Discussion Papers In Economics And Business

Local Independence, Monotonicity, Incentive Compatibility and Axiomatic Characterization of Price-Money Message Mechanism

Ken Urai
Hiromi Murakami

Discussion Paper 17-08

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Local Independence, Monotonicity, Incentive Compatibility and Axiomatic Characterization of Price-Money Message Mechanism

Ken Urai
Hiromi Murakami

Discussion Paper 17-08

April 2017

Graduate School of Economics and
Osaka School of International Public Policy (OSIPP)
Osaka University, Toyonaka, Osaka 560-0043, JAPAN

Local Independence, Monotonicity, Incentive Compatibility and Axiomatic Characterization of Price-Money Message Mechanism*

Ken Urai†
Hiromi Murakami‡

Abstract

To characterize money in a static economic model, it is known to be important to consider the agent-commodity double-infinity settings, i.e., the overlapping-generations framework. There does not seem to exist abundant literature, however, treating the axiomatic characterization problems for such monetary Walras allocations under the social choice and/or mechanism design settings. We show that the monetary Walras allocation for the economy with double infinities is characterized by *weak Pareto-optimality*, *individual rationality* and *local independence* or the *monotonicity*, or the *incentive compatibility* conditions of *social choice correspondence* among the *allocation mechanisms with messages* under the *category theoretic* approach in Sonnenschein (1974). We utilize Sonnenschein's market extension axiom for *swamped* economies that is closely related to the *replica stability* axiom of Thomson (1988). We can see how these conditions characterize the price-money message mechanism *universally* among a wide class of mechanisms, and *efficiently* in the sense that it has the minimal message spaces (*price-money dictionary theorems*). Moreover, by using the category theoretic framework, we can obtain the up-to-isomorphism uniqueness for such a dictionary object (*isomorphism theorems*).

Keywords : Resource Allocation Mechanism, Social Choice Correspondence, Overlapping-Generations Economy, Monetary Walras Allocation, Local Independence, Monotonicity, Incentive Compatibility, Universal Mapping Property

JEL Classification: C60, D50, D51, D71, E00

---

*We wish to express special gratitude to professor Ryoichi Nagahisa (Kansai University) for his important comments and advice. We are also grateful for comments by professors Shigehiro Serizawa (Osaka University), Tomoyuki Kamo (Kyoto Sangyo University) and Kazuhiko Hashimoto (Osaka University of Economics), and to participants in the Joint Seminar of Mathematical Economics at Osaka University (August 7, 2015), Poster Session at Sophia University (Octover 10, 2015, Autumn Meeting of Japanese Economic Association) and the 21st Decentralization Conference at Keio University (Octover 12, 2015). We would also like to thank Professor Ivan Brenes (Osaka University) for his kind help. A part of this research was supported by the Grant-in-Aid for Scientific Research from the Japan Society for the Promotion of Science in 2013 (No.25380227) and in 2015 (No.15J01034).

†Graduate School of Economics, Osaka University, E-mail: urai@econ.osaka-u.ac.jp
‡Faculty of Commerce, Doshisha University, E-mail: murakami@math.econ.osaka-u.ac.jp

1 Introduction

To introduce *money* in a static general equilibrium model, the overlapping-generations model with the double infinity of commodities and agents is known to be the most fundamental framework. The model was firstly introduced by Samuelson in 1958 (Samuelson 1958). It generated various discussions because of its outstanding feature that *competitive equilibria may not necessarily be Pareto-optimal*. Although Samuelson characterized the role of fiat money as a certain kind of *social contract* that leads to Pareto improvement, his argument was not accurate enough. In the 1970s and the 1980s, many papers on monetary general equilibria make certain that the existence of fiat money may not necessarily cause Pareto improvements nor necessarily assure the existence of monetary equilibria that may or may not be Pareto-optimal (see, for example, Shell 1971, Hayashi 1976 and Okuno and Zilcha 1980). Moreover, a monetary equilibrium is unstable from the cooperative game theoretic viewpoint (Esteban 1986) as well as from the viewpoint of equilibrium dynamics (Gale 1973). The only affirmative result that we have for a characterization of equilibrium in overlapping-generations economies with money is the relation between *weakly Pareto-optimality* and valuation equilibrium (see Balasko and Shell 1980 and Esteban 1986).

Recently, the authors presented a replica finite core equivalence and characterization for the monetary Walrasian correspondence under the overlapping-generations framework and provided an axiomatic characterization for it.[1] Except for our works, however, there do not seem to exist papers treating such axiomatic characterization problems for the double-infinity monetary equilibrium allocations under the social choice and mechanism design settings. In this paper, we treat this problem through the setting of the *allocation mechanism with messages* like Sonnenschein (1974), Hurwicz (1960), Mount and Reiter (1974), Osana (1978) and Jordan (1982). In order to facilitate a typical infinite dimensional treatment for message for message spaces on this problem, we especially follow the strongly structured response-function approach in Sonnenschein (1974) and Sonnenschein's market extension axiom for *swamped* economies which is closely related to the *replica stability* axiom of Thomson (1988) and Nagahisa (1994) for the social choice framework. We show that the monetary equilibrium allocation and the price-money message mechanism are possibly characterized axiomatically by using well-known social choice theoretic normative criteria, especially the local independency, the monotonicity and the incentive compatibility, through the category theoretic approach of Sonnenschein (1974).

By treating the number of agents as a variable and allowing for an arbitrarily general class of message spaces including the class of finite dimensional topological manifolds as in Jordan (1982), Sonnenschein's category theoretic framework together with his axiom S enables us to obtain a stronger type of an axiomatic characterization, which we call the universal implementability, than the simple uniqueness and/or efficiency theorems provided in the traditional informational efficiency arguments like Hurwicz (1960), Mount and Reiter (1974), Osana (1978) and Jordan (1982).

Both Sonnenschein (1974) and the traditional informational efficiency arguments treat an allocation mechanism with messages, although Sonnenschein describes an allocation as a response to (a function of) a message for each economy instead of describing a general relation of the message process among these variables, an allocation, an economy, and a message. Under Axiom S on the concept of an equilibrium relative to an increase in the number of economic agents, Sonnenschein (1974) shows that among an arbitrarily general class of messages, every message together with a response to it (a process using the message) can uniquely be realized through a certain price message together with the optimal demand

[1] See Urai and Murakami (2016a) and Urai and Murakami (2016b).

response to the price (the demand process of agents using the price). Let us call this property as the dictionary property of the price mechanism. Then, the price mechanism will be characterized as the unique mechanism having such a dictionary property. The uniqueness and the minimality (efficiency) results on the message space of prices are also given simultaneously as the unique up to isomorphism feature of the dictionary object obtained through an immediate mathematical property on the response commutative diagrams among processes.

Mathematically, the above characterization is known as a universal mapping property and the unique object defined under the isomorphism feature is called the solution of the universal mapping problem. An object defined like a solution of a universal mapping problem is quite important since such an object is completely characterized by relations (commutativity of several mappings) and can be grasped through a finite simple algebra among such relations (mappings) without referring anything to the concrete specific object itself. To obtain and/or define a concept in a simple, general and mathematically accurate way as much as possible, the setting like the universal mapping problem under the category theoretic framework is known to be one of the most appropriate and desirable manners from the viewpoint of the scientific universal language.

If we identify each economic agent's consumption set with its strategy set and a message space for an economy with a strategy set of an additional imaginary agent, the above message process (by considering the response functions as constraint correspondences) defines a generalized game for the economy in exactly the same way as defining the abstract economy for the competitive market. It would be natural, therefore, to call the above message process in the Sonnenschein's response function form as a mechanism (in a generalized sense) or a message mechanism.

Theorem 1 shows in the domain of double infinity exchange economies, sufficient and/or necessary relations between the social choice correspondence satisfying the property of *weak Pareto-optimality* (WPO), *individual rationality* (IR), the *local independence* (LI) and the correspondence that allocate for each economy its monetary Walrasian equilibria, the *monetary Walrasian correspondence*. Based on this knowledge, Theorem 2 asserts that every *allocation mechanism with messages* whose equilibrium results are compatible with the three conditions, WPO, IR and LI, can *universally* and *uniquely* be identified with a part of monetary Walrasian (price-money) message mechanism. We can also obtain Theorem 3 that assures the *uniqueness* of such a message space as a solution to the *universal mapping problem*. Theorems 4 and 5 (6 and 7, resp.) show that the price-money dictionary theorem (Theorem 2) and the isomorphism theorem (Theorem 3) can also be obtained through the *monotonicity* (*incentive compatibility*, resp.) instead of the *local independency*.

