique *resource allocation mechanism* that can be referenced (i) uniquely under many-to-one configurations by (ii) many allocation mechanisms satisfying the several important axioms, where (i) means a certain kind of *efficiency* and (ii) means a *universality* for the price message mechanism. His result uses Debreu-Scarf's replica core equivalence theorem and a similar characterization under the social choice framework is given as a replica stability condition in characterizing Walrasian mechanism like Thomson (1988) and Nagahisa (1994). Also, his result can be related to the informational efficiency problem like Hurwicz (1960), Mount and Reiter (1974), Osana (1978) and Jordan (1982).[1]

In the double infinity economies including overlapping-generations ones, the *non-negative wealth transfer* or *outside money* plays an essential role for equilibrium states, so it is important to characterize the competitive market mechanism as the allocation mechanism with a message space not only of prices but also of wealth transfers, the *price-money message mechanism.* We use in this paper the method of Sonnenschein (1974) since his categorical treatment is appropriate for our infinite dimensional and partially economy-dependent (monetary) message space settings. It should also be noted that in the approach of Sonnenschein, the Cobb-Douglas dependent argument is not necessary. Recently the authors have shown a core limit theorem for monetary overlapping-generations economies (Urai and Murakami 2015). Our limit theorem together with Sonnenschein's axioms and category theoretic framework is sufficient to characterize the *efficiency* and *universality* properties of the price-money message mechanism.

2 Economies

Let $\boldsymbol{N}$ be the set of all positive integers and $\boldsymbol{R}$ be the set of real numbers. A *pure exchange overlapping-generations economy*, or more simply, an *economy*, $\boldsymbol{\mathcal{E}}$, is comprised of the following list:

[1] See Sonnenschein (1974, Propositions 3–6).

(E.1) $\{I_t\}_{t=1}^{\infty}$: a countable family of mutually disjoint finite subsets of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} I_t \subset \boldsymbol{N}$, $I_1 \neq \emptyset$ and for each $t \in \boldsymbol{N}$, if $I_t = \emptyset$, then $I_{t+1} = \emptyset$. I_t is the index set of agents in *generation* t.

(E.2) $\{K_t\}_{t=1}^{\infty}$: a countable family of non-empty finite intervals, $K_t = \{k(t), k(t)+1, \cdots, k(t)+\ell(t)\}$ such that $k(t) \in \boldsymbol{N}$, $\ell(t) \in \boldsymbol{N}$, $\bigcup_{t=1}^{\infty} K_t = \boldsymbol{N}$, $k(t) < k(t+1) \leqq k(t)+\ell(t)$ for all $t \in \boldsymbol{N}$. K_t is the index set of commodities available to generation t.

(E.3) $\{(\succsim_i,\ \omega^i)\}_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$: countably many agents, where $\succsim_i$ is a continuous, strictly monotonic, strictly convex and rational weak preference relation on the commodity space for each generation, $\boldsymbol{R}_+^{K_t}$, for each $i \in I_t$. The *initial endowment* of i, ω^i, is an element of $\boldsymbol{R}_{++}^{K_t} = \{x \mid x : K_t \to \boldsymbol{R}_{++}\}$ for each $i \in I_t$.

It is convenient to identify the commodity space for each generation $\boldsymbol{R}_+^{K_t}$ with a subset of $\boldsymbol{R}^{\boldsymbol{N}}$, which is the set of all functions from $\boldsymbol{N}$ to $\boldsymbol{R}$, by considering $x \in \boldsymbol{R}_+^{K_t}$ a function that takes value 0 on $\boldsymbol{N} \setminus K_t$. Then we can define the total commodity space for economy $\oplus_{t=1}^{\infty} \boldsymbol{R}_+^{K_t}$ as the set of all finite sums among the points in the commodity spaces of the generations. Clearly, $\oplus_{t=1}^{\infty} \boldsymbol{R}_+^{K_t}$ can be identified with a subset of direct sum $\boldsymbol{R}_\infty$, the set of all finite real sequences, which is a subspace of the set of all real sequences, $\boldsymbol{R}^\infty \approx \boldsymbol{R}^{\boldsymbol{N}}$ with pointwise convergence topology.

Given an economy, $\mathcal{E} = (\{I_t\}_{t=1}^{\infty},\ \{K_t\}_{t=1}^{\infty},\ \{(\succsim_i,\ \omega^i)\}_{i \in \bigcup_{t\in \boldsymbol{N}} I_t})$, the *price space* for $\mathcal{E}$, $\mathcal{P}(\mathcal{E})$, is defined as the set of all p in $\boldsymbol{R}_+^{\boldsymbol{N}}$ such that under the duality between $\boldsymbol{R}_\infty$ (with relative topology) and $\boldsymbol{R}^\infty$ (with pointwise convergence topology), p positively evaluates all the agents' initial endowments:

$$\mathcal{P}(\mathcal{E}) = \{p \in \boldsymbol{R}_+^{\boldsymbol{N}} \mid p \cdot \omega^i > 0 \text{ for all } i \in I_t, \text{ for all } t \in \boldsymbol{N}\}. \tag{1}$$

Since for all $i \in I_t$, ω^i belongs to $\boldsymbol{R}_{++}^{K_t}$ for all $t \in \boldsymbol{N}$, the price space of $\mathcal{E}$ always includes $\boldsymbol{R}_{++}^{\boldsymbol{N}}$ for all $\mathcal{E}$ in $\boldsymbol{\mathcal{E}con}$, where $\boldsymbol{\mathcal{E}con}$ denotes the set of all economies satisfying conditions (E.1), (E.2) and (E.3).

