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Abstract

This paper provides a unified viewpoint on some axioms in social choice theory and a setting for the allocation mechanism with messages in the informational efficiency problem. In particular, our arguments are concerned with the category theoretical axiomatic method in Sonnenschein (1974) and the replica stability axiom in the social choice arguments like Thomson (1988) and Nagahisa (1994). The unified view enables us to obtain an extension of Sonnenschein's axiomatic characterization of the price mechanism as an *agent-characteristics form dictionary property* to a *utility form economy-dependent universal implementability theorem*.

Keywords: Price Mechanism, Axiomatic Characterization, Category Theory, Informational Efficiency, Universal Implementability, Message Mechanism

JEL classification: C60, D50, D71

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1 Introduction

This paper provides a new perspective or a unified viewpoint on some axioms in social choice theory and a setting for the informational efficiency problem on allocation mechanisms with messages like Hurwicz (1960), Mount and Reiter (1974), Sonnenschein (1974), and so forth. Particularly, our arguments are concerned with the category theoretical axiomatic method in Sonnenschein (1974) and the replica stability axiom in the social choice arguments as in Thomson (1988) and Nagahisa (1994). Sonnenschein's axiomatic argument through his expansion possibility axiom (Axiom S in Sonnenschein 1974) and the social choice argument through the replica stability axiom (see Thomson 1988) have closely related theoretical structures as a characterization of the price mechanism. The unified view enables us to obtain an extension of Sonnenschein's axiomatic characterization of the price mechanism as an agent-characteristics form of *dictionary property* to an economy-dependent utility form of *universal implementability theorem.*

In this paper, we see that the price mechanism can be characterized as the unique mechanism that *implements* resource allocations of all other message mechanisms satisfying several important axioms (core property, replica stability of responses, Sonnenschein's expansion possibility, etc.). The replica stability social choice axiom is reformulated into an axiom not on equilibria but on responses for an allocation mechanism with messages. The Sonnenschein's axiom, on the other hand, is weakened so as to incorporate economy-dependent utility form representation.[1]

Our result as an extension of Sonnenschein's theorem (Propositions 1 and 7 in Sonnenschein 1974) characterize an interesting feature of the price mechanism. By rigorously treating the role of Sonnenschein's expansion possibility axiom while separating from it the replica stability property of responses, we can find out that the condition that uniquely and universally characterize the price mechanism is nothing but the feature that *there exists a possibility that no agents can be better off when they admit an expansion of the economy.* In other words, the price can be characterized as a message that prepares for each small economy, $\mathcal{E}$, a sufficiently good allocation at least in the following sense: under the message, we have to expect a possibility that the utility level of every member is not improved for a certain kind of large extension of $\mathcal{E}$.[2] Although it is true that the price mechanism is a universal rule for the global economy, the property ensuring its universality (universal implementability) is rather its *stability* feature as a good message for each small local economy.

2 The Basic Model

In this paper, I, I', $\cdots$ etc. represent finite index sets of *agents.* Note that such an index will be used independently with the notations for economies like $\mathcal{E}$, $\mathcal{E}'$, $\cdots$, so I is sometimes used as a set of agents in both $\mathcal{E}$ and $\mathcal{E}'$. Let I be the set of agents in economy $\mathcal{E}$. Economy $\mathcal{E}$ consists of the *feasible consumption set*, the *preference preordering* and the initial endowment for $i \in I$, denoted respectively by X_i, $\precsim_i$ and ω_i. For each $i \in I$, we define X_i as $X_i = R^\ell_+$. The preference preordering of i in economy $\mathcal{E}$, $\precsim_i$, is a subset of $X_i \times X_i$ and the initial endowment of i in $\mathcal{E}$, ω_i, is an element of R^ℓ_{++}.