2 The Model

As in our previous paper (Urai and Murakami 2016b), we define the general overlapping-generations settings under the duality between $\boldsymbol{R}_\infty$ and $\boldsymbol{R}^\infty$. Since we are concerned with one shot (perfect foresight) equilibrium states, this kind of overlapping-generations model (one good for each period and $(\ell(t)+1)$-periods lifetime-span for each generation t) is sufficiently general to include all types with ℓ-goods and n-periods lifetime for each generation t.[2]

[2] With respect to the topological structure for this kind of double infinity economies, we also use the simplest case of the duality between the direct limit of finite dimensional commodity spaces and the inverse limit of finite dimensional price spaces (see Aliprantis et al. 1989, Urai 1990 and Urai 1994), which is equivalent to treating the duality between $\boldsymbol{R}_\infty (\subset \boldsymbol{R}^\infty)$ and $\boldsymbol{R}^\infty$ under the ordinary product topology.

We denote by $\boldsymbol{N}$ the set of all positive integers and by $\boldsymbol{R}$ the set of real numbers. An *overlapping-generations economy*, or an *economy*, $\mathcal{E}$, is a list of:

(OG1) $\{I_t\}_{t=1}^{\infty}$; a countable family of mutually disjoint finite subsets of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} I_t = \boldsymbol{N}$, where $I_t \neq \emptyset$ for each $t \in \boldsymbol{N}$. I_t is the index set of agents in *generation* t.

(OG2) $\{K_t\}_{t=1}^{\infty}$; a countable family of non-empty finite intervals, $K_t = \{k(t), k(t)+1, \cdots, k(t)+\ell(t)\}$ where $k(t)$ and $\ell(t)$ are elements of $\boldsymbol{N}$ such that $\bigcup_{t=1}^{\infty} K_t = \boldsymbol{N}$ and $k(t) \leqq k(t+1) \leqq k(t) + \ell(t)$ for all $t \in \boldsymbol{N}$, and $\{t \mid n \in K_t\}$ is finite for each $n \in \boldsymbol{N}$. K_t is the index set of *commodities* available to generation t.

(OG3) $\{(\succsim_i,\ \omega_i)\}_{i \in \bigcup_{t\in \boldsymbol{N}} I_t}$; countably many agents, where $\succsim_i$ is a rational weak preference on the commodity space, $\boldsymbol{R}^{K_t}$, of $i \in I_t$ for $t \in \boldsymbol{N}$. Every preference, $\succsim_i$, can be represented by a continuous utility function, $u_i : \boldsymbol{R}^{K_t} \to \boldsymbol{R}$, that is strictly quasi-concave and strictly monotonic. The *initial endowment* of i, ω_i, is an element of $\boldsymbol{R}_{++}^{K_t} = \{x \mid x : K_t \to \boldsymbol{R}_{++}\}$ for each $i \in I_t$.

The commodity space for each generation, $\boldsymbol{R}_{+}^{K_t}$, can be recognized as the subset of $\boldsymbol{R}^{\boldsymbol{N}}$, the set of all functions from $\boldsymbol{N}$ to $\boldsymbol{R}$, by identifying $x \in \boldsymbol{R}_{+}^{K_t}$ with the function that takes value 0 on $\boldsymbol{N}\backslash K_t$. The total commodity space for an economy is, therefore, the set of all finite sums in $\boldsymbol{R}^{\boldsymbol{N}}$ among points in commodity spaces of some generations, $\oplus_{t=1}^{\infty}\boldsymbol{R}_{+}^{K_t} \subset \boldsymbol{R}^{\boldsymbol{N}}$. Clearly, $\oplus_{t=1}^{\infty}\boldsymbol{R}_{+}^{K_t}$ can also be identified with a subset of the direct sum, $\boldsymbol{R}_{\infty}$, the set of all finite real sequences, which is a subspace of the set of all real sequences, $\boldsymbol{R}^{\infty} \approx \boldsymbol{R}^{\boldsymbol{N}}$.

Given an economy, $\mathcal{E} = (\{I_t\}_{t=1}^{\infty},\ \{K_t\}_{t=1}^{\infty},\ \{(\succsim_i,\ \omega_i)\}_{i\in\bigcup_{t\in\boldsymbol{N}} I_t})$, the *price space* for $\mathcal{E}$, $\mathcal{P}(\mathcal{E})$, is defined as the set of all p in $\boldsymbol{R}_{+}^{\boldsymbol{N}}$ such that under the duality between $\boldsymbol{R}_{\infty}$ and $\boldsymbol{R}^{\infty}$, p evaluates all agents' initial endowments positively, i.e.,

(1) $$\mathcal{P}(\mathcal{E}) = \{p \in \boldsymbol{R}_{+}^{\boldsymbol{N}} \mid p \cdot \omega_i > 0 \text{ for all } i \in I_t, \text{ for all } t \in \boldsymbol{N}\}.$$

Since for all $i \in I_t$, $\omega_i \in \boldsymbol{R}_{++}^{K_t}$, for all $t \in \boldsymbol{N}$, the price space of $\mathcal{E}$ always includes $\boldsymbol{R}_{++}^{\boldsymbol{N}}$ for all $\mathcal{E}$ in $\mathcal{E}con$, the set of all economies satisfying conditions (OG1), (OG2) and (OG3).

For each $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega_i)\}) \in \mathcal{E}con$, sequence $(x_i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ is called an *allocation* for $\mathcal{E}$. An allocation $x = (x_i)_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ is said to be *feasible* if

(2) $$\sum_{t\in\boldsymbol{N}}\sum_{i\in I_t} x_i = \sum_{t\in\boldsymbol{N}}\sum_{i\in I_t} \omega_i,$$

where the summability in $\boldsymbol{R}^{\boldsymbol{N}}$ of both sides of the equation is assured by (OG2).[3] The list of a price vector $p^* \in \mathcal{P}(\mathcal{E})$, a *non-negative wealth transfer function* $M_{\mathcal{E}}^* : \boldsymbol{N} = \bigcup_{t=1}^{\infty} I_t \to \boldsymbol{R}_{+}$, and a feasible allocation $(x_i^* \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$ is called a *monetary Walras allocation* for $\mathcal{E}$, if for each $t \in \boldsymbol{N}$ and $i \in I_t$, x_i^* is a $\succsim_i$-greatest element in the set $\{x_i \in \boldsymbol{R}^{K_t} \mid p^* \cdot x_i \leqq p^* \cdot \omega_i + M_{\mathcal{E}}^*(i)\}$ (see, e.g., Balasko and Shell 1981 and Esteban and Millán 1990). We denote the set of all monetary Walras allocations by $\mathcal{MWalras}(\mathcal{E})$.

An allocation, $x = (x_i)_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$, for economy $\mathcal{E} = (\{I_t\},\ \{K_t\},\ \{(\succsim_i,\ \omega_i)\}) \in \mathcal{E}con$ is said to be *weakly Pareto-optimal* (WPO), if there is no allocation $y = (y_i)$ with the property $\sum_{t\in\boldsymbol{N}}\sum_{i\in I_t} y_i = \sum_{t\in\boldsymbol{N}}\sum_{i\in I_t} x_i$, $y_i = x_i$ except for a finite number of i, and $y_i \succsim_i x_i$ with at least one strict preference

[3] In the following, we treat a sequence, $(x_i \in \boldsymbol{R}^{K_t})_{i\in\bigcup_{t\in\boldsymbol{N}} I_t}$, as a vector and denote it by a single letter, e.g., (x_i) by x, (y_i) by y and so forth.

$\succ_i$ for $i \in \bigcup_{t \in \boldsymbol{N}} I_t$.[4] Moreover, we say that allocation x is *individually rational* (IR) if $x_i \succsim_i \omega_i$ for all $i \in \bigcup_{t \in \boldsymbol{N}} I_t$.

Social choice correspondence g on domain $\mathcal{E}con$ is a correspondence that assigns some allocations for each $\mathcal{E}$, $g : \mathcal{E}con \ni \mathcal{E} \mapsto g(\mathcal{E}) \subset \boldsymbol{R}^{\boldsymbol{N}}$. Let $K(s) = \bigcup_{t=1}^{s} K_t$ and $I(s) = \bigcup_{t=1}^{s} I_t$ for each $s \in \boldsymbol{N}$. We say that social choice correspondence g satisfies the condition of *local independence* (LI) if $x \in g(\mathcal{E})$ implies $x \in g(\mathcal{E}')$ whenever $\mathcal{E}$ is $(\{I_t\}, \{K_t\}, \{(\succsim_i, \omega_i)\})$, $\mathcal{E}'$ is $(\{I_t\}, \{K_t\}, \{(\succsim'_i, \omega_i)\})$, $\succsim'_i = \succsim_i$ except for finite agents, and if there exists $s \in \boldsymbol{N}$ such that $I(s)$ includes all such finite agents and a unique supporting hyperplane $H_s \subset \boldsymbol{R}^{K(s)}$ at $x \in g(\mathcal{E})$ of every better set $\{y_i | y_i \succ_i x_i\}$ of $i \in I(s)$ also supports every $\{y_i | y_i \succ'_i x_i\}$ of $i \in I(s)$.[5] Social choice correspondence g is said to satisfy the condition of *monotonicity* if $x \in g(\mathcal{E})$ implies $x \in g(\mathcal{E}')$ whenever $\mathcal{E}$ is $(\{I_t\}, \{K_t\}, \{(\succsim_i, \omega_i)\})$, $\mathcal{E}'$ is $(\{I_t\}, \{K_t\}, \{(\succsim'_i, \omega_i)\})$, $\succsim'_i = \succsim_i$ except for finite agents, and there exists $s \in \boldsymbol{N}$ such that $I(s)$ includes all such finite agents and every better set $\{y_i | y_i \succ_i x_i\}$ at x of $i \in I(s)$ includes $\{y_i | y_i \succ'_i x_i\}$ of each $i \in I(s)$.[6] It is possible to consider a weak version of monotonicity by replacing the condition for finite agents with for a singele agent. We also define an *incentive compatibility* condition for social choice correspondence g as follows: if $\mathcal{E}$ is $(\{I_t\}, \{K_t\}, \{(\succsim_i, \omega_i)\})$, $\mathcal{E}'$ is $(\{I_t\}, \{K_t\}, \{(\succsim'_i, \omega_i)\})$, $\succsim'_i = \succsim_i$ except for a single agent, i^*, $x = (x_i) \in g(\mathcal{E})$ and $x' = (x'_i) \in g(\mathcal{E}')$, then we have $x_{i^*} \succsim_{i^*} x'_{i^*}$.

