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Abstract

Dividend equilibrium, defined by Aumann and Drèze (1986), is one of the most general competitive equilibrium concepts for the market, including satiated consumers. Konovalov (2005) shows a core equivalence theorem to the dividend equilibrium using the concept of *rejective core*. Konovalov's argument, however, is based on the setting of an atomless large economy and the core limit problem for dividend equilibrium remains unsolved. In a previous paper, Urai and Murakami (2015), we provided a generalization of the Debreu-Scarf core limit theorem (Debreu and Scarf 1963) for monetary overlapping generations economies under an extended concept of replica core allocation. In this paper, we show that the concept and method also provide a core limit theorem for economies with satiation.

Keywords: Dividend Equilibrium, Core Equivalence, Replica Economy, Economy with Satiation

JEL Classification: C71, D50

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1 Introduction

Dividend equilibrium or *equilibrium with slack*, defined by Aumann and Drèze (1986), is one of the most general competitive equilibrium concepts for the market, including satiated consumers. For an economy with satiated consumers, a competitive equilibrium might fail to exist, and "the existence of an equilibrium can be restored if we give consumers appropriate extra amounts of income to spend" (Mas-Colell 1992). Konovalov (2005) shows a core equivalence theorem to the dividend equilibrium using the concept of *rejective core*. Konovalov's argument, however, is based on the setting of an atomless large economy and the core limit problem for dividend equilibrium remains unsolved.[1] In a previous paper, Urai and Murakami (2015), we provided a generalization of the Debreu-Scarf core limit theorem (Debreu and Scarf 1963) for monetary overlapping generations economies under an extended concept of replica core allocation. In this paper, we show that the concept and method also provide a core limit theorem for economies with satiation.

2 The Model

We use R as the set of real numbers. Let I be the non-empty finite set of agent indices and let K be the non-empty finite set of commodity indices. Each agent $i \in I$ is represented by $(\succsim_i,\ \omega_i)$, where $\succsim_i$ is the preference relation on consumption set R^K_+ for each agent $i \in I$ satisfying reflexivity, transitivity, completeness, continuity, and strict convexity.[2] The preferences, therefore, can be represented by utility functions. In addition, for each agent, the preference relation is allowed to be satiated and is locally non-satiated at every point except for the maximal satiation point that is unique, if it exists, under strict convexity.

To prove a core limit theorem for an economy with satiation, we use the next assumption as one of the simplest ways for ensuring the resource relatedness condition among agents:

> (**SNS**: socially non-satiated preference configuration)
> For each commodity $k \in K$ and allocation $x = (x_i)_{i\in I}$, there exists at least one agent $i \in I$ such that i's utility strictly increases as i's consumption amount of commodity k strictly increases.

The initial endowment of each $i \in I$ is represented by ω_i and is assumed to be an element of R^K_{++}. For economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i\in I})$, if allocation $(x_i \in R^K_+)_{i\in I}$ satisfies

$$\sum_{i\in I} x_i = \sum_{i\in I} \omega_i, \tag{1}$$

we say that $(x_i)_{i\in I}$ is *feasible*.

2.1 Dividend Equilibria

We define the *dividend equilibrium allocation* for economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i\in I})$ based on a feasible allocation that establishes the utility maximization for each agent under given price vector $p = (p_k)_{k\in K} \in R^K_+$ and non-negative dividends $d = (d_i)_{i\in I} \in R^I_+$. The list of price vector $p^* \in R^K_+$, dividends d^* and

[1] Aumann and Drèze (1986) gives a limit characterization for the dividend equilibrium using a Sharpley value, but fails to provide a limit theorem based on such game-theoretic solution concepts as core.
[2] We use R^K instead of $R^{\sharp K}$ to represent $\sharp K$-dimensional vector space since the set can be regarded as the set of functions on K to R.

feasible allocation $(x_i^*)_{i\in I}$ is called a *dividend equilibrium* for $\mathcal{E}$, if for each $i \in I$, x_i^* is the $\succsim_i$-greatest element in the following set:

(2) $$\{x_i \in R^K \mid p^* \cdot x_i \leqq p^* \cdot \omega_i + d_i^*\}.$$

A dividend equilibrium is also called an *equilibrium with slack*, and we denote the set of all dividend equilibrium allocations for economy $\mathcal{E}$ by $\boldsymbol{SWalras}(\mathcal{E})$.