[1] The setting in this paper also provides a general perspective on the relation between the arguments on the informational efficiency problem for the allocation mechanism and axiomatic characterization through fundamental social choice axioms like individual rationality, Pareto-optimality, local independency, monotonicity, incentive compatibility and so on. Based on our papers, Urai and Murakami (2016a) and Murakami and Urai (2017a), on the replica core limit theorem, we are preparing for axiomatic characterization results as Urai and Murakami (2016b) of the price-money mechanisms and Murakami and Urai (2017b) of the price-dividends mechanism. On the monotonicity and the incentive compatibility, we are also preparing a paper, Urai and Murakami (2017).

[2] For every economy, $\mathcal{E}$, an extension $\hat{\mathcal{E}}$ of $\mathcal{E}$ exists such that the message is an equilibrium for $\hat{\mathcal{E}}$ and the utility levels of all members of $\mathcal{E}$ are not improved under the equilibrium of $\hat{\mathcal{E}}$ (see Axiom (C_3') in section 3).

We can write an *economy*, $\mathcal{E}$, as $\mathcal{E} = (\precsim_i, \omega_i)_{i\in I}$. Suppose that the preference, $\precsim_i$, is represented by a utility function of each individual, $u_i : X_i \to R$, and each u_i satisfies continuity, strict monotonicity and strict quasi-concavity (strict convexity in the sense of Debreu 1959). We write the set of economies as $\mathcal{E}\!con$.

For each $\mathcal{E} = (\precsim_i, \omega_i)_{i\in I}$, sequence $(x_i \in X_i)_{i\in I}$ is called an *allocation* for $\mathcal{E}$. Allocation $(x_i \in X_i)_{i\in I}$ is said to be *feasible* if

$$\sum_{i\in I} x_i = \sum_{i\in I} \omega_i. \tag{1}$$

A *coalition* in economy $\mathcal{E} = (\succsim_i, \omega_i)_{i\in I}$ is a set of agents $S \subset I$. Feasible allocation x is said to be the *core allocation* if there are no coalition S and no $y = (y_i)_{i\in S}$, satisfying (a) $\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$ and (b) $y_i \succsim x_i$ for all $i \in S$ and $y_i \succ x_i$ for at least one $i \in S$. We call the set of all core allocations the *core* of economy $\mathcal{E}$ and denote it by $\mathcal{C}ore(\mathcal{E})$. Allocation x is said to be *blocked* by coalition S if conditions (a) and (b) hold.

Next, we define a message mechanism on an economy. Let A be a set. Given a *message*, $a \in A$, we assume that for each economy $\mathcal{E} = (\succsim_i, \omega_i)_{i\in I}$, allocation $f(a, \mathcal{E}) = (f_i(a, \mathcal{E}))_{i\in I} \in \prod_{i\in I} X_i$ is defined. We call f on $A \times \mathcal{E}\!con$ such that $(a, \mathcal{E}) \mapsto f(a, \mathcal{E}) \in \prod_{i\in I} X_i$ a *response function*. In addition, we consider an *equilibrium correspondence* $\mu : \mathcal{E}\!con \ni \mathcal{E} \mapsto \mu(\mathcal{E}) \subset A$. As in Sonnenschein (1974), given a correspondence, g, that defines for each economy $\mathcal{E} = (\succsim_i, \omega_i)_{i\in I}$ a subset of its feasible allocations, we call the triple (A, μ, f) as a *message mechanism* (an resource allocation mechanism with messages) based on a social choice correspondence, g, if

$$g(\mathcal{E}) = \{(f_i(a, \mathcal{E}))_{i\in I} \mid a \in \mu(\mathcal{E})\}. \tag{2}$$

For economies, $\mathcal{E} = (\succsim_i, \omega_i)_{i\in I}$ and $\mathcal{E}' = (\succsim'_i, \omega'_i)_{i\in I'}$, we write $\mathcal{E} \subset \mathcal{E}'$ to mean that (i) $I \subset I'$, (ii) for each $i \in I$, the preference of i, $\precsim_i$ is equal to $\precsim'_i$, and (iii) for each $i \in I$, $\omega_i = \omega'_i$.