For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega^i)\}) \in \boldsymbol{\mathcal{E}con}$, sequence $(x^i \in \boldsymbol{R}_+^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$ is called an *allocation* for $\mathcal{E}$. Allocation $(x^i \in \boldsymbol{R}_+^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$ is said to be *feasible* if

$$\sum_{t\in \boldsymbol{N}} \sum_{i \in I_t} x^i = \sum_{t\in \boldsymbol{N}} \sum_{i \in I_t} \omega^i, \tag{2}$$

where the summability in $\boldsymbol{R}^{\boldsymbol{N}}$ of both sides of the inequality is assured by (E.2). The list of price vector $p^* \in \mathcal{P}(\mathcal{E})$, *non-negative wealth transfer function* $M_{\mathcal{E}}^* : \bigcup_{t=1}^{\infty} I_t \to \boldsymbol{R}_+$, and feasible allocation $(x_*^i \in \boldsymbol{R}_+^{K_t})_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$, is called a *monetary Walras allocation* for $\mathcal{E}$, if for each $t \in \boldsymbol{N}$ and $i \in I_t$, x_*^i is a $\succsim_i$-greatest element in set $\{x^i \in \boldsymbol{R}_+^{K_t} \mid p^* \cdot x^i \leqq p^* \cdot \omega^i + M_{\mathcal{E}}^*(i)\}$. Since the non-negative wealth transfer is an abstraction of the money supply in perfect-foresight overlapping-generations economies, we denote the set of all monetary Walras allocations by $\boldsymbol{\mathcal{MW}alras}(\mathcal{E})$.

A *coalition* in economy $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega^i)\}) \in \boldsymbol{\mathcal{E}con}$ is a set of consumers $S \subset \bigcup_{t=1}^{\infty} I_t$. Allocation x for economy $\mathcal{E}$ is said to be *blocked* by coalition S if it is possible to find commodity bundles y^i for all $i \in S$ such that $\sum_{i\in S}(y^i - \omega^i) = 0$ and $y^i \succsim_i x^i$ for all $i \in S$, and $y^i \succ_i x^i$ for at least one $i \in S$. For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega^i)\}) \in \boldsymbol{\mathcal{E}con}$, the set of all feasible allocations that cannot be blocked by any coalition is said to be the *core* of economy $\mathcal{E}$ and is denoted by $\boldsymbol{\mathcal{C}ore}(\mathcal{E})$. Element $x \in \boldsymbol{\mathcal{C}ore}(\mathcal{E})$ is called a *core allocation.* The set of all feasible allocations that cannot be blocked by any finite coalition is called the *finite core* of economy $\mathcal{E}$ and is denoted by $\boldsymbol{\mathcal{F}core}(\mathcal{E})$. Element $x \in \boldsymbol{\mathcal{F}core}(\mathcal{E})$ is called a *finite core allocation* for $\mathcal{E}$.

3 Replica Core Equivalence Theorem

The replica core equivalence theorem for overlapping-generations economy was proved in Aliprantis and Burkinshaw (1990). Their result, however, does not include the case with monetary (non-negative wealth transfer) equilibrium. Our previous paper (Urai and Murakami 2015) treats this problem through the following replica finite core concept.

Given economy $\mathcal{E} = (\{I_t\}, \{K_t\}, \{(\precsim_i, \omega^i)\}) \in \boldsymbol{\mathcal{E}con}$ and its feasible allocation $x = (x^i \in \boldsymbol{R}_+^{K_t})_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}$ we denote by $\mathcal{E}(x)$ the economy whose initial endowment allocation, $\omega = (\omega^i)$, is replaced by $x = (x^i)$.[2] Hence, we have $\mathcal{E} = \mathcal{E}(\omega)$. Let us consider the replica economy

$$\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega), \tag{3}$$

consisting of all members of the m-fold replica economy of $\mathcal{E}(x)$ and all members of the n-fold replica economy of $\mathcal{E}(\omega)$ for each m and n in $\boldsymbol{N}$. Note that when the number of agents of $\mathcal{E}$ is finite, $\boldsymbol{\mathcal{C}ore}(\mathcal{E}^{m+n})$ is equal to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^{m+n})$ and for each feasible allocation x of $\mathcal{E}$, if its $(m+n)$-fold replica allocation belongs to $\boldsymbol{\mathcal{C}ore}(\mathcal{E}^{m+n})$, then the replica allocation also belongs to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$.[3] Moreover, as we see below, every $(m+n)$-fold replica allocation of x belongs to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ if and only if x is a monetary Walras allocation. Therefore the standard replica core limit theorem (Debreu and Scarf 1963) can be restated as follows (because there is no difference between our monetary Walras allocation and the Walras allocation for finite economies).