3 A Preliminary Theorem

Social choice correspondence $g : \mathcal{E}con \to \boldsymbol{R}^{\boldsymbol{N}}$ is said to be *monetary Walrasian* if $g(\mathcal{E}) = \mathcal{MW}alras(\mathcal{E})$. We have the next theorem that characterizes the monetary Walrasian social choice correspondence through the LI condition.

Theorem 1 (LI Characterization Theorem) : (i) Assume that social choice correspondence g is WPO and IR. If g satisfies LI and all better sets of $i \in I(s)$ at $x \in g(\mathcal{E})$ are supported by a unique price p in $\boldsymbol{R}^{K(s)}$, then for all $i \in I(s)$, their initial endowments are evaluated less than or equal to the value of x_i under p. In particular, if g satisfies LI and all better sets of agents at $x \in g(\mathcal{E})$ is supported by a unique price, x is a monetary Walras allocation (g is monetary Walrasian). (ii) On the other hand, monetary Walrasian social choice correspondence g is WPO and IR valued, and if the non-negative wealth transfer $M_{\mathcal{E}}$ does not depend on the preferences of agents, it satisfies the LI condition.[7]

Proof : Assume that an allocation $x \in g(\mathcal{E})$ is WPO and IR, and social choice correspondence g satisfies the LI. By the weak Pareto-optimality, we have a price, $p \in \boldsymbol{R}^{\infty}$, such that for each $i \in \bigcup_{t=1}^{\infty} I_t$, $x'_i \succ_i x_i$ implies that $p \cdot x'_i > p \cdot x_i$ (Balasko and Shell 1980). If $p \cdot x_i \geqq p \cdot \omega_i$ for all i, by defining $M_{\mathcal{E}}(i)$ as $p \cdot x_i - p \cdot \omega_i \in \boldsymbol{R}_+$, we can identify x as a monetary Walras allocation. Hence, the latter part of the assertion (i) follows from the former. Suppose that for some $i \in I(s)$, $p \cdot x_i < p \cdot \omega_i$ and all better sets of $i \in I(s)$ at x are supported by a unique price $p \in \boldsymbol{R}^{K(s)}$. Then, it is possible to change the preference of i to $\succ'_i$ so that $\succ'_i$ satisfies $\omega_i \succ'_i x_i$ and all assumptions in (OG3), and p remains to be a supporting

[4] See Balasko and Shell (1980).

[5] The definition was originally given and used in Nagahisa (1991) for finite and differentiable economies.

[6] Note that the above monotonicity is weaker than the local independence condition as long as we ignore the condition of the existence of a unique supporting hyperplane at allocation x. The local independence condition also enables us to obtain a simple characterization of monetary Walrasian allocation in the next section (Theorem 1).

[7] To obtain an analogous result based on the monotonicity and/or incentive compatibility, we need Axiom S in the next section.

hyperplane of the better set of i under $\succ'_i$ at x. Under the LI, x should be in the value of the social choice for such an economy, which is impossible, however, since $\omega_i \succ'_i x_i$ contradicts the IR.

On the other hand, monetary Walras allocation is obviously IR and is well known to be WPO (see Balasko and Shell 1980 and Esteban 1986). Moreover, for each allocation $x \in g(\boldsymbol{\mathcal{E}})$ having a unique supporting hyperplane $H_s \in \boldsymbol{R}^{K(s)}$ for better sets of agents in $I(s)$ for some $s \in \boldsymbol{N}$, we can confirm that the monetary Walrasian social choice correspondence g satisfies the LI. Indeed, as long as each non-negative wealth transfer does not depend on preferences, any preference changes from economy $\boldsymbol{\mathcal{E}}$ to $\boldsymbol{\mathcal{E}}'$ to check the LI condition do not affect the property of allocation x to be agents' individual price-wealth maximands. ■

Note that in the definition of LI, the uniqueness property of the supporting hyperplane for better sets of agents at allocation x of g is important. The LI condition does not say anything for allocations that do not have this uniqueness property. For differentiable class of economies (treated in section 4 Theorem 3), the uniqueness property is satisfied at every allocations.

4 Axiomatic Characterization of the Price-Money Message Mechanism

Three conditions in Theorem 1, the weak Pareto-optimality (WPO), the individual rationality (IR), and the local independence (LI), enable us to provide an axiomatic characterization of the monetary Walrasian (price-money) message mechanism through the category theoretic framework as in Sonnenschein (1974). We formulate, especially, the local independence condition and, interchangeably, the monotonicity condition as axioms for such allocation mechanisms.

At first, we reformulate the concepts in Sonnenschein (1974) into the social choice settings. A *WPO-IR compatible social choice correspondence* associates with each economy $\boldsymbol{\mathcal{E}}$ a set of allocations which are WPO and IR allocations for $\boldsymbol{\mathcal{E}}$. An *allocation mechanism with messages* in our model or an *(abstract) message mechanism* based on such a social choice correspondence, g, is a triple, (A, μ, f): the set A is a *message domain*, μ is a correspondence which indicates for each economy $\boldsymbol{\mathcal{E}}$ the set $\mu(\boldsymbol{\mathcal{E}}) \subset A$ of *equilibrium messages for* $\boldsymbol{\mathcal{E}}$, and f is a function which defines for each agent, i, and each message, a, the *response*, $f^i(\boldsymbol{\mathcal{E}}, a)$, of the agent to the message, satisfying that $g(\boldsymbol{\mathcal{E}}) = \{(f^i(\boldsymbol{\mathcal{E}}, a))_{i=1}^{\infty} \mid a \in \mu(\boldsymbol{\mathcal{E}})\}$.[8] The *list of equilibrium responses* associated with $a \in \mu(\boldsymbol{\mathcal{E}})$ assigns to each agent in $\boldsymbol{\mathcal{E}}$ his response to the message a.

The monetary Walrasian social choice correspondence associates with each economy the monetary Walras allocations of the economy. The standard message mechanism is such that $A = \boldsymbol{R}_+^{\boldsymbol{N}} \times \{M | \, M : \boldsymbol{\mathcal{E}con} \to \boldsymbol{R}_+^{\boldsymbol{N}}\}$, $\mu(\boldsymbol{\mathcal{E}})$ is the set of equilibrium prices with non-negative wealth transfer of $\boldsymbol{\mathcal{E}}$, and f gives the excess demand function of each consumer relative to price-money messages. Let us consider the following axioms.

Axiom S (Sonnenschein): For each finite list of the economies and the members, $(i_1, \boldsymbol{\mathcal{E}}_1)$, $(i_2, \boldsymbol{\mathcal{E}}_2)$, $\ldots$, $(i_m, \boldsymbol{\mathcal{E}}_m)$, each message $a \in A$ and each list of responses $(f_{i_s}(\boldsymbol{\mathcal{E}}_s, a))_{s=1}^m$, there exists an economy $\boldsymbol{\mathcal{E}}_*$ including $\{i_1, i_2, \ldots, i_m\}$ such that a is an equilibrium message for $\boldsymbol{\mathcal{E}}_*$ satisfying that the equilibrium list, $(f_i(\boldsymbol{\mathcal{E}}_*, a))_{i=1}^{\infty}$, is an extension of $(f_{i_s}(\boldsymbol{\mathcal{E}}_s, a))_{s=1}^m$.

[8] For this concept, Sonnenschein (1974) uses the word *private representation*. The word "private," however, is not appropriate for our setting, since the responses of agents to the messages are partly dependent on the economy.

The above condition is closely related to the replica stability axiom of Thomson (1988) (see Urai and Murakami 2016b). Note that since the non-negative wealth transfer may be different among agents having the same individual characteristics, it would be desirable to treat general messages that are *partly* economy-dependent.[9] Hence the finite agents in the previous axioms should be listed with the economies to which they belong.

The following axioms redefine the local independency and the monotonicity conditions in the previous section through the terms in the allocation mechanism with messages.