The dividend equilibrium, which is one of the most general equilibrium concepts for a market economy, allows preferences to be satiated. It includes the *coupons equilibrium* under price rigidities and quantity rationing (see Drèze and Müller 1980 and Aumann and Drèze 1986). The coupons equilibrium is associated with the problem of quantity adjustments in the temporary equilibrium of Grandmont (1977).[3] Aumann and Drèze (1986) pointed out that the coupons equilibrium can be identified with a dividend equilibrium by regarding a fixed price as a 1-dimensional additional constraint and considering $(\sharp K - 1)$-dimensional modified commodity space with non-negative dividends.

3 Core and Replica Economy

In this paper, the core in the replica economy has special roles and meanings. First, let us define the concepts of the standard core and the rejective core of Konovalov (2005). Then we provide the concept of re-negotiation in replica economies that plays an essential role in proving the core limit theorem of this paper.

3.1 Core

A *coalition* in economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i\in I})$ is a set of agents $S \subset I$. Feasible allocation x is said to be the *core allocation* if there is no coalition S, and no feasible allocation y satisfies the following conditions:

(a) $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$.

(b) $y_i \succsim x_i$ for all $i \in S$ and $y_i \succ x_i$ for at least one $i \in S$.

We call the set of all core allocations the *core* of economy $\mathcal{E}$ and denote it by $\boldsymbol{Core}(\mathcal{E})$. Allocation x is said to be *blocked* by coalition S if conditions (a) and (b) hold. When we strengthen condition (b) to condition (b$'$) $y_i \succ x_i$ for all $i \in S$, the set of feasible allocations that cannot be blocked by such a coalition is called a *weak core* and is denoted by $\boldsymbol{Wcore}(\mathcal{E})$.

3.2 Rejective Core

Konovalov (2005) shows the equivalence theorem between the competitive equilibrium and the rejective core in the large economy. The concept of rejective core can easily be translated for finite economies. For economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i\in I})$, feasible allocation x is said to be a *rejective core allocation* if there is no coalition S with partition (S_1, S_2), and no feasible allocation y satisfies the following conditions:[4]

[3] For a dividend equilibrium allocation, the budget constraint for each agent is defined as (2). On the other hand, for a coupons equilibrium allocation, the budget constraint is defined as $\{x_i \in R^K \mid \bar{p} \cdot x_i = \bar{p} \cdot \omega_i$ and $q \cdot x_i \leqq q \cdot \omega_i + c_i\}$ under a certain fixed price, $\bar{p} \in R_+^K$, a coupons price vector, $q \in R^K$, and a coupons endowment, $c_i \in R$.

[4] Since Konovalov (2005) treats the equivalence theorem in the limit, he uses only strict preferences and his definition of rejective core is based on the weak core concept, and condition (d) is weakened as the above (b$'$). In this paper, we consider the standard core concept of Debreu and Scarf (1963).

(c) $\sum_{i \in S} y_i = \sum_{i \in S_1} \omega_i + \sum_{i \in S_2} x_i$.
(d) $y_i \succsim x_i$ for all $i \in S$ and $y_i \succ x_i$ for at least one $i \in S$.
(e) $y_i \succsim \omega_i$ for all $i \in I \setminus S$.

We call the set of all rejective core allocations the *rejective core* of economy $\mathcal{E}$. Note that if $S_2 = \emptyset$, we can neglect condition (e) since it is always possible to define y_i as ω_i for all $i \in I \setminus S$, so the definition of rejective core allocation becomes that of the standard core allocation. When allocation x is blocked by coalition S in the sense of the standard core, it is also blocked by $S = S_1$ (with $S_2 = \emptyset$) in the sense of the rejective core. Hence a rejective core allocation is also a core allocation.

3.3 Replica Economy and Re-Negotiation Replica Economy

In Urai and Murakami (2015), based on the concept of re-negotiation in replica economies, we showed the core equivalence theorem for monetary general equilibria in overlapping-generations economies. Now, we introduce the concept of re-negotiation in a replica economy to this paper and give the necessary settings for our arguments.

For economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i \in I})$, $\mathcal{E}^n$ represents the *n-fold replica economy* with n-times replica agents of economy $\mathcal{E}$. For each feasible allocation $x = (x_i \in R^K_+)_{i \in I}$ for economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i \in I})$, we denote by $\mathcal{E}(x)$ an *economy where initial endowment allocation* $\omega = (\omega_i)_{i \in I}$ *is replaced by* $x = (x_i)_{i \in I}$. The difference between $\mathcal{E}(x)$ and $\mathcal{E}$ is only the initial endowment and the other settings like preferences are identical. Then, we can write $\mathcal{E} = \mathcal{E}(\omega)$. Consider the following replica economy,

$$\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x), \tag{3}$$

which consists of all the members of the m-fold replica economy of $\mathcal{E}(\omega)$ and the n-fold replica economy of $\mathcal{E}(x)$ for non-negative integers m and n. We call this economy, $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$, the *$(m+n)$-fold re-negotiation replica economy* (**RNR** economy) of $\mathcal{E}$. For allocation y of economy $\mathcal{E}(x)$, y^n represents the n-fold replica allocation of y for n-fold replica economy $\mathcal{E}^n(x)$ such that each replica agent is assigned the same allocation under y in original economy $\mathcal{E}(x)$. In the same way, for common allocation y in economies $\mathcal{E}(\omega)$ and $\mathcal{E}(x)$, y^{m+n} represents the $(m+n)$-fold replica allocation of y for $(m+n)$-fold RNR economy $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$ such that each replica agent is assigned the same allocation under y in original economies $\mathcal{E}(\omega)$ or $\mathcal{E}(x)$.

4 Re-Negotiation Replica Core Limit Theorem

In our main theorem, we show that the core of RNR economy $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$ converges to the dividend equilibria. Before its proof, we check that the theorem includes an equivalence theorem with a rejective core.

Lemma: Let x be a feasible allocation of economy $\mathcal{E}$, let m be a positive integer, and let n be a non-negative integer. If $(m+n)$-fold replica allocation x^{m+n} is a rejective core allocation of $(m+n)$-fold replica economy $\mathcal{E}^{m+n}$, then x^{m+n} is a core allocation of $(m+n)$-fold RNR economy $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$. The lemma is also true for cases with a weak core and a weak rejective core.

Proof: For feasible allocation x' of economy $\mathcal{E}$, if there exists agent j with $\omega_j \succ_j x'_j$, x' is blocked by the coalition consisting of the single agent j in the sense of standard core blocking definition. Hence, the $(m+n)$-fold replica allocation of x' cannot be a core allocation of the RNR economy $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$ nor a rejective core allocation of replica economy $\mathcal{E}^{m+n}$. Thus we can assume that the allocation x of the lemma satisfies $x_i \succsim_i \omega_i$ for all $i \in I$, i.e., the individual rationality.

For any $m \geqq 1$ and $n \geqq 0$, assume that $(m+n)$-fold allocation x^{m+n} is blocked by coalition S in $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$, where x is feasible and satisfying the individual rationality in $\mathcal{E}$. Coalition S is a union of S_1 and S_2 where S_1 is the set of agents belonging to the m-fold replica economy $\mathcal{E}^m(\omega)$ and S_2 is the set of agents belonging to the n-fold replica economy $\mathcal{E}^n(x)$. Let N be the set of all agents belonging to the RNR economy $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$. Consider feasible allocation y of $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$ where the agents belonging to the coalition S are assigned the blocking allocation and the lest agents belonging to $N \setminus S$ are simply assigned the initial endowments ω_i if i is a member of $\mathcal{E}^m(\omega)$ or x_i if i is a member of $\mathcal{E}^n(x)$.

First, for the coalition $S = S_1 \cup S_2$, by regarding this S, S_1 and S_2 as the coalition S and the partition (S_1, S_2) in the definition of the rejective core, we can check that conditions (c) and (d) hold. Next, for each agent $i \in N \setminus S$ of the economy $\mathcal{E}^m(\omega)$, $y_i = \omega_i$ and $y_i \succsim_i \omega_i$ holds evidently. For each agent $i \in N \setminus S$ of the economy $\mathcal{E}^n(x)$, $y_i = x_i$ and $y_i \succsim_i \omega_i$ also follows from the individual rationality of x. Hence, the third condition (e) of the definition of the rejective core is also satisfied.