> A feasible allocation x for $\mathcal{E}$ is a Walras allocation if and only if its $(m+n)$-fold replica allocation belongs to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ for all $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$ sufficiently large.

In general, the number of agents in $\mathcal{E}$ is not finite, and $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ gives us a complete characterization of monetary Walras allocations. That is, an element of $\boldsymbol{\mathcal{MW}alras}(\mathcal{E})$ has a certain kind of stability-under-replication property with respect to the form $\mathcal{E}^m(x)$ and $\mathcal{E}^n(\omega)$ for the finite core solution concept. The next theorem is proved in Urai and Murakami (2015) showing the above replica finite core stability condition for the element of $\boldsymbol{\mathcal{MW}alras}(\mathcal{E})$. Thus a replica stability condition like Thomson (1988) and Nagahisa (1994) is satisfied automatically in our price-money message mechanism.

Theorem 1 (Replica Core Equivalence Theorem): A feasible allocation x for $\mathcal{E}$ is a monetary Walras allocation if and only if its $(m+n)$-fold replica allocation belongs to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ for every $m \in \boldsymbol{N}$ and $n \in \boldsymbol{N}$.

Let $\mathbb{C}^{m \oplus n}(\mathcal{E})$ be the set of allocations x for $\mathcal{E}$ such that the $(m+n)$-fold replica allocation of x belongs to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$. Then $\mathbb{C}^{m \oplus n}(\mathcal{E}) \supset \mathbb{C}^{m' \oplus n'}(\mathcal{E}) \supset \cdots$ where $m' \geqq m$ and $n' \geqq n$, and $\boldsymbol{\mathcal{MW}alras}(\mathcal{E}) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \mathbb{C}^{m \oplus n}(\mathcal{E})$. Thus we have obtained an extension of Debreu-Scarf replica core limit theorem to monetary Walras allocations in economies including overlapping-generations settings.

[2] In the following, the subscript, $i \in \bigcup_{t \in \boldsymbol{N}} I_t$, for an allocation is sometimes omitted as long as there is no risk of confusion.

[3] Assume that an $(m+n)$-fold replica allocation x^{m+n} for $\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega)$ does not belong to $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$. If the number of agents of $\mathcal{E}$ is finite, then allocation x is feasible for the finite set of all agents of $\mathcal{E}$. It is possible, therefore, to extend every finite coalition blocking x^{m+n} in $\boldsymbol{\mathcal{F}core}(\mathcal{E}^m(x) \oplus \mathcal{E}^n(\omega))$ to a finite coalition including all members of m-fold replica economy of $\mathcal{E}$, i.e., to a finite coalition blocking x^{m+n} in $\mathcal{E}^{m+n}(\omega)$.

4 An Axiomatic Characterization of the Price-Money Message Mechanism

The replica finite core limit theorem in the above section, Theorem 1, enables us to provide an axiomatic characterization of the price-money message mechanism through the universal mapping problem as in Sonnenschein (1974). Our Theorem 2 on the price-money dictionary is corresponding to Sonnenschein's Proposition 1 and Theorem 3 on the isomorphism property is to his Proposition 7.

A *social choice correspondence* $g : \mathcal{E}\!\mathit{con} \to (\boldsymbol{R}_\infty)^{\boldsymbol{N}}$ is said to be *compatible with* a finite-core and weakly Pareto-optimal allocation if and only if $g(\mathcal{E}) \subset \mathcal{F}\!\mathit{core}(\mathcal{E})$ and $g(\mathcal{E}) \subset \{x | \, x \in \mathcal{F}\!\mathit{core}(\mathcal{E}(x))\}$ for all $\mathcal{E} \in \mathcal{E}\!\mathit{con}$. An abstract *message mechanism* (an allocation mechanism with messages under the framework of Sonnenschein 1974) based on social choice correspondence g is a triple (A, μ, f), where A is a set that is called a *message domain*, μ is a correspondence that indicates for each economy $\mathcal{E}$ the set $\mu(\mathcal{E}) \subset A$ of *equilibrium messages for* $\mathcal{E}$, and f is a function that defines for each agent, i, and each message, a, the *response*, $f^i(\mathcal{E}, a)$, of the agent in $\mathcal{E}$ to the message, satisfying that $g(\mathcal{E}) = \{(f^i(\mathcal{E}, a))_{i=1}^\infty \mid a \in \mu(\mathcal{E})\}$.

The monetary Walrasian social choice correspondence associates with each economy the monetary Walras allocations of the economy. In this section, we treat allocations that are compatible with finite-core and weakly Pareto-optimal only. For the standard message mechanism, let A be the product of $\boldsymbol{R}_+^{\boldsymbol{N}}$ and $\{M | \, M : \mathcal{E}\!\mathit{con} \to \boldsymbol{R}_+^{\boldsymbol{N}}\}$, $\mu(\mathcal{E})$ be the set of equilibrium prices with the non-negative wealth transfers of $\mathcal{E}$, and f be the excess demand function of each consumer relative to price-money messages (more precisely, see below). We use the following idempotency axiom.