Now we restate the category theoretic argument in Sonnenschein (1974) while reconstructing it by explicitly treating several implicit settings and assumptions as independent axioms. Let us consider the following conditions on f and μ.

Axiom (C_1) : Responses are invariant for the extension of the economy. (Messages are not economy dependent.) That is, $\forall a \in A$, $\forall \mathcal{E} \in \mathcal{E}\!con$, $\forall \mathcal{E}' \in \mathcal{E}\!con$, $\mathcal{E} \subset \mathcal{E}'$,

$$f(a, \mathcal{E}) \text{ is a restriction of } f(a, \mathcal{E}') \text{ on members of } \mathcal{E}. \tag{3}$$

Axiom (C_2) : Equilibrium responses are core compatible. That is, $\forall a \in A$, $\mathcal{E} \in \mathcal{E}\!con$,

$$a \in \mu(\mathcal{E}) \Longrightarrow f(a, \mathcal{E}) \in \mathcal{C}ore(\mathcal{E}). \tag{4}$$

Axiom (C_3) : Mechanism satisfies Sonnenschein's Axiom S. That is, for each economy $\mathcal{E}$ and each message $a \in A$, there exists an economy $\mathcal{E}' \supset \mathcal{E}$ such that a is an equilibrium message for $\mathcal{E}'$.

Define the set of price vectors, P, as $P = \{(p_1, \cdots, p_\ell) \in R^\ell_+ \mid \sum_{k=1}^{\ell} p_k = 1\}$. The *price mechanism* is an allocation mechanism with messages, (P, π, e), where for each $\mathcal{E} = (\precsim_i, \omega_i)_{i\in I} \in \mathcal{E}\!con$, $\pi(\mathcal{E}) \subset P$ denotes

$$
\begin{tikzcd}
R^{\infty} & \mathfrak{C} \times P \arrow[l, "e"'] \\
 & \mathfrak{C} \times A \arrow[ul, "f"] \arrow[u, "1_{\mathfrak{C}} \times h"']
\end{tikzcd}
$$

Figure 1: Commutative Diagram for the Universal Mapping Problem in Sonnenschein (1974)

the set of all competitive equilibrium prices for $\mathcal{E}$ and for each $p \in P$, $e(p) = (e_i(p))_{i \in I} \in \prod_{i \in I} X_i$ is the list of the value of each agent's excess demand function.[3]

The commutative diagram in Figure 1 with respect to the class of agents' characteristics, $\mathfrak{C}$, information sets, excess demand structure and any equilibrium structures satisfying Axioms (C_1), (C_2) and (C_3) was proved in Sonnenschein (1974) as the next proposition.

Proposition (Sonnenschein 1974; Propositions 1 and 7) : If (A, μ, f) is a message mechanism based on a social choice correspondence, and if (A, μ, f) satisfies Axioms (C_1), (C_2) and (C_3), then there exists a unique function $h : A \to P$ such that the triangle in Figure 1 commutes (Dictionary Property). Function h can be taken as continuous (or differentiable) when the domain of message spaces are restricted on topological spaces (resp. differentiable manifolds) with continuous (resp. differentiable) response functions. Moreover, the mechanism that can play the above dictionary property is unique up to isomorphism (Universal Mapping Property).

3 Generalized Axioms and Theorems on Universal Implementability

Now, we consider more general conditions to characterize the price mechanism under the globalization or the economic expansion framework. In the following, Axioms (C_1), (C_2) and (C_3) are reformulated by making allowance for an expansion concept of economies. The economy independent axiom of messages, (C_1), will be replaced with the following replica stability axiom for responses, (C_1').[4] Moreover, Sonnenschein's Axiom S is generalized by incorporating an economy dependent response function together with a utility form characterization as the following, (C_3').