From the above arguments, the allocation y blocks the replica allocation x^{m+n} of the replica economy $\mathcal{E}^{m+n}$. We can prove the lemma for cases with the weak core and weak rejective core in exactly the same way. ■

From the lemma, if the replica allocation is a rejective core allocation, it becomes the core allocation of the RNR economy. Thus we can have the replica core equivalence theorem of the rejective core by showing the following core limit theorem of an RNR economy.

Theorem 1: Feasible allocation x for $\mathcal{E}$ belongs to $\mathcal{SWalras}(\mathcal{E})$ iff its $(m+n)$-fold replica allocation belongs to $\mathcal{Core}(\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x))$ for every $m \geqq 1$ and $n \geqq 0$.[5] The theorem is also true for cases with a weak core and a weak rejective core.

Proof: **[Necessity]** Let $\bar{x} = (\bar{x}_i)_{i \in I}$ be a feasible allocation for economy $\mathcal{E} = (I, \{(\succsim_i,\ \omega_i)\}_{i \in I})$ such that every $(m+n)$-fold replica allocation of $\bar{x}$ belongs to $\mathcal{Core}(\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x))$ for all $m \geqq 1$ and $n \geqq 0$. Define for each $i \in I$, Γ_i as $\Gamma_i = \{\beta_i z_i^1 + (1-\beta_i) z_i^2 \mid z_i^1 + \omega_i \succ_i \bar{x}_i,\ z_i^2 + \bar{x}_i \succ_i \bar{x}_i,\ 0 \leqq \beta_i \leqq 1\} \subset R^K$. Then, take the convex hull Γ of finite union $\bigcup_{i \in I} \Gamma_i \subset R^K$. Since Γ_i is convex for every i and is non-empty for at least one agent by the SNS condition, Γ becomes a non-empty convex set. Let I' be the set of agents $i \in I$ such that $\Gamma_i \neq \emptyset$, we have $\bigcup_{i \in I'} \Gamma_i = \Gamma$. Then Γ consists of all vectors z that can be written as $z = \sum_{i \in I'} \alpha_i (\beta_i z_i^1 + (1-\beta_i) z_i^2)$, with $\alpha_i \geqq 0$, $\sum_{i \in I'} \alpha_i = 1$, where $z_i^1 + \omega_i \succ_i \bar{x}_i$ and $z_i^2 + \bar{x}_i \succ_i \bar{x}_i$ for each $i \in I'$.

We will show $0 \notin \Gamma$ in the similar way as in the proof of Debreu and Scarf (1963; Theorem 3). Let us suppose that 0 belongs to Γ. Then, one can write $\sum_{i \in I'} \alpha_i (\beta_i z_i^1 + (1-\beta_i) z_i^2) = 0$, with $\alpha_i \geqq 0$ and $\sum_{i \in I'} \alpha_i = 1$, where $z_i^1 + \omega_i \succ_i \bar{x}_i$ and $z_i^2 + \bar{x}_i \succ_i \bar{x}_i$ for each $i \in I'$. For sufficiently large κ, let $a_i^{1\kappa}$ and $a_i^{2\kappa}$ be the smallest integers greater than $\kappa \alpha_i \beta_i$ and $\kappa \alpha_i (1-\beta_i)$ respectively. Also, let J be the set of all

[5] In the case with standard core, the proof of sufficiency part strongly depends on the strict convexity of preferences, i.e., the fact that indifference curves are thin. However, in the case with weak core, the sufficiency part can be proved without the strict convexity of preferences.

$i \in I'$ for which $\alpha_i > 0$. For each $i \in J$, we define $z_i^{1\kappa}$ as $\frac{\kappa\alpha_i\beta_i}{a_i^{1\kappa}} z_i^1$, and $z_i^{2\kappa}$ as $\frac{\kappa\alpha_i(1-\beta_i)}{a_i^{2\kappa}} z_i^2$. Observe that $z_i^{1\kappa} + \omega_i$ belongs to the segment $[\omega_i,\ z_i^1 + \omega_i]$ and $z_i^{2\kappa} + \bar{x}_i$ belongs to the segment $[\bar{x}_i,\ z_i^2 + \bar{x}_i]$.