Axiom I (Idempotency): For each economy $\mathcal{E} \in \mathcal{E}\!\mathit{con}$ and message $a \in A$, $f(\mathcal{E}(f(\mathcal{E}, a)), a) = f(\mathcal{E}, a)$.

In the above, we use the notation $\mathcal{E}(x)$ in section 3. Moreover, let us consider the following axiom of Sonnenschein (1974).

Axiom S (Sonnenschein): For each finite list of agents and economies, $(i_1, \mathcal{E}^1), (i_2, \mathcal{E}^2), \ldots, (i_m, \mathcal{E}^m)$, each message $a \in A$ and each list of responses $(f^{i_s}(\mathcal{E}^s, a))_{s=1}^m$, there exists an economy $\mathcal{E}_*$ including $\{i_1, i_2, \ldots, i_m\}$ such that a is an equilibrium message for $\mathcal{E}_*$ for which the equilibrium list $(f^i(\mathcal{E}_*, a))_{i=1}^\infty$ is an extension of $(f^{i_s}(\mathcal{E}^s, a))_{s=1}^m$.

It follows that we do not treat messages that cannot constitute an equilibrium state for any sufficiently large extension of a certain list of agents and economies. The above axiom together with the replica equivalence theorem (Theorem 1) is closely related to the replication stability axiom of Thomson (1988).

We assume in the following that the commodity structure $\{K_t\}_{t=1}^\infty$ is fixed. The set of all economies with the commodity structure $\{K_t\}_{t=1}^\infty$ is denoted by $\mathcal{E}\!\mathit{con}^*$. Denote by $I(t)$ the set of all agents in generations from 1 to t, i.e., $I(t) = \bigcup_{s=1}^t I_s$, and by $K(t)$ the set of all commodities that are available for agents in $I(t)$, i.e., $K(t) = \bigcup_{s=1}^t K_s$. For each t, denote by $\Delta^{K(t)}$ the unit simplex in $\boldsymbol{R}^{K(t)}$ and by $\Delta_{++}^{K(t)}$ its relative interior, $\boldsymbol{R}_{++}^{K(t)} \cap \Delta^{K(t)}$. Let us consider the projective system $(\Delta_{++}^{K(t')}, \varrho_{t't})_{t',t \in \boldsymbol{N}}$ and projective limit $\Delta_{++} = \varprojlim (\Delta_{++}^{K(t')}, \varrho_{t't})$, where $\varrho_{t't} : \Delta_{++}^{K(t)} \to \Delta_{++}^{K(t')}$ is defined as $\varrho_{t't}(p) = \frac{\mathrm{pr}_{K(t')}\, p}{\|\,\mathrm{pr}_{K(t')}\, p\|}$.[4] We note that Δ_{++} can be recognized as a subset of $\boldsymbol{R}_{++}^\infty$ by identifying the equivalence class $[(x_t)_{t=1}^\infty]$ of $(x_t)_{t=1}^\infty \in \prod_{t=1}^\infty \Delta_{++}^{K(t)}$ with the element $p \in \boldsymbol{R}_{++}^\infty$ such that $\mathrm{pr}_{K(1)}\, p = x_1$ and $\frac{\mathrm{pr}_{K(t)}\, p}{\|\,\mathrm{pr}_{K(t)}\, p\|} = x_t$ for

[4] The projective limit is non-empty since $t \in \boldsymbol{N}$ is countable and every $\varrho_{t't}$ is surjective (see Bourbaki 1939, p.198, Proposition 5).

all $t = 2, 3, \cdots$.[5] We can take the price and non-negative wealth transfer domain as $\mathcal{P} \times \mathcal{M} = \{p \in \boldsymbol{R}^{\infty} | \, \exists[(x_t)_{t=1}^{\infty}] \in \Delta_{++}, \mathrm{pr}_{K(1)}\, p = x_1, \frac{\mathrm{pr}_{K(t)}\, p}{\| \mathrm{pr}_{K(t)}\, p \|} = x_t, \text{ for each } t = 1, 2, \cdots\} \times \{M | \, M : \mathcal{E}con^* \ni \mathcal{E} \mapsto M_{\mathcal{E}} \in R_{+}^{\boldsymbol{N}}\}$. The excess demand function e^i for the i-th consumer $(\succsim_i, \omega_i)$ in $\mathcal{E} \in \mathcal{E}con^*$ is defined as $e^i : \mathcal{P} \times \boldsymbol{R}_{+}^{\boldsymbol{N}} \ni (p, M_{\mathcal{E}}) \mapsto e^i(p, M_{\mathcal{E}}) \in \boldsymbol{R}_{\infty}$, where $e^i(p, M_{\mathcal{E}})$ is identified with the points of the consumption set $\boldsymbol{R}_{+}^{K_t}$, such that $e^i(p, M_{\mathcal{E}})$ is the $\succsim_i$-greatest point in $\{x^i \in \boldsymbol{R}^{K_t} | \, p \cdot x^i \leqq p \cdot \omega^i + M_{\mathcal{E}}(i)\}$ for each $i \in I_t$ and each $t \in \boldsymbol{N}$.