Axiom LI (Local Independency) : For each economy $\mathcal{E}$ and message a, if there exist generation $s \in \boldsymbol{N}$ and a unique hyperplane $H_s \subset \boldsymbol{R}^{K(s)}$ that supports the better set at $f_i(\mathcal{E}, a)$ of every $i \in I(s)$, then for each economy $\mathcal{E}'$ having the same indices of agents and commodities, endowments and possibly different preferences of agents in $I(s)$ of economy $\mathcal{E}$, such that H_s is also a supporting hyperplane of the better set at $f_i(\mathcal{E}, a)$ of every $i \in I(s)$, we have $f(\mathcal{E}, a) = f(\mathcal{E}', a)$.[10]

Axiom (Monotonicity) : For each economy $\mathcal{E}$ and message a, if for some $s \in \boldsymbol{N}$, $\mathcal{E}'$ is an economy having the same indices and endowments of agents in $\mathcal{E}$ together with the same preferences except for agents in $I(s)$ such that every better set at $f_i(\mathcal{E}, a)$ in $\mathcal{E}$ includes the better set at the same point of i in $\mathcal{E}'$ for each $i \in I(s)$, then we have $f(\mathcal{E}, a) = f(\mathcal{E}', a)$.

For our result, the following single-agent formed weak version of the monotonicity condition is sufficient.

Axiom M (Single Agent Monotonicity) : For each economy $\mathcal{E}$ and message a, if $\mathcal{E}'$ is an economy having the same indices and endowments of agents in $\mathcal{E}$ together with the same preferences except for an agent, i, such that the better set of i at $f_i(\mathcal{E}, a)$ in $\mathcal{E}$ includes the better set of i at the same point in $\mathcal{E}'$, then we have $f_i(\mathcal{E}, a) = f_i(\mathcal{E}', a)$.

An economic interpretation of Axiom LI is that for a fixed message a, agents' responses do not change as long as their local evaluations on the relative importance of each commodity do not change. In the same way, the monotonicity axioms are interpreted that for message a, agents do not change their responses as long as they think that the importance of their choices do not decrease.[11]

In relation to the above monotonicity conditions, we also use the next *incentive compatibility* or *no incentive to misrepresent* condition to obtain an axiomatic characterization of the price mechanism.

Axiom IC (Incentive Compatibility) : For each economy $\mathcal{E}$ and message a, if $\mathcal{E}'$ is an economy having the same indices and endowments of agents in $\mathcal{E}$ together with the same preferences except for an agent, i, then we have $f_i(\mathcal{E}, a) \succsim_i f_i(\mathcal{E}', a)$, where $\succsim_i$ denotes the preference relation of i in $\mathcal{E}$.

[9] In the sense that the message is related not only to each agent's characteristics, i.e., the initial endowment and the preference, but also to their places in the economy to which they belong.

[10] It is possible to weaken Axiom LI by restricting the condition to the messages satisfying $a \in \mu(\mathcal{E})$ as long as we use the axiom with Axiom S. From the viewpoint of independency among axioms, it is desirable to define Axioms LI and M as conditions not on equilibria but merely on responses.

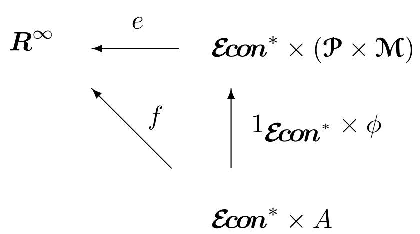
[11] The importance of their choices do not decrease in the sense that the better sets at their choice points do not increase. In view of an axiom for the message mechanism, it can be said that we request messages to be self-convincing for agents having such preference changes.

From the viewpoint of an axiom for a message mechanism, the condition says that messages are such that for agents who mistake their own preferences, their responses fails to be better ones.[12]

Consumer i is a pair $(\precsim_i, \omega_i)$, where $\precsim_i$ and ω_i satisfy the conditions in (OG3). We assume in the following the commodity structure, $\{K_t\}_{t=1}^{\infty}$, is fixed, and identify the set of all economies, $\mathcal{E}con^*$, with the set of those in $\mathcal{E}con$ with the commodity structure $\{K_t\}_{t=1}^{\infty}$. Denote by $I(t)$ the set of all agents in generations from 1 to t, i.e., $I(t) = \bigcup_{s=1}^{t} I_s$, and by $K(t)$ the set of all commodities that are available for agents in $I(t)$, i.e., $K(t) = \bigcup_{s=1}^{t} K_s$. For each t, by $\Delta^{K(t)}$, we denote the unit simplex in $\boldsymbol{R}^{K(t)}$ and by $\Delta_{++}^{K(t)}$ its relative interior, $\boldsymbol{R}_{++}^{K(t)} \cap \Delta^{K(t)}$. Let us consider projective system $(\Delta_{++}^{K(t')}, \varrho_{t't})_{t',t\in\boldsymbol{N}}$ and projective limit $\Delta_{++} = \varprojlim (\Delta_{++}^{K(t')}, \varrho_{t't})$, where $\varrho_{t't} : \Delta_{++}^{K(t)} \to \Delta_{++}^{K(t')}$ is defined as $\varrho_{t't}(p) = \frac{\mathrm{pr}_{K(t')}\, p}{\|\, \mathrm{pr}_{K(t')}\, p\|}$. Note that Δ_{++} can be recognized as a subset of $\boldsymbol{R}_{++}^{\infty}$ by identifying the equivalence class $[(x^t)_{t=1}^{\infty}]$ of $(x^t)_{t=1}^{\infty} \in \prod_{t=1}^{\infty} \Delta_{++}^{K(t)}$ with the element $p \in \boldsymbol{R}_{++}^{\infty}$ such that $\mathrm{pr}_{K(1)}\, p = x^1$ and $\frac{\mathrm{pr}_{K(t)}\, p}{\|\, \mathrm{pr}_{K(t)}\, p\|} = x^t$ for all $t = 2, 3, \ldots$. We take the price and non-negative wealth transfer domain as $\mathcal{P} \times \mathcal{M} = \{p \in \boldsymbol{R}^{\infty} \,|\, \exists [(x^t)_{t=1}^{\infty}] \in \Delta_{++}, \mathrm{pr}_{K(1)}\, p = x^1, \frac{\mathrm{pr}_{K(t)}\, p}{\|\, \mathrm{pr}_{K(t)}\, p\|} = x^t, \text{ for each } t = 1, 2, \ldots\} \times \{M \,|\, M : \mathcal{E}con^* \ni \mathcal{E} \mapsto M_{\mathcal{E}} \in R_{+}^{\boldsymbol{N}}\}$. The excess demand function of the i-th consumer, $(\precsim_i, \omega_i)$, in $\mathcal{E} \in \mathcal{E}con^*$ is defined as $e_i : \mathcal{P} \times \mathcal{M} \ni (p, M) \mapsto e_i(p, M_{\mathcal{E}}) \in \boldsymbol{R}_{\infty}$, where $e_i(p, M_{\mathcal{E}})$ is the $\precsim_i$-greatest point in $\{x_i \in \boldsymbol{R}^{K_t} \,|\, p \cdot x_i \leqq p \cdot \omega_i + M_{\mathcal{E}}(i)\}$, for each $i \in I_t$ and $t \in \boldsymbol{N}$.

Define $e : \mathcal{E}con^* \times (\mathcal{P} \times \mathcal{M}) \to \boldsymbol{R}^{\infty}$ by $e(\mathcal{E}, \precsim_i, \omega_i, p, M) = (e_i(p, M_{\mathcal{E}})_{i\in I_t})_{t\in\boldsymbol{N}}$. If for each $\mathcal{E} \in \mathcal{E}con^*$, $\pi(\mathcal{E})$ denotes the set of all price-money equilibrium messages, then $(\mathcal{P} \times \mathcal{M}, \pi, e)$ is a message mechanism based on WPO-IR compatible social choice correspondence $\mathcal{MW}alras(\mathcal{E})$. It is called the *price-money message mechanism.* Note that $(\mathcal{P} \times \mathcal{M}, \pi, e)$ does not satisfy Axioms S and L.[13]

Theorem 2 (Price-Money Dictionary Theorem under Axiom LI): If (A, μ, f) is a message mechanism based on a WPO-IR compatible social choice correspondence, and if (A, μ, f) satisfies Axioms S and LI, then (i) there exists a unique function $\phi : A \to \mathcal{P} \times \mathcal{M}$, such that the following triangle commutes, and (ii) on $\phi(A) \subset \mathcal{P} \times \mathcal{M}$, the price-money message mechanism satisfies Axioms S and LI.