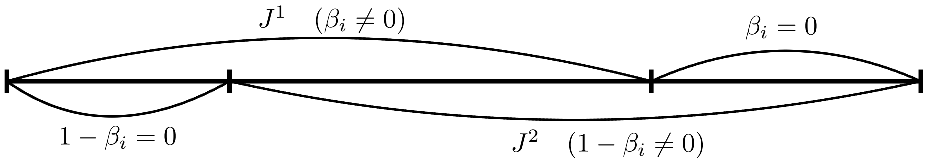

Figure 1: $J = J^1 \cup J^2$ with $\alpha_i > 0$ and $0 \leqq \beta_i \leqq 1$.

Let J^1 be the set of $i \in J$ such that $\beta_i \neq 0$, and J^2 be the set of $i \in J$ such that $1 - \beta_i \neq 0$. Note that $J^1 \cup J^2 = J$ (see Figure 1). For $i \in J^1$, $z_i^{1\kappa} + \omega_i$ tends to $z_i^1 + \omega_i$, and for $i \in J^2$, $z_i^{2\kappa} + \bar{x}_i$ tends to $z_i^2 + \bar{x}_i$ as κ tends to infinity. The continuity assumption on preferences implies that $z_i^{1\kappa} + \omega_i \succ_i \bar{x}_i$ for all $i \in J^1$ and $z_i^{2\kappa} + \bar{x}_i \succ_i \bar{x}_i$ for all $i \in J^2$ for all κ sufficiently large. Select one of such κ. Then we have

$$
\begin{aligned}
(4) \qquad 0 &= \kappa \sum_{i \in J} \alpha_i(\beta_i z_i^1 + (1-\beta_i) z_i^2) = \sum_{i \in J} \kappa\alpha_i(\beta_i z_i^1 + (1-\beta_i) z_i^2) \\
&= \sum_{i \in J^1 \setminus J^2} \kappa\alpha_i\beta_i z_i^1 + \sum_{i \in J^1 \cap J^2} \kappa\alpha_i(\beta_i z_i^1 + (1-\beta_i) z_i^2) + \sum_{i \in J^2 \setminus J^1} \kappa\alpha_i(1-\beta_i) z_i^2 \\
&= \sum_{i \in J^1 \setminus J^2} a_i^{1\kappa} z_i^{1\kappa} + \sum_{i \in J^1 \cap J^2} (a_i^{1\kappa} z_i^{1\kappa} + a_i^{2\kappa} z_i^{2\kappa}) + \sum_{i \in J^2 \setminus J^1} a_i^{2\kappa} z_i^{2\kappa}.
\end{aligned}
$$

Let us consider the RNR economy $\boldsymbol{\mathcal{E}}^m(\omega) \oplus \boldsymbol{\mathcal{E}}^n(x)$ with $m = \max_{i \in J}\ a_i^{1\kappa}$ and $n = \max_{i \in J}\ a_i^{2\kappa}$. Take the coalition composed of $a_i^{1\kappa}$ replica members of i for each $i \in J^1$ to each one of whom we assign $z_i^{1\kappa} + \omega_i$, and $a_i^{2\kappa}$ replica members of i for each $i \in J^2$ to each one of whom we assign $z_i^{2\kappa} + \bar{x}_i$. This coalition blocks the allocation $(\bar{x}^i)$ as equation (4) and the fact that $z_i^{1\kappa} + \omega_i \succ_i \bar{x}_i$ for each $i \in J^1$ and $z_i^{2\kappa} + \bar{x}_i \succ_i \bar{x}_i$ for all $i \in J^2$ show.[6] This is a contradiction to the definition of $\boldsymbol{Core}(\boldsymbol{\mathcal{E}}^m(\omega) \oplus \boldsymbol{\mathcal{E}}^n(x))$. Hence, we have established $0 \notin \Gamma$.