Define $e : \mathcal{E}con^* \times (\mathcal{P} \times \mathcal{M}) \to (\boldsymbol{R}_{\infty})^{\boldsymbol{N}}$ by $e(\mathcal{E}, p, M) = (e^i(p, M_{\mathcal{E}}))_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}$.[6] If $\pi(\mathcal{E})$ denotes the set of all *monetary Walrasian equilibrium messages* (the set of market equilibrium price and non-negative wealth transfer messages) for each $\mathcal{E} \in \mathcal{E}con^*$, then $(\mathcal{P} \times \mathcal{M}, \pi, e)$ is a message mechanism based on social choice correspondence $\mathcal{MW}alras(\mathcal{E})$. This is called the *price-money message mechanism.*

Theorem 2 (Price-Money Dictionary Theorem): If (A, μ, f) is a message mechanism based on social choice correspondence g that is compatible with the finite-core and weakly Pareto-optimal allocations, and if (A, μ, f) satisfies Axioms I and S, then (i) there exists a unique function $\phi : A \to \mathcal{P} \times \mathcal{M}$ such that the following triangle commutes, and (ii) on $\phi(A) \subset \mathcal{P} \times \mathcal{M}$, the price-money message mechanism satisfies axioms I and S.

$$\begin{tikzcd}
(\boldsymbol{R}_{\infty})^{\boldsymbol{N}} & \mathcal{E}con^* \times (\mathcal{P} \times \mathcal{M}) \arrow[l, "e"'] \\
& \mathcal{E}con^* \times A \arrow[ul, "f"] \arrow[u, "1_{\mathcal{E}con^*} \times \phi"']
\end{tikzcd}$$

Proof: (i) Assume that (A, μ, f) is a message mechanism based on social choice correspondence g satisfying Axioms I and S, and let a be an element of A. Define for each $t \in \boldsymbol{N}$, $h^{(t)}(x, \succsim_i)$ for each consumption $x \in \boldsymbol{R}^{K_s}$ for agent $i \in I_s \subset I(t)$ of an economy $\mathcal{E} \in \mathcal{E}con^*$ as $h^{(t)}(x, \succsim_i) = \{p \in \Delta^{K(t)} | \, y \succ_i x \text{ implies } p \cdot y \geqq p \cdot x\}$, where every $\boldsymbol{R}^{K_s}$ is canonically identified with a subspace of $K(t)$. We first show that $\bigcap h^{(t)}(f^i(\mathcal{E}, a), \succsim_i)$ is non-empty for each $t \in \boldsymbol{N}$, where the intersection is over all consumers and economies in $\mathcal{E}con^*$, and $f^i(\mathcal{E}, a)$ is a response of $i \in I(t)$ in $\mathcal{E}$ to message a in (A, μ, f). Because $\Delta^{K(t)}$ is compact, and because each of the sets in the collection of which we are forming the intersection is closed, it is sufficient to show that $\bigcap_{s=1}^{m} h^{(t)}(f^{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty for any $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$. Given the list $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$ of agents in $I(t)$ and economies, by Axiom S there exists $\mathcal{E}_* \in \mathcal{E}con^*$ containing $\{i_1, i_2, \ldots, i_m\}$ and $a \in \mu(\mathcal{E}_*)$, such that the equilibrium list, $(f^i(\mathcal{E}_*, a))_{i=1}^{\infty}$, is an extension of $(f^{i_s}(\mathcal{E}^s, a))_{s=1}^{m}$. Because $(f^i(\mathcal{E}_*, a))_{i=1}^{\infty}$ is weakly Pareto-optimal, by Balasko and Shell (1980) and Esteban (1986), it is supported by a price as a price-wealth equilibrium, and thus $\bigcap_{s=1}^{m} h^{(t)}(f^{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty. Moreover, because for some economy and its agents, $\bigcap_{i=1}^{\infty} h^{(t)}(f^i(\mathcal{E}, a), \succsim_i)$ is singleton and is an element of $\Delta_{++}^{K(t)}$, it follows that $\bigcap h^{(t)}(f^i(\mathcal{E}, a), \succsim_i)$ is composed of a single point $p(t)$.

By definition of $h^{(t)}$, for all $t' \leqq t$, $p(t') = \varrho_{t't}(p(t))$, and we obtain a unique element $p \in \mathcal{P}$ by identifying it with the unique element of the projective limit $\varprojlim \bigcap_{s=1}^{m} h^{(t)}(f^{i_s}(\mathcal{E}_s, a), \succsim_{i_s}) \subset \Delta_{++}$. Let us denote that

[5] For example, if $K(1) = \{1, 2\}$, $K(2) = \{1, 2, 3\}$, $K(3) = \{1, 2, 3, 4\}$, $\cdots$, then the element $((1/2, 1/2), (1/3, 1/3, 1/3), (1/4, 1/4, 1/4, 1/4), \cdots)$ of Δ_{++} can be identified with $(1/2, 1/2, 1/2, 1/2, 1/2, \cdots)$ of $\boldsymbol{R}_{++}^{\infty}$.