Proof: (i) Assume that (A, μ, f) is a message mechanism based on a social choice correspondence satisfying Axioms S and LI, and let a be an element of A. Define for each $t \in \boldsymbol{N}$, $h^{(t)}(x, \precsim_i)$ for each consumption $x \in \boldsymbol{R}^{K_s}$ for agent $i \in I_s \subset I(t)$ of an economy $\mathcal{E} \in \mathcal{E}con^*$ as $h^{(t)}(x, \precsim_i) = \{p \in \Delta^{K(t)} \,|\, y \succ_i x \text{ implies } p \cdot y \geqq p \cdot x\}$, where every $\boldsymbol{R}^{K_s}$ is canonically identified with a subspace of $K(t)$. We first show that $\bigcap h^{(t)}(f_i(\mathcal{E}, a), \precsim_i)$ is non-empty for each $t \in \boldsymbol{N}$, where the intersection is over all consumers and economies in $\mathcal{E}con^*$, and $f_i(\mathcal{E}, a)$ is a response of $i \in I(t)$ in $\mathcal{E}$ to message a in (A, μ, f). Because $\Delta^{K(t)}$ is

[12] In social choice theory, it is known that if we restrict the domain of preferences to the class of linear orderings, the incentive compatibility necessarily means the monotonicity (see Mas-Colell et al. 1995, p.811, Proof of Theorem 21.E.2). For a general equilibrium setting, we cannot restrict preferences to be an element of such a narrow domain. For a message mechanism argument, however, since we can fix a certain message a, Axiom S enable us to use Axiom IC in the same way with Axiom M (see the proof of Theorem 6, below).

[13] For Axiom LI, see (2) of Theorem 1. For Axiom S, see footnote 7 of Urai and Murakami (2016b).

compact, and because each of the sets in the collection from which we are forming the intersection is closed, it is sufficient to show that $\bigcap_{s=1}^{m} h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty for any $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$. Given the list $[(i_1, \mathcal{E}_1), (i_2, \mathcal{E}_2), \ldots, (i_m, \mathcal{E}_m)]$ of agents in $I(t)$ and economies, by Axiom S there exists $\mathcal{E}_* \in \mathcal{E}con^*$ containing $\{i_1, i_2, \ldots, i_m\}$ and $a \in \mu(\mathcal{E}_*)$, such that the equilibrium list, $(f_i(\mathcal{E}_*, a))_{i=1}^{\infty}$, is an extension of $(f_{i_s}(\mathcal{E}_s, a))_{s=1}^{m}$. Because $(f_i(\mathcal{E}_*, a))_{i=1}^{\infty}$ is an element of $g(\mathcal{E})$, the allocation is weakly Pareto-optimal, so by Balasko and Shell (1980) and Esteban (1986), it is supported by a price as a price-wealth equilibrium, and thus $\bigcap_{s=1}^{m} h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s})$ is non-empty. Moreover, because for some economy and its agents in generations in 1 to t, their intersection of $h^{(t)}(f_i(\mathcal{E}, a), \succsim_i)$ is a singleton and its unique element belongs to $\Delta_{++}^{K(t)}$, it follows that $\bigcap h^{(t)}(f_i(\mathcal{E}, a), \succsim_i)$ is composed of a single point $p(t) \in \Delta_{++}^{K(t)}$.

By definition of $h^{(t)}$, we have $p(t') = \varrho_{t't}(p(t))$ for all $t' \leqq t$, and obtain a unique element $p \in \mathcal{P}$ by identifying it with the unique element of the projective limit $\varprojlim \bigcap_{s=1}^{m} h^{(t)}(f_{i_s}(\mathcal{E}_s, a), \succsim_{i_s}) \subset \Delta_{++}$. Let us denote that point, p, by $\phi^1(a)$, and define $\phi^2(a) = M$, $M : \mathcal{E}con^* \ni \mathcal{E} \mapsto M_{\mathcal{E}} \in \boldsymbol{R}^{\boldsymbol{N}}$, as $M_{\mathcal{E}}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}, a) - \omega_i)$, which will be proved as non-negative in the following by Axioms S, LI and Theorem 1. Let $\phi(a)$ be $(\phi^1(a), \phi^2(a)) \in \mathcal{P} \times \mathcal{M}$. In order to establish the theorem, it is sufficient to show that for each economy $\mathcal{E}_* \in \mathcal{E}con^*$ and $a \in A$, an allocation $y^* = (y_i^*) = (f_i(\mathcal{E}_*, a))$ is such that for each i, $y_i^* = f_i(\mathcal{E}_*, a)$ satisfies $M_{\mathcal{E}_*}(i) = \phi^1(a) \cdot (f_i(\mathcal{E}_*, a) - \omega_i) \geqq 0$. Fix a member i of $\mathcal{E}_*$. By using Axiom S, let $\mathcal{E}_{**}$ be an economy including i such that a is an equilibrium message for $\mathcal{E}_{**}$ and the response $y_i^{**} = f_i(\mathcal{E}_{**}, a)$ is equal to $y_i^* = f_i(\mathcal{E}_*, a)$. Without loss of generality we can assume that $\mathcal{E}_{**}$ has, at least to the generation s of member i, one consumer in each generation whose supporting hyperplane for any better set at their individually rational point is unique.[14] We show that response $y^{**} = f(\mathcal{E}_{**}, a)$ is a monetary Walras allocation. This allocation is IR and WPO. Moreover the social choice rule defined by a and its responses for all economies in $\mathcal{E}con^*$ having the same indices of agents in $\mathcal{E}$ satisfies the LI condition by Axiom LI (see footnote 10). So by (i) of Theorem 1, the allocation belongs to $\mathcal{MW}alras(\mathcal{E}_{**})$.

(ii) One can observe in the above argument, $y^{**} = f(\mathcal{E}_{**}, a) = e(\mathcal{E}_{**}, \phi(a))$ is a monetary Walras allocation, which proves that Axiom S is satisfied on $\phi(A)$. Moreover, it is straightforward that the commutativity of the diagram with Axiom LI for (A, μ, f) means that Axiom LI is satisfied on $\phi(A)$. ■

By Theorem 2, from every message mechanism (A, μ, f) based on a WPO-IR compatible social choice correspondence satisfying Axiom S and LI , there exists a unique *price dictionary function*, $\phi : A \to \mathcal{P} \times \mathcal{M}$. In other words, the results of such message mechanisms can be realized *universally* and *efficiently* through the price-money message mechanism $(\mathcal{P} \times \mathcal{M}, \pi, e)$. Thus we have obtained the price-money dictionary theorem as in our previous paper (Theorem 2 of Urai and Murakami 2016b: based on a WPO and finite core compatible social choice correspondence).

We can also obtain an isomorphism theorem for the price-money message mechanism (Theorem 3 of Urai and Murakami 2016b) as follows. Denote by $\mathcal{PM}_{LI}^*$ the set of all $(p, M) \in \mathcal{P} \times \mathcal{M}$ which is an image of ϕ for some (A, μ, f) in Theorem 2 satisfying Axioms S and LI. The following axiom on the dependence of monetary messages on the economic structure is necessary to show the second assertion.

Axiom D (Dependency of Money Supply on $\mathcal{E}$): If $\mathcal{E} = (\{I_t\}_{t=1}^{\infty}, \{K_t\}_{t=1}^{\infty}, \{(\succsim_i, \omega_i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ and $\mathcal{E}' = (\{I'_t\}_{t=1}^{\infty}, \{K'_t\}_{t=1}^{\infty}, \{(\succsim'_i, \omega'_i)_{i \in \bigcup_{t \in \boldsymbol{N}} I_t}\})$ are such that $\{I_t\}_{t=1}^{\infty} = \{I'_t\}_{t=1}^{\infty}$, $\{K_t\}_{t=1}^{\infty} = \{K'_t\}_{t=1}^{\infty}$ and $\omega_i = \omega'_i$ for all $i \in \bigcup_{t \in \boldsymbol{N}} I_t$, then $M_{\mathcal{E}} = M_{\mathcal{E}'}$ for all $M \in \mathcal{M}$.

[14] It is always possible to add finite agents in constructing economy $\mathcal{E}_*$ in Axiom S.

Theorem 3 (Isomorphism Theorem under Axiom LI): Consider the restriction of price-money message mechanism $(\mathcal{PM}^*_{LI}, \pi, e)$. Let (P', π', e') be a message mechanism based on a WPO-IR compatible social choice correspondence on $\mathcal{E}con^*$. If (P', π', e') satisfies Axioms S and LI, and if, for every message mechanism (A, μ, f) satisfying Axioms S and LI, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}^*_{LI} \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. (ii) Moreover, if monetary messages satisfy Axiom D, we can restrict the problem on spaces with topological (resp. differentiable on each inverse-system component) structures and continuous mappings (resp. differentiable coordinate mappings) to obtain continuous (resp. differentiable on each component space) unique ϕ in the previous theorem, so that isomorphism h' can be taken as a homeomorphism (resp. diffeomorphism for each component space).[15]

Proof: Because $(\mathcal{PM}^*_{LI}, \pi, e)$ is now assumed to be a message mechanism based on a WPO-IR compatible social choice correspondence satisfying Axioms S and LI, we have the next diagram by assumption.

$$\begin{array}{ccc} (\boldsymbol{R}_\infty)^{\boldsymbol{N}} & \xleftarrow{\;e'\;} & \mathcal{E}con^* \times P' \\ & \nwarrow^{e} & \uparrow 1_{\mathcal{E}con^*} \times \phi' \\ & & \mathcal{E}con^* \times \mathcal{PM}^*_{LI} \end{array}$$

Moreover, because (P', π', e') is also a message mechanism based on a social choice correspondence satisfying Axiom S and LI, the previous theorem shows that we have the next diagram.

$$\begin{array}{ccc} (\boldsymbol{R}_\infty)^{\boldsymbol{N}} & \xleftarrow{\;e\;} & \mathcal{E}con^* \times \mathcal{PM}^*_{LI} \\ & \nwarrow^{e'} & \uparrow 1_{\mathcal{E}con^*} \times \phi \\ & & \mathcal{E}con^* \times P' \end{array}$$

Since the identity mapping is the unique mapping for P' to P' satisfying $e' = e' \circ id$ and $\mathcal{PM}^*_{LI}$ to $\mathcal{PM}^*_{LI}$ satisfying $e = e \circ id$, we have $\phi' \circ \phi = id$ and $\phi \circ \phi' = id$, which means that ϕ and ϕ' are bijectives. Let us define h' as $h' = \phi'$, then we have the first assertion.