Let π be the set of prices such that $\pi = \{p \in R^K \cap \Delta \mid p \cdot z \geqq 0$ for all $z \in \Gamma\}$, where Δ represents the standard $(\sharp K - 1)$-dimensional simplex of R^K, i.e., $\Delta = \{p \mid p = (p_1, p_2, \cdots, p_K) \in R_+^K,\ \sum_{k=1}^K p_k = 1\}$. Set π is closed in R_+^K and is non-empty since there exists $p \in R^K \setminus \{0\}$ by the separating hyperplane theorem.[7]

From $p \in \pi$ and $\omega_i \in R_{++}^K$, we have $p \cdot \omega_i > 0$ for all $i \in I$. If a price of some commodity $k \in K$, p_k, is zero, we have a contradiction as follows. From the SNS condition, there exist some agent who demands the commodity k at $\bar{x}$. Then we call one such agent as i. Consider first the case that $p \cdot \bar{x}_i = 0$. Then, since $p \cdot \omega_i > 0$, let $\delta \in R_{++}$ be sufficiently small value such that $p \cdot \omega_i > p_k\delta$. A vector $\bar{x}_i + (0, \cdots, 0, +\delta, 0, \cdots, 0) - \omega_i$ such that $\bar{x}_i + (0, \cdots, 0, +\delta, 0, \cdots, 0)$ is strictly preferred to $\bar{x}_i$, where

[6] This is the only part that the proof depends on the definition of the core. Since in the coalition, $a_i^{1\kappa}$ replica members of i for each $i \in J^1$ and $a_i^{2\kappa}$ replica members of i for each $i \in J^2$, the utility of each member increases strictly, we can easily check that this proof can also follow in the case of the weak core.

[7] For example, consider any element $z \in \Gamma$ such that $\beta_i = 0$ for each i. For the element $z \in \Gamma$, in the non-negative direction of every coordinate, there exist $\bar{x}_i + z_i + e^k$ that is preferred to $\bar{x}_i + z_i$ by some agent, and there also exist $z + e^k \in \Gamma$ from the SNS condition. Note that e^k is a unit vector $e^k = (0, \cdots, 0, 1, 0, \cdots, 0)$ of R^K where the k-th coordinate is 1. Hence, from the convexity of Γ, Γ has interior points. For the separating hyperplane theorem, see, for example, Schaefer (1971; p.46, Theorem 3.1).

$+\delta > 0$ is the k-th coordinate of a commodity, will not be non-negatively supported by p. This is a contradiction to the definition of Γ. Secondly, if $p \cdot \bar{x}_i > 0$, we have $p_k = 0$ and there exist a commodity $k' \neq k$ such that $p_{k'} > 0$ and $\bar{x}_{ik'} > 0$. Then, a vector $\bar{x}_i + (0, \cdots, 0, +\epsilon, 0, \cdots, 0, -\eta, 0, \cdots, 0)$ such that $\bar{x}_i + (0, \cdots, 0, +\epsilon, 0, \cdots, 0, -\eta, 0, \cdots, 0)$ is strictly preferred to $\bar{x}_i$,[8] where $+\epsilon > 0$ is the k-th coordinate of a commodity and $-\eta < 0$ is the k'-th coordinate of a commodity, will not be non-negatively supported by p. This is a contradiction to the definition of Γ. Hence, $p \in R^K_{++}$ holds for each $p \in \pi$. Let us choose one of such p arbitrarily and denote it by p^*.

For each $i \in I'$, since $x_i \succ_i \bar{x}_i$ means that $x_i - \omega_i$ and $x_i - \bar{x}_i$ belong to Γ_i, we have $p^* \cdot x_i \geqq p^* \cdot \omega_i$ and $p^* \cdot x_i \geqq p^* \cdot \bar{x}_i$. Moreover, for each $i \in I'$, since p^* is non-negative and the local non-satiation holds on this point $\bar{x}_i$, we can take x_i arbitrarily near to $\bar{x}_i$. Then we have $p^* \cdot \bar{x}_i \geqq p^* \cdot \omega_i$.

Define $d^*_i \geqq 0$ as $d^*_i = p^* \cdot \bar{x}_i - p^* \cdot \omega_i$ for all $i \in I'$. Then, we have $p^* \cdot \bar{x}_i = p^* \cdot \omega_i + d^*_i$. In addition, since $\omega_i \in R^K_{++}$ for all $i \in I$ and p^* is strictly positive, $p^* \cdot \omega_i > 0$. Since $x_i \succ_i \bar{x}_i$ means that $p^* \cdot x_i \geqq p^* \cdot \bar{x}_i$, the continuity of preference together with $p^* \cdot \omega_i + d^*_i > 0$ implies that for every $i \in I'$, $\bar{x}_i$ is an individual maxima under price p^* and dividend d^*_i.