[6] For the argument in the following, we use the property that for each price-money message (p, M) there is an economy in $\mathcal{E}con^*$ such that (p, M) is the unique price-money equilibrium message supporting the equilibrium allocation. For this, the condition that $\mathcal{E}con^*$ includes the class of economies consisting of agents with Cobb-Douglas utility functions, is sufficient but not necessary.

point, p, by $\phi^1(a)$, and define $\phi^2(a) = M$, $M : \boldsymbol{\mathcal{E}con}^* \ni \boldsymbol{\mathcal{E}} \mapsto M_{\mathcal{E}} \in \boldsymbol{R}_+^{\boldsymbol{N}}$, where $M_{\mathcal{E}}(i) = \phi^1(a) \cdot (f^i(\boldsymbol{\mathcal{E}}, a) - \omega^i)$, which we prove as follows that it is non-negative under Axioms S, I and Theorem 1. Let $\phi(a)$ be $(\phi^1(a), \phi^2(a)) \in \boldsymbol{\mathcal{P}} \times \{M |\, M : \boldsymbol{\mathcal{E}con}^* \ni \boldsymbol{\mathcal{E}} \mapsto M_{\mathcal{E}} \in R_+^{\boldsymbol{N}}\} = \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$. To establish the theorem, it is sufficient to show that for each economy $\boldsymbol{\mathcal{E}}_* \in \boldsymbol{\mathcal{E}con}^*$ and each $a \in A$, allocation $y_* = (y_*^i)_{i=1}^\infty = (f^i(\boldsymbol{\mathcal{E}}_*, a))_{i=1}^\infty$ is such that, for each i, $y_*^i = f^i(\boldsymbol{\mathcal{E}}_*, a)$ can be extended (through Axiom S) to a monetary Walras allocation under the message a for a large economy including i. To see this, by using Axiom S, let $\boldsymbol{\mathcal{E}}_{**}$ be an economy including i such that a is an equilibrium message for $\boldsymbol{\mathcal{E}}_{**}$ and response $y_{**}^i = f^i(\boldsymbol{\mathcal{E}}_{**}, a)$ is equal to $y_*^i = f^i(\boldsymbol{\mathcal{E}}_*, a)$. We show that the list of responses, $y_{**} = f(\boldsymbol{\mathcal{E}}_{**}, a)$, is a monetary Walras allocation. Assume that it is not. Then, by Theorem 1, there are m and n such that $(m+n)$-fold replica allocation of $y_{**} = (y_{**}^j)_{j=1}^\infty$ is not an element of $\boldsymbol{\mathcal{F}core}(\boldsymbol{\mathcal{E}}_{**}^m(y_{**}) \oplus \boldsymbol{\mathcal{E}}_{**}^n(\omega))$. Then there are $(k_1 + k_2)$ members of $\boldsymbol{\mathcal{E}}_{**}^m(y_{**}) \oplus \boldsymbol{\mathcal{E}}_{**}^n(\omega)$ blocking y_{**}, where k_1 members belong to $\boldsymbol{\mathcal{E}}_{**}^m(y_{**})$ and k_2 members belong to $\boldsymbol{\mathcal{E}}_{**}^n(\omega)$. Under Axiom I, by applying Axiom S again, we obtain an economy $\boldsymbol{\mathcal{E}}_{***}$ including the (k_1+k_2) members of $\boldsymbol{\mathcal{E}}_{**}^m(y_{**}) \oplus \boldsymbol{\mathcal{E}}_{**}^n(\omega)$ such that a is a solution message for $\boldsymbol{\mathcal{E}}_{***}$ satisfying that the equilibrium list, $(f^i(\boldsymbol{\mathcal{E}}_{***}, a))_{i=1}^\infty$, a finite core allocation, is an extension of those of the k_1 members of $\boldsymbol{\mathcal{E}}_{**}^m(y_{**})$ (that are by Axiom I, $f^i(\boldsymbol{\mathcal{E}}_{***}, a) = f^i(\boldsymbol{\mathcal{E}}_{**}(y_{**}), a) = f^i(\boldsymbol{\mathcal{E}}_{**}, a)$, equal to their original responses) and the k_2 members of $\boldsymbol{\mathcal{E}}_{**}^n(\omega)$ under $a \in A$. However, this is impossible, because the (k_1+k_2) members can improve upon it.

(ii) For the latter assertion, one can observe in the above argument, $y_{**} = f(\boldsymbol{\mathcal{E}}_{**}, a) = e(\boldsymbol{\mathcal{E}}_{**}, \phi(a))$ is a monetary Walras allocation, which proves that axiom S is satisfied on $\phi(A)$. Moreover, it is straightforward that the commutativity of the diagram with axiom I for (A, μ, f) means that axiom I is satisfied on $\phi(A)$ since $e(\boldsymbol{\mathcal{E}}(e(\boldsymbol{\mathcal{E}}, \phi(a))), \phi(a)) = e(\boldsymbol{\mathcal{E}}(f(\boldsymbol{\mathcal{E}}, a)), \phi(a)) = f(\boldsymbol{\mathcal{E}}(f(\boldsymbol{\mathcal{E}}, a)), a) = f(\boldsymbol{\mathcal{E}}, a) = e(\boldsymbol{\mathcal{E}}, \phi(a))$. ■