Axiom (C_1') : $\forall \mathcal{E} \in \boldsymbol{\mathcal{E}con}$, $\forall a \in \mu(\mathcal{E})$, $\forall n$, $f(a, \mathcal{E}^n) = (f(a, \mathcal{E}))^n$.[5]

Axiom (C_3') : For each $\mathcal{E} \in \boldsymbol{\mathcal{E}con}$ and message $a \in A$, there exists economy $\mathcal{E}^* = (\precsim_i^*, \omega_i^*)_{i \in I^*}$, $\mathcal{E} \subset \mathcal{E}^*$, such that a is an equilibrium message for $\mathcal{E}^*$ and $u_i(f_i(a, \mathcal{E})) \geqq u_i(f_i(a, \mathcal{E}^*))$ for each agent i in $\mathcal{E}$.

[3] The excess demand function exists since each agent's utility function is strictly quasi-concave.

[4] See Thomson (1988), Nagahisa (1994), etc. In those frameworks the replica stability concept was applied to a social choice result or an equilibrium. Here, we use the same concept on the structure of agents' responses to the messages.

[5] For each allocation x in economy $\mathcal{E}$, we denote by x^n the n-fold replica allocation in the n-fold replica economy, $\mathcal{E}^n$, for $n = 1, 2, , \cdots$.

In the previous section, (C_1) indicates that the responses of agents do not depend on the scale of the economy. The above extended axioms, (C_1') and (C_3'), generalize the settings in Sonnenschein (1974) in the sense that responses are allowed to be economy-dependent except for the replica extension with equilibrium messages. Note that condition (C_3') is given in the utility form. The framework enables us to discuss problems such as relation between the stability of price mechanism and an expansion possibility of economies like the globalization .

To obtain an extension theorem of Sonnenschein (1974; Propositions 1 and 7), we describe the commutative diagram in Figure 1 by considering the dependence of responses on economy $\mathcal{E}$. To this purpose, we focus our attention on equilibrium messages.[6] Moreover, for the sake of simplicity, we restrict the class of all economies $\mathcal{E}con$ as the set of economies in which there is at least one agent whose utility function is differentiable.

$$\begin{tikzcd}
R^\infty & \mathcal{E}con \times P \arrow[l, "e"'] \\
 & \mathcal{E}con \times A \arrow[u, "1_{\mathcal{E}con} \times h"'] \arrow[ul, "f"']
\end{tikzcd}$$

Figure 2: A Commutative Diagram for Economy Dependent Message Mechanisms

Theorem 1: *If a message mechanism based on a social choice correspondence, (A, μ, f), satisfies Axioms (C_1'), (C_2) and (C_3'), then there exists a unique function $h : A \to P$ such that the triangle in Figure 2 commutes for each $\mathcal{E} \in \mathcal{E}con$ and $a \in \mu(\mathcal{E})$* (Universal Implementability). *Function h can be taken as continuous (or differentiable) when the domain of message spaces are restricted on topological spaces (resp. differentiable manifolds) with continuous (resp. differentiable) response functions. Moreover, the mechanism that can play the above universal implementability property is unique up to isomorphism* (Universal Mapping Property) *among the class of mechanisms satisfying Axioms (C_1'), (C_2) and (C_3').*

Proof: [**The first assertion: Universal Implementability**] Let a and $\mathcal{E}^1$ be an arbitrary pair of a message and an economy satisfying $\mathcal{E}^1 = (\precsim_i^1, \omega_i^1)_{i \in I^1} \in \mathcal{E}con$ and $a \in \mu(\mathcal{E}^1)$. Since a is an equilibrium message for $\mathcal{E}^1$, $f(a, \mathcal{E}^1)$ is Pareto-optimal. By considering that at least one member of $\mathcal{E}^1$ is differentiable, we obtain a unique supporting price p for allocation $f(a, \mathcal{E}^1)$.