For the second assertion, by considering on $\mathcal{PM}^*_{LI}$, the relativised topology of the product topology of $R^\infty_+ \times \prod_{\mathcal{E} \in \mathcal{E}con}(R^{\sharp\mathcal{E}})$, where $\sharp\mathcal{E}$ denotes the cardinal number of the set of agents in $\mathcal{E}$, we can easily check the continuity (resp. differentiability on each component space) of the mapping ϕ in Theorem 2 through the local homeomorphism (resp. diffeomorphism) property at each point of the domain of the

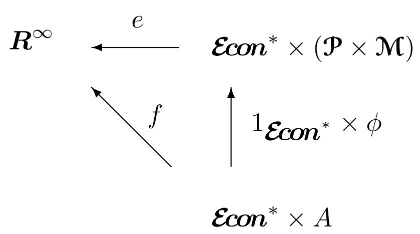
[15] In this paper, the price-money message space has been treated as an inverse limit of finite dimensional domains of coordinate functions of e. The differentiability for e and a differentiable structure on its domain, however, is appropriate to be treated on each of its coordinate functions, e_i, whose domain is always possible to be identified with a finite dimensional subspace of $\mathcal{P} \times \mathcal{M} \supset \mathcal{PM}^*_{LI}$. More precisely, under the definitions of $e = (e_i)_{i=1}^\infty$ and $e' = (e'_i)_{i=1}^\infty$ with Axiom D, the bijection h' gives an algebraic isomorphism between the domain of e_i and e'_i for each i, to which the diffeomorphism argument can be applied. We can construct (as a subspace of $\mathcal{PM}^*_{LI}$ under the identification of $\boldsymbol{R}_\infty \subset \boldsymbol{R}^\infty$) a direct limit of the finite dimensional projection, P_t, of the domain of $(e_i)_{i \in I(t)}$ for each $t \in \boldsymbol{N}$, so that the bijection h' gives an algebraic isomorphism between the domains P_t of $(e_i)_{i \in I(t)}$ and P'_t of $(e'_i)_{i \in I(t)}$ for each t. In this sense, each restriction of h' gives a diffeomorphism between the direct systems, $(P_t)_{t=1}^\infty$ and $(P'_t)_{t=1}^\infty$.

excess demand function that is appropriately chosen for each $\boldsymbol{\mathcal{E}}$ by using Axiom D. Then the proof of the first assertion is sufficient for assuring the isomorphism poroperty of h'. ■

Hence, Theorem 3 asserts that if we restrict the domain of the price-money messages to where Axioms S and LI are satisfied, the price-wealth formed monetary message mechanism is essentially the only object having the above universality and efficiency as a solution to the universal mapping problem.

The above price-money dictionary theorem (Theorem 2) and the isomorphism theorem (Theorem 3) can also be obtained through the monotonicity axiom (Axiom M) instead of the local independency axiom (Axiom LI).

Theorem 4 (Price-Money Dictionary Theorem under Axiom M): If (A, μ, f) is a message mechanism based on a WPO-IR compatible social choice correspondence, and if (A, μ, f) satisfies Axioms S and M, then (i) there exists a unique function $\phi : A \to \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$, such that the following triangle commutes, and (ii) on $\phi(A) \subset \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$, the price-money message mechanism satisfies Axioms S and M.



Proof: (i) We can repeat the argument in the first paragraph in the proof of Theorem 2 and obtain the single point $p(t)$ in $\bigcap h^{(t)}(f_i(\boldsymbol{\mathcal{E}}, a), \succsim_i) \subset \Delta_{++}^{K(t)}$, where the intersection is over all consumers in generations 1 to t and economies in $\boldsymbol{\mathcal{E}con}^*$. Also by definition of $h^{(t)}$, for all $t' \leqq t$, we have $p(t') = \varrho_{t't}(p(t))$, and obtain a unique element $p \in \boldsymbol{\mathcal{P}}$ by identifying it with the unique element of the projective limit $\varprojlim \bigcap_{s=1}^{m} h^{(t)}(f_{i_s}(\boldsymbol{\mathcal{E}}_s, a), \succsim_{i_s}) \subset \Delta_{++}$. Let us define $\phi^1(a)$ as this unique element, p, and define $\phi^2(a) = M$, $M : \boldsymbol{\mathcal{E}con}^* \ni \boldsymbol{\mathcal{E}} \mapsto M_{\boldsymbol{\mathcal{E}}} \in \boldsymbol{R}^{\boldsymbol{N}}$, as $M_{\boldsymbol{\mathcal{E}}}(i) = \phi^1(a) \cdot (f_i(\boldsymbol{\mathcal{E}}, a) - \omega_i)$, which will be proved as non-negative in the following by Axioms S and M.

Let $\phi(a)$ be $(\phi^1(a), \phi^2(a)) \in \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{M}}$. To establish the theorem, it is sufficient to show that for each economy $\boldsymbol{\mathcal{E}}_* \in \boldsymbol{\mathcal{E}con}^*$ and $a \in A$, an allocation $y^* = (y_i^*) = (f_i(\boldsymbol{\mathcal{E}}_*, a))$ is such that for each i, $y_i^* = f_i(\boldsymbol{\mathcal{E}}_*, a)$ satisfies $M_{\boldsymbol{\mathcal{E}}_*}(i) = \phi^1(a) \cdot (f_i(\boldsymbol{\mathcal{E}}_*, a) - \omega_i) \geqq 0$.

Assume the contrary, that is, that there is a member i of $\boldsymbol{\mathcal{E}}_*$ such that at $y_i^* = f_i(\boldsymbol{\mathcal{E}}_*, a)$ we have $\phi^1(a) \cdot (f_i(\boldsymbol{\mathcal{E}}_*, a) - \omega_i) < 0$. By using Axiom S, let $\boldsymbol{\mathcal{E}}_{**}$ be an economy including i such that a is an equilibrium message for $\boldsymbol{\mathcal{E}}_{**}$ and the response $y_i^{**} = f_i(\boldsymbol{\mathcal{E}}_{**}, a)$ is equal to $y_i^* = f_i(\boldsymbol{\mathcal{E}}_*, a)$, and there is at least one agent $j \neq i$ such that i and j are in the same generation s and the supporting hyperplane at $f_j(\boldsymbol{\mathcal{E}}_{**}, a)$ for the better set of j is unique (that is necessarily equal to $\phi^1(a)$).

Fix the indifference surface of i in $\boldsymbol{\mathcal{E}}_{**}$ at $y_i^* = y_i^{**} = f_i(\boldsymbol{\mathcal{E}}_{**}, a)$ and change the preference $\succsim_i$ of i to $\succsim_i'$ that is obtained through the homothetic transformation of the surface at y_i^* under $\succsim_i$ (see Figure 1).[16]

[16] Let $U(y_i^*)$ be the indifferent surface at y_i^* of i under $\succsim_i$. By IR, $y_i^{**} \neq 0$. By the strict monotonicity, $\alpha U(y_i^*)$, $\alpha > 0$ covers the $\boldsymbol{R}_+^{K_s}$. Moreover for each $x \in \boldsymbol{R}_+^{K_s}$, there exist $\alpha(x)$ such that $x \in \alpha(x) U(y_*^i)$ and such $\alpha(x)$ is unique under the strong monotonicity. Thus by defining $u(x)$ as $u(x) = \alpha(x)$, we have the preference, $\succsim_i'$, satisfying all the conditions in (OG3).