The above theorem shows that the price-money message mechanism represents all the message mechanisms based on social choice correspondences compatible with finite-core and weakly Pareto-optimal allocations under axioms I and S. The theorem, therefore, can be interpreted as a *price-money message mechanism representation* theorem. We call ϕ the *price-money dictionary.*

A non-negative price-wealth transfer message, (p, M), has a huge scope. In particular, the domain of M is $\boldsymbol{\mathcal{E}con}^*$, and it must specify all non-negative wealth transfers for all economies. Such a message may become too complicated to be called "simple" in the sense of Sonnenschein (1974) and one can easily verify that the price-money message mechanism may not satisfy axioms I and S.[7] However, if we restrict the domain of non-negative price-wealth transfer messages to the class satisfying Axioms I and S, $\boldsymbol{\mathcal{PM}}^* = \{(p, M) \in \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}} |\, (p, M)$ is an image of ϕ for some $(A,\ \mu,\ f)$ in Theorem 2 satisfying axioms I and S.$\}$, we have the following isomorphism theorem. For this assertion, we use the next axiom on the dependency of monetary messages on economic structures.

Axiom D (Dependency on the Economic Structure) : If $\boldsymbol{\mathcal{E}} = (\{I_t\}_{t=1}^\infty,\ \{K_t\}_{t=1}^\infty,\ \{(\succsim_i,\ \omega_i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ and $\boldsymbol{\mathcal{E}}' = (\{I_t'\}_{t=1}^\infty,\ \{K_t'\}_{t=1}^\infty,\ \{(\succsim_i',\ \omega_i')_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ are such that $\{I_t\}_{t=1}^\infty = \{I_t'\}_{t=1}^\infty$, $\{K_t\}_{t=1}^\infty = \{K_t'\}_{t=1}^\infty$ and $\omega_i = \omega_i'$ for all $i \in \bigcup_{t \in \boldsymbol{N}} I_t$, then $M_{\mathcal{E}} = M_{\mathcal{E}'}$ for all $M \in \boldsymbol{\mathcal{M}}$.

[7] Axiom I will be satisfied if we restrict our attention to monetary mesages that intend to realize a certain desirable level of income for each person independently from their initial endowments for each economy $\boldsymbol{\mathcal{E}}$, but in general, it will not be satisfied. To obtain a counter example for axiom S, assume that $\{K_t\}_{t=1}^\infty$ is the two period one commodity overlapping-generation structure. Consider M such that amount $p \cdot \omega^i + M_{\mathcal{E}}(i) > 0$ for each $\boldsymbol{\mathcal{E}}$, $i \in I(1)$, is greater than the amounts of the total value of their initial endowments under $(1, 1, 1, \cdots) \in \boldsymbol{R}_+^\infty$ that is greater than all equilibrium prices for commodities in $K(1)$. (Note that a price $(p_1^*, p_2^*, p_3^*, \cdots)$ is assumed to satisfy $p_1^* + p_2^* = 1$. See definition of $\boldsymbol{\mathcal{P}}$.) Thus, we cannot expect the above message to be an equilibrium of any $\boldsymbol{\mathcal{E}} \in \boldsymbol{\mathcal{E}con}$. In Sonnenschein (1974), the mnemonic S is used because the condition states that "any finite number of individuals can be swamped" and it "restricts the messages of a message mechanism to be simple."

Theorem 3 (Isomorphism Theorem): Consider the restriction of price-monetary message mechanism $(\mathcal{PM}^*, \pi, e)$. Let (P', π', e') be a message mechanism based on social choice correspondence g on $\mathcal{E}con^*$ compatible with the finite core and weakly Pareto-optimal allocations. If (P', π', e') satisfies axioms I and S, and if, for every message mechanism (A, μ, f) satisfying axioms I and S, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}^* \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. (ii) Moreover, assume that monetary messages satisfy Axiom D. If we can restrict the problem on spaces with topological (resp. on each component space of an inverse system with differentiable) structures and continuous mappings (resp. differentiable coordinate mappings), then the isomorphism can be taken as the homeomorphism (resp. diffeomorphism for each component space).[8]

Proof: Because $(\mathcal{PM}^*, \pi, e)$ is now assumed to be a message mechanism based on social choice correspondence g satisfying axioms I and S, we have the next diagram by assumption.

$$\begin{tikzcd}
(\boldsymbol{R}_\infty)^{\boldsymbol{N}} & \mathcal{E}con^* \times P' \arrow[l, "e'"'] \\
 & \mathcal{E}con^* \times \mathcal{PM}^* \arrow[ul, "e"] \arrow[u, "1_{\mathcal{E}con^*} \times \phi'"']
\end{tikzcd}$$

Moreover, because (P', π', e') is also a message mechanism based on social choice correspondence g, the previous theorem shows that we have the next diagram.

$$\begin{tikzcd}
(\boldsymbol{R}_\infty)^{\boldsymbol{N}} & \mathcal{E}con^* \times \mathcal{PM}^* \arrow[l, "e"'] \\
 & \mathcal{E}con^* \times P' \arrow[ul, "e'"] \arrow[u, "1_{\mathcal{E}con^*} \times \phi"']
\end{tikzcd}$$

Since the identity mapping is the unique mapping for P' to P' satisfying $e' = e' \circ id$ and $\mathcal{PM}^*$ to $\mathcal{PM}^*$ satisfying $e = e \circ id$, we have $\phi' \circ \phi = id$ and $\phi \circ \phi' = id$, which means that ϕ and ϕ' are bijectives. Let us define h' as $h' = \phi'$, then we have the first assertion.