Next, we show that under p, $f_i(a, \mathcal{E}^1)$ satisfies the budget constraint for all $i \in I^1$. Assume not. Then, the allocation $f(a, \mathcal{E}^1)$ is not a Walras allocation. By Debreu-Scarf core limit theorem (Debreu and Scarf 1963; Theorem 3), there exist a positive n such that the n-fold replica allocation of $f(a, \mathcal{E}^1)$ cannot be in the core of the n-fold replica economy of $\mathcal{E}^1$, $(\mathcal{E}^1)^n$. Hence, we have a set G of the individuals in $(\mathcal{E}^1)^n$ who can block the n-fold replica allocation of $f(a, \mathcal{E}^1)$. Under (C_1'), we have $(f(a, \mathcal{E}^1))^n = f(a, (\mathcal{E}^1)^n)$. Furthermore, by applying (C_3') on $(\mathcal{E}^1)^n$, there exists $\mathcal{E}^*$ such that $a \in \mu(\mathcal{E}^*)$, $(\mathcal{E}^1)^n \subset \mathcal{E}^*$ and $u_i(f_i(a, \mathcal{E}^*)) \geqq u_i(f_i(a, (\mathcal{E}^1)^n))$ for all i of $(\mathcal{E}^1)^n$. It follows that, by (C_2), $f(a, \mathcal{E}^*)$ is an element of $\mathcal{C}ore(\mathcal{E}^*)$ satisfying $u_i(f_i(a, \mathcal{E}^*)) \geqq u_i(f_i(a, (\mathcal{E}^1)^n))$ for all i of $(\mathcal{E}^1)^n$. But this is impossible because G in $(\mathcal{E}^1)^n$ blocks the utility allocation under $(f(a, \mathcal{E}^1))^n$. Therefore, $p \cdot \left(f_i(a, \mathcal{E}^1) - \omega_i^1\right) = 0$ for all $i \in I^1$.

[6] Our approach does not restrict the domain of messages as long as for every message $a \in A$, there exists at least one economy $\mathcal{E}$ such that $a \in \mu(\mathcal{E})$. Such convention is also sufficient for our purpose to characterize the universal implementability of the price mechanism for all of other message mechanisms and equilibrium allocations.

It is easy to check that such h is necessarily continuous (or differentiable) when we restrict the class of message spaces on topological spaces (resp. differentiable manifolds) with continuous (resp. differentiable) response functions. Indeed, for each $a \in A$, it is quite easy (e.g., by considering an economy with Cobb-Douglas utility agents,) to choose an economy such that near $h(a)$, a coordinate of the list of excess demand functions, e, provides a local homeomorphism (resp. diffeomorphism). The composition of such local homeomorphism (resp. diffeomorphism) and a coordinate response function of f gives an another local representation of h that is necessarily continuous (resp. differentiable).

[**The second assertion: Universal Mapping Property**] Though it would be possible to obtain this result as a corollary to the fundamental mathematical theorem on the universal mapping (see, e.g., Bourbaki 1971), we present a direct proof since (i) the completeness of this paper is also desirable and (ii) the restriction of our commutative diagram argument to equilibrium messages requires some special treatments for a direct application of the universal mapping theorem.

Note that Axiom (C_3') ensures that every message $a \in A$ can be an equilibrium message for at least one economy in $\mathcal{E}con$. Suppose that (P, π, e) and (P', π', e') are two message mechanisms satisfying Axioms (C_1'), (C_2) and (C_3') and having the above universal implementability property. Then, by the first assertion of this theorem, we obtain two mappings $h : P' \to P$ and $h' : P \to P'$ such that the diagram in Figure 2 commutes for each equilibrium messages. Moreover, the uniqueness of such mappings also implies that both $h \circ h' : P \to P$ and $h' \circ h : P' \to P'$ are identity mappings. It follows that h and h' must be bijections such that $h^{-1} = h'$. The requests on h and h' to be continuous (or differentiable) imply that both h and h' are homeomorphisms (resp. diffeomorphisms). ■

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