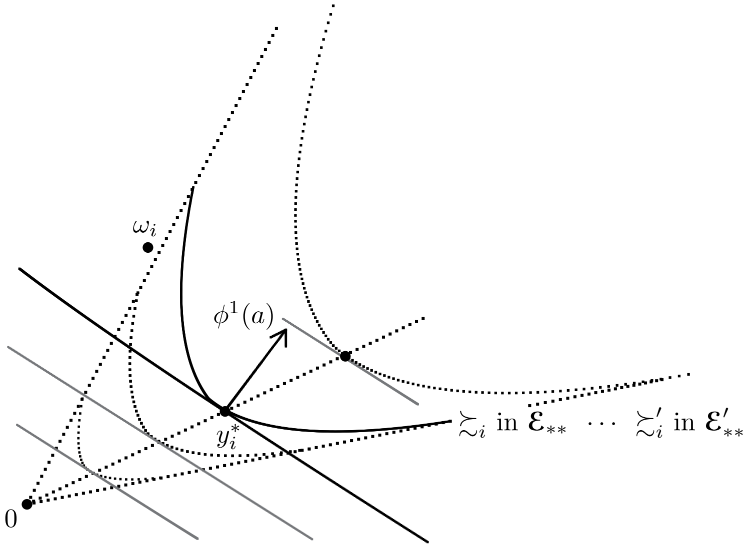


Figure 1: Homothetic Transformation 1

Consider the economy $\boldsymbol{\mathcal{E}}'_{**}$ such that the preference of i in $\boldsymbol{\mathcal{E}}_{**}$ is replaced with $\succsim'_i$. By Axiom M, $f_i(\boldsymbol{\mathcal{E}}'_{**}, a) = y_i^*$. Change the preference of i to $\hat{\succsim}'_i$ satisfying condition (OG3) such that $\hat{\succsim}'_i$ satisfies $\omega_i \hat{\succsim}'_i y_i^*$ and $y_i^* \hat{\not\succsim}'_i \omega_i$, the better set of i at y_i^* under $\succsim'_i$ is included by that of i at y_i^* under $\hat{\succsim}'_i$, and $\phi^1(a)$ remains to be a supporting hyperplane of the better set (see Figure 2).

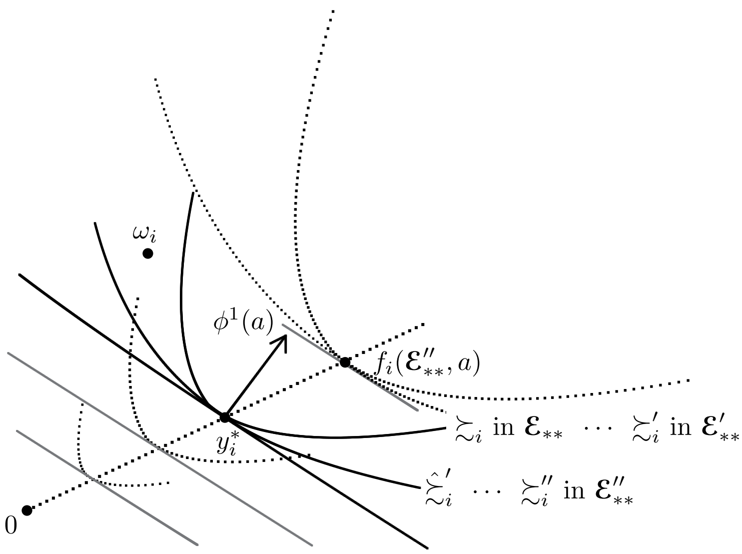


Figure 2: Homothetic Transformation 2

Moreover, consider the homothetic transformation of the preference, $\hat{\succsim}'_i$, of i by using the indifference surface at y_i^*, and call it $\succsim''_i$. Let us denote by $\boldsymbol{\mathcal{E}}''_{**}$ the economy where we replace the preference $\succsim'_i$ of i with $\succsim''_i$.

By Axiom S, we have an economy $\boldsymbol{\mathcal{E}}_{***}$ including i in $\boldsymbol{\mathcal{E}}''_{**}$ and j in $\boldsymbol{\mathcal{E}}_{**}$, and a is an equilibrium message for $\boldsymbol{\mathcal{E}}_{***}$. Since $\phi^1(a)$ must support the better sets of i and j at $f_i(\boldsymbol{\mathcal{E}}''_{**}, a)$ and $f_j(\boldsymbol{\mathcal{E}}_{**}, a)$, respectively, by the WPO property of $f(\boldsymbol{\mathcal{E}}_{***}, a)$, it follows that $f_i(\boldsymbol{\mathcal{E}}''_{**}, a)$ must be different from y_i^* by the IR property of $f(\mathcal{E}_{***}, a)$ under $\succsim''_i$. The point, $f_i(\mathcal{E}''_{**}, a)$, should, however, be equal to some point z_i at which the indifference surface under $\succsim''_i$ of the point is supported by $\phi^1(a)$. That is, $f_i(\mathcal{E}''_{**}, a) = z_i \neq y^*_i = f_i(\mathcal{E}'_{**}, a)$. This is a contradiction since $f_i(\mathcal{E}'_{**}, a)$ is equal to $f_i(\mathcal{E}''_{**}, a)$ under Axiom M.

(ii) Repeat the arguments in the proof (ii) of Theorem 2 (replace Axiom LI with Axiom M). ■

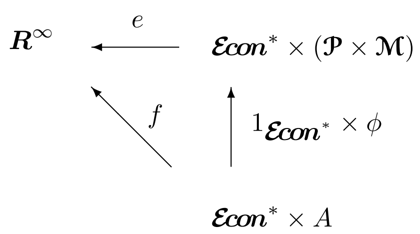
Denote by $\mathcal{PM}^*_M$ the set of all $(p, M) \in \mathcal{P} \times \mathcal{M}$ which is an image of ϕ for some (A, μ, f) in Theorem 4 satisfying Axioms S and M.

Theorem 5 (Isomorphism Theorem under Axiom M): Consider the restriction of price-money message mechanism $(\mathcal{PM}^*_M, \pi, e)$. Let (P', π', e') be a message mechanism based on a WPO-IR compatible social choice correspondence on $\mathcal{E}con^*$. If (P', π', e') satisfies Axioms S and M, and if, for every message mechanism (A, μ, f) satisfying Axioms S and M, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}^*_M \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. (ii) Moreover, if monetary messages satisfy Axiom D, we can restrict the problem on spaces with topological (resp. differentiable on each inverse-system component) structures and continuous mappings (resp. differentiable coordinate mappings) to obtain continuous (resp. differentiable on each component space) unique ϕ in the previous theorem, so that isomorphism h' can be taken as a homeomorphism (resp. diffeomorphism for each component space).

Proof: Repeat the proof of Theorem 3 (replace Axiom LI and $\mathcal{PM}^*_{LI}$ respectively with Axiom M and $\mathcal{PM}^*_M$). ■

We can also obtain the dictionary and the isomorphism theorems under Axiom IC instead of M. See the following Theorems 6 and 7.

Theorem 6 (Price-Money Dictionary Theorem under Axiom IC): If (A, μ, f) is a message mechanism based on a WPO-IR compatible social choice correspondence, and if (A, μ, f) satisfies Axioms S and IC, then (i) there exists a unique function $\phi : A \to \mathcal{P} \times \mathcal{M}$, such that the following triangle commutes, and (ii) on $\phi(A) \subset \mathcal{P} \times \mathcal{M}$, the price-money message mechanism satisfies Axioms S and IC.



Proof: (i) Follow the arguments in the first four paragraphs of the proof of Theorem 4. By appropriately changing the reference to Axiom M with Axiom IC, obtain the homothetically transformed preference, $\succsim'_i$, of the indifferent surface at y^*_i of $\succsim_i$.

Consider the economy $\mathcal{E}'_{**}$ such that the preference of i in $\mathcal{E}_{**}$ is replaced with $\succsim'_i$ (see Figure XX). Let y^{**}_i be $f_i(\mathcal{E}'_{**}, a)$. Consider an economy $\mathcal{E}_{***}$ including $i \in \mathcal{E}'_{**}$ and $i \in \mathcal{E}_{**}$ as different agents such that message a is an equilibrium message for $\mathcal{E}_{***}$ under responses y^{**}_i and y^*_i through Axiom S. By the WPO property of equilibrium allocations, y^{**}_i cannot belong to the indifference surface of y^*_i in $\mathcal{E}_{**}$ that is necessarily equal to the indifference surface of y_i^* in $\mathcal{E}'_{**}$ except for the case that $y_i^* = y_i^{**}$. Indeed, if $y_i^* \neq y_i^{**}$, allocation among these two agents can be improved by replacing y_i^* and y_i^{**} with $\frac{y_i^*+y_i^{**}}{2}$ in $\mathcal{E}_{***}$ without changing any other agents' allocations. Hence, if $y_i^* \neq y_i^{**}$, there are only two cases: (i-i) $y_i^{**} = f_i(\mathcal{E}'_{**}, a)$ is preferred to y_i^* for i in both $\mathcal{E}_{**}$ and $\mathcal{E}'_{**}$, and (i-ii) y_i^* is preferred to $y_i^{**} = f_i(\mathcal{E}'_{**}, a)$ for i in both $\mathcal{E}_{**}$ and $\mathcal{E}'_{**}$. In Case (i-i), for i in $\mathcal{E}_{**}$, there is an incentive to misrepresent his/her preference as the one in $\mathcal{E}'_{**}$, and in Case (i-ii), for i in $\mathcal{E}'_{**}$, there is an incentive to misrepresent his/her preference as the one in $\mathcal{E}_{**}$. It follows that we must have $y_i^{**} = f_i(\mathcal{E}'_{**}, a) = y_i^*$ under Axiom IC.