For the second assertion, for each $(p, M) \in \mathcal{PM}^*$, each economy $\mathcal{E} \in \mathcal{E}con^*$ and each generation t, consider two agents i_s and j_s, $s \in \{1, \ldots, t\}$ such that $e = (\cdots, e_{i_1}, \cdots, e_{j_1}, \cdots, e_{i_2}, \cdots, e_{j_2}, \cdots)$ on $\mathcal{P} \times \mathcal{M}$ is one to one, continuous and/or differentiable.[9] ■

[8] In this paper, the price-money message space can be identified with an inverse limit of the finite dimensional domains of each coordinate function e_i for e and the differentiable structure can be discussed under such finite dimensional domains. For example, we can take the domain of e_i for $i \in I(t)$ as $\Delta_{++}^{K(t)} \times \boldsymbol{R}_+^{I(t)}$ and identify it with a subset of $\mathcal{PM}^*$ under the canonical identification $\boldsymbol{R}_\infty \subset \boldsymbol{R}^\infty$. Then, we can also obtain under h', the image of $\Delta_{++}^{K(t)} \times \boldsymbol{R}_+^{I(t)}$ in P', the domain of all e'_i, $i \in I(t)$, and the diffeomorphism between them.

[9] If generation $s \in \{1, \ldots, t\}$ of $\mathcal{E}$ consists of a single member, alternatively consider an economy $\hat{\mathcal{E}}$ including all members of generations $1, \cdots, t$ of economy $\mathcal{E}$ such that every generation s of $\hat{\mathcal{E}}$ consists of at least two members and (p, M) is an equilibrium message of it by Axiom S. To obtain a concrete example for such a one-to-one, continuous and/or differentiable mapping, take a pair of Cobb-Douglas and Leontief utility agents for each generation. Note that the Leontief type utility is not differentiable, but the demand function induced from it can be differentiable.

By Theorem 3, we see that an allocation mechanism with messages that can play the role of the dictionary property described in Theorem 2 is essentially unique. The result can also be restated through the universal mapping theorem (Bourbaki 1939, p.284) that assures the existence of such an object unique up to isomorphism as a solution to the universal mapping problem.

5 Conclusion

1. We have seen that for every private representation (A, μ, f) of a social choice correspondence g compatible with the finite core and weakly Pareto-optimal allocations satisfying Axioms I and S, there exists a *dictionary function*, $\phi : A \to \mathcal{P} \times \mathcal{M}$ uniquely (Theorem 2). In other words, the result of such a private representation (an allocation mechanism with message) can be *universally* realized through the price-money message mechanism $(\mathcal{P} \times \mathcal{M}, \pi, e)$ *uniquely* (the universality and the efficiency property).

2. If we restrict the domain of an abstract message mechanism to those satisfying axioms I and S, the price-money message mechanism has the unique (up to isomorphism) minimum size of the information space satisfying the dictionary property (Theorem 3).

3. Based on the differentiable structure, to develop the alternative informational efficiency argument to our framework is also desirable. By considering the differentiable structure on the message and commodity spaces, we will be able to obtain the same results as Propositions 3–6 in Sonnenschein (1974) that will provide an informational efficiency theorem like Hurwicz (1960), Mount and Reiter (1974) and Osana (1978), as well as a uniqueness theorem of the competitive message mechanism like Jordan (1982). These will be, however, the subjects of further investigations.

4. It should also be noted that our monetary equilibrium concept (a competitive equilibrium with non-negative wealth transfer) is related to the concept of "dividend equilibrium" or "equilibrium with slack" (see, e.g., Aumann and Drèze 1986) and the role of our replica finite core concept can also be identified with that of "rejective core" in Konovalov (2005) on the limit core characterization for such equilibria. Therefore, if we generalize the preference satiation structure for the double infinity economy (see our Urai and Murakami 2015, footnote 15), our results are possible to be identified with an another replica core characterization for such dividend equilibria.

5. Thomson (1988) and Nagahisa (1994) use the *replication stability* axiom like Axiom S of Sonnenschein (1974) to characterize Walrasian correspondences in a *direct* way for the standard social choice setting. It will be interesting to apply their arguments for our double infinity monetary economic situation, since various social choice axioms and frameworks would be desirable for describing the institution like money.

6. The method of Sonnenschein is appropriate for our setting of (i) the infinite dimensional commodity and generation structure and (ii) the partially economy-dependent property of messages. Moreover, his strongly structured response function setting has an advantage over the message-process argument in Hurwicz (1960), Mount and Reiter (1974), Osana (1978) and Jordan (1982) since it does not necessarily depend on (iii) the Cobb-Douglas utility assumption of the agents.

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