For agent $i \in I \setminus I'$, note that $\bar{x}_i$ is a satiation point, and we can define d^*_i as follows. Let I'' be the set of all agents belonging to $I \setminus I'$ such that $p^* \cdot \bar{x}_i - p^* \cdot \omega_i > 0$. Then, we can define $d^*_i \geqq 0$ as $d^*_i = p^* \cdot \bar{x}_i - p^* \cdot \omega_i$ for each $i \in I''$. For the other agents, i.e., for each $i \in I \setminus (I' \cup I'')$ with $p^* \cdot \bar{x}_i - p^* \cdot \omega_i \leqq 0$, define d^*_i as $d^*_i = 0$. From these definitions, $\bar{x}_i$ satisfies the budget constraint and is an individual maxima under price p^* and dividen d^*_i. Hence, allocation $\bar{x}$ is an element of $\boldsymbol{\mathcal{SW}alras}(\mathcal{E})$.

[Sufficiency] Let $x^* = (x^*_i)_{i \in I}$ be an element of $\boldsymbol{\mathcal{SW}alras}(\mathcal{E})$ under price p^* and non-negative dividends d^*. Assume that $S = S_1 \cup S_2$ is a finite coalition of $\mathcal{E}^m(\omega) \oplus \mathcal{E}^n(x)$ for some $m \geqq 1$ and $n \geqq 0$ blocking the $(m+n)$-fold replica allocation of x^*, where S_1 is a coalition in $\mathcal{E}^m(\omega)$ and S_2 is a coalition in $\mathcal{E}^n(x^*)$. Then, by definition, there is an allocation $(x_i)_{i \in S}$ such that $\sum_{i \in S} x_i = \sum_{i \in S_1} \omega_i + \sum_{i \in S_2} x^*_i$, $x_i \succsim_i x^*_i$ for all $i \in S$ and $x_j \succ_j x^*_j$ for some $j \in S$. Note that the equilibrium price p^* is strictly positive under the SNS condition of preferences.[9] Hence, the equilibrium condition means that $x_j \succ_j x^*_j$ implies $p^* \cdot x_j > p^* \cdot x^*_j$. In addition, for each agent $i \in S$ with $x_i \succsim_i x^*_i$, if $p^* \cdot x_i < p^* \cdot x^*_i$, we have $x_i \neq x^*_i$. Then, all the points belong to the segment $[x_i, x^*_i]$ other than x_i and x^*_i are strictly preferred to x^*_i from the strict convexity of the preference and satisfies the budget constraint. This contradicts to the assumption that x^* is a dividend equilibrium allocation. Hence, $p^* \cdot x_i \geqq p^* \cdot x^*_i$ for all $i \in S$ and $p^* \cdot x_j > p^* \cdot x^*_j$ at least for $j \in S$. It follows that $p^* \cdot (\sum_{i \in S_1} x_i + \sum_{i \in S_2} x_i) > p^* \cdot \sum_{i \in S_1} x^*_i + p^* \cdot \sum_{i \in S_2} x^*_i \geqq p^* \cdot (\sum_{i \in S_1} x^*_i + \sum_{i \in S_2} \omega_i)$, a contradicts to $\sum_{i \in S} x_i = \sum_{i \in S_1} x^*_i + \sum_{i \in S_2} \omega_i$.

[The Case with Weak Core] For cases with the weak core, the necessity part of the proof is completely the same as we note in footnote 6. For the sufficiency part, we can show the same kind of contradiction without using the strict convexity of preferences, since the condition, $p^* \cdot x_j \succ_j p^* \cdot x^*_j$, for agent j holds for all agents belonging to S. ■

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[8] From the SNS condition, we have $(\bar{x}_i + (0, \cdots, 0, +\epsilon, 0, \cdots, 0)) \succ_i \bar{x}_i$ for each $+\epsilon > 0$. Then, from the continuity of preferences, we also have $\bar{x}_i + (0, \cdots, 0, +\epsilon, 0, \cdots, 0, -\eta, 0, \cdots, 0) \succ_i \bar{x}_i$ for sufficiently small $\eta > 0$.

[9] See the 6th paragraph of the necessity part.

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