Now, change the preference of i to $\hat{\precsim}'_i$ satisfying (OG3) such that $\hat{\precsim}'_i$ satisfies $\omega_i \hat{\precsim}'_i y_i^*$ and $y_i^* \hat{\not\precsim}'_i \omega_i$, the better set of i at y_i^* under $\precsim'_i$ is included by that of i at y_i^* under $\hat{\precsim}'_i$, and $\phi^1(a)$ remains to be a supporting hyperplane of the better set (see Fig. xxx). Moreover, consider the homothetic transformation of the preference, $\hat{\precsim}'_i$, of i by using the indifference surface at y_i^*, and call it $\precsim''_i$. Let us denote by $\mathcal{E}''_{**}$ the economy where we replace the preference $\precsim'_i$ of i with $\precsim''_i$.

By Axiom S, we have an economy $\mathcal{E}_{***}$ including i in $\mathcal{E}''_{**}$ and j in $\mathcal{E}_{**}$, and a is an equilibrium message for $\mathcal{E}_{***}$. Point $f_i(\mathcal{E}''_{**}, a)$ must be different from and strictly preferred to y_i^* under $\precsim''_i$ by the IR property of $f(\mathcal{E}_{***}, a)$. That is, $f_i(\mathcal{E}''_{**}, a) \succ''_i f_i(\mathcal{E}'_{**}, a) = y_i^*$, which also means that $f_i(\mathcal{E}''_{**}, a) \succ'_i y_i^*$. Indeed, $f_i(\mathcal{E}''_{**}, a)$ that is known to be different from $y_i^* = y_i^{**} = f_i(\mathcal{E}'_{**}, a)$, must be a part of weakly Pareto optimal allocation in $\mathcal{E}_{***}$ and have $\phi^1(a)$ as a supporting price for the $\succ''_i$-better set. Since $\succ''_i$ is homothetic, point $f_i(\mathcal{E}''_{**}, a)$ must belong to the closed half line from 0 containing y_i^* satisfying the condition "$\precsim''_i$-preferred to y_i^*." By the monotonicity of preferences, the subset of closed half line from 0 containing y_i^* satisfying the condition "$\precsim''_i$-preferred to y_i^*" is equal to the subset of closed half line from 0 containing y_i^* satisfying the condition "$\precsim'_i$-preferred to y_i^*." It follows that $f_i(\mathcal{E}''_{**}, a) \succ'_i y_i^*$. This, however, implies that there is an incentive for agent i in $\mathcal{E}'_{**}$ to misrepresent his/her preference as the one in $\mathcal{E}''_{**}$, a contradiction under Axiom IC.

(ii) Repeat the arguments in the proof (ii) of Theorem 2 (replace Axiom LI with Axiom IC). ■

Denote by $\mathcal{PM}^*_{IC}$ the set of all $(p, M) \in \mathcal{P} \times \mathcal{M}$ which is an image of ϕ for some (A, μ, f) in Theorem 6 satisfying Axioms S and IC.

Theorem 7 (Isomorphism Theorem under Axiom IC): Consider the restriction of price-money message mechanism $(\mathcal{PM}^*_{IC}, \pi, e)$. Let (P', π', e') be a message mechanism based on a WPO-IR compatible social choice correspondence on $\mathcal{E}con^*$. If (P', π', e') satisfies Axioms S and IC, and if, for every message mechanism (A, μ, f) satisfying Axioms S and IC, there exists a unique mapping $\phi' : A \to P'$ such that $f(\mathcal{E}, a) = e' \circ [1_{\mathcal{E}con^*} \times \phi'](\mathcal{E}, a)$, then (i) there exists an isomorphism (bijection) h' such that $h' : \mathcal{PM}^*_M \to P'$ and $e = e' \circ [1_{\mathcal{E}con^*} \times h']$. (ii) Moreover, if monetary messages satisfy Axiom D, we can restrict the problem on spaces with topological (resp. differentiable on each inverse-system component) structures and continuous mappings (resp. differentiable coordinate mappings) to obtain continuous (resp. differentiable on each component space) unique ϕ in the previous theorem, so that isomorphism h' can be taken as a homeomorphism (resp. diffeomorphism for each component space).

Proof: Repeat the proof of Theorem 3 (replace Axiom LI and $\mathcal{PM}^*_{LI}$ respectively with Axiom IC and $\mathcal{PM}^*_{IC}$). ■

5 Conclusion

We have thus obtained three kinds of *price-money dictionary* theorems (Theorem 2, 4 and 6) and the *isomorphism* theorems (Theorem 3, 5 and 7). It can be said that the *price-money dictionary theorem*, which asserts that the price-money message mechanism can be referenced *uniquely* and *universally* among the category satisfying the axioms in question (a property *for itself*), together with a restriction of price-money messages to the place where all such axioms are satisfied (a property *on itself*), enables us to show the *isomorphism theorem* which says that the price-money message mechanism can be characterized as an *essentially* unique mechanism (up to isomorphism between the message spaces) in the category of allocation mechanisms with messages satisfying those axioms.

REFERENCES

Aliprantis, C. D., Brown, D. J., and Burkinshaw, O. (1989): *Existence and Optimality of Competitive Equilibria.* Springer-Verlag, New York/Berlin.

Balasko, Y. and Shell, K. (1980): "The overlapping-generations model. I. The case of pure exchange without money," *Journal of Economic Theory 23*, 281–306.

Balasko, Y. and Shell, K. (1981): "The overlapping-generations model. II. The case of pure exchange with money," *Journal of Economic Theory 24*, 112–142.

Esteban, J. (1986): "A characterization of core in overlapping-generations economies," *Journal of Economic Theory 39*, 439–456.

Esteban, J. and Millán, T. (1990): "Competitive equilibria and the core of overlapping generations economies," *Journal of Economic Theory 50*, 155–174.

Gale, D. (1973): "Pure exchange equilibrium of dynamic economic models," *Journal of Economic Theory 6*, 12–36.

Hayashi, T. (1976): "Monetary equilibrium in two classes of stationary economies," *Review of Economic Studies 43*, 269–284.

Hurwicz, L. (1960): "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences 1959*, (Arrow, K. J., Karlin, S., and Suppes, P. ed) , pp. 27–46, Stanford University Press, Stanford. Also in *Readings in Welfare Economics*, edited by K.J.Arrow and T.Scitovsky. Irwin, New York, 1969.

Jordan, J. S. (1982): "The Competitive Allocation Process is Informationally Efficient Uniquely," *Journal of Economic Theory 28*, 1–18.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995): *Microeconomic Theory.* Oxford University Press, New York.

Mount, K. and Reiter, S. (1974): "The Informational Size of Message Spaces," *Journal of Economic Theory 8*, 161–192.

Nagahisa, R. (1991): "A Local Independence Condition for Characterization of Wairasian Allocations Rule," *Journal of Economic Theory 54*, 106–123.

Nagahisa, R. (1994): "A Necessary and Sufficient Condition for Walrasian Social Choice," *Journal of Economic Theory 62*, 186–208.

Okuno, M. and Zilcha, I. (1980): "On the efficiency of a competitive equilibrium in infinite horizon monetary economies," *Review of Economic Studies 47*, 797–807.

Osana, H. (1978): "On the Informational Size of Message Spaces for Resource Allocation Processes," *Journal of Economic Theory 17*, 66–78.

Samuelson, P. A. (1958): "An exact consumption loans model of interest with or without social contrivance of money," *Journal of Political Economy 66*(6), 467–482.

Shell, K. (1971): "Notes on the economies of infinity," *Journal of Political Economy 79*, 1002–1011.

Sonnenschein, H. (1974): "An axiomatic characterization of the price mechanism," *Econometrica 42*(3), 425–433.

Thomson, W. (1988): "A Study of Choice Correspondences in Economies with a Variable Number of Agents," *Journal of Economic Theory 46*, 237–254.

Urai, K. (1990): "On the Existence of Pareto-Optima and Interest Competitive Equilibria in the Generation-Overlapping Exchange Economy," *The Economic Studies Quarterly 41*(3), 235–246.

Urai, K. (1994): "On the existence of equilibria in economies with infinitely many agents and commodities: The direct system of economies," *Journal of Mathematical Economics 23*, 339–359.

Urai, K. and Murakami, H. (2016a): "Replica Core Equivalence Theorem: An Extension of the Debreu-Scarf Limit Theorem to Double Infinity Monetary Economies," *Journal of Mathematical Economics 66*, 83–88.

Urai, K. and Murakami, H. (2016b): "An Axiomatic Characterization of the Price-Money Message Mechanism," Discussion Paper No. 15-31-Rev., Graduate School of Economics and Osaka School of International Public Policy, Osaka University.