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Abstract

This article presents a dynamic general equilibrium model of mixed oligopoly. In this model, public firms engage in manufacturing differentiated goods and cost-reducing R&D investments. This paper shows that public firms suddenly disappear from the mixed oligopoly market when the parameter of competitiveness exceeds the threshold value. This paper also shows that subsidies are larger when public firms regard a representative household as more important.

Keywords: Mixed oligopoly, Public firm, R&D investment, Dynamic general equilibrium model
JEL classification: L13, L32, O30

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1 Introduction

Privatization programs have been executed mainly in developed countries for the improvement of public firms' efficiency and reconstruction of public finances since the 1980s. However, mixed oligopolies are still common, especially in developing countries. This gives the impression that the maturation of an industry makes public firms appear to be troublemakers in the market. The aim of this paper is to discuss the effects technology improvements of public firms have on the economy.

This paper presents a dynamic general equilibrium model of mixed oligopoly. Since De Fraja and Delbono (1989), many studies investigated mixed oligopolies, where public firms maximize social welfare and private firms maximize their profits. Among them, Nishimori and Ogawa (2002) and Cato (2008) considered the mixed oligopoly market where only public firms invest in research and development (R&D) activities. They have shown that investment in public monopolies is higher than that in mixed oligopolies. Mixed oligopolies are common in the industries of education, finance, infrastructure, and resource development. Managements of public firms in these industries are required to have a long-term perspective. This implies that public firms should take the future benefits of economic agents into account. Hence, public firms in the present study consider the future benefits of economic agents.

This paper takes an intertemporal optimization approach when considering strategies of public firms. Public firms determine their production and investments for cost reduction considering social benefits. We apply Matsumura's (1998) approach, the first research that examined partial privatization, to our objective functions of public firms to consider mixture ownership.

Our main findings are as follows. First, the steady state is classified into two cases according to the degree of differentiation between public firms and private firms, and the difficulty of cost reduction. If public firms manufacture similar products to those of private firms and their R&D activities are efficient, there are two kinds of steady state: one exhibits mixed oligopoly and the other exhibits pure oligopoly. However, the steady states of the mixed oligopoly may not exist if public firms manufacture different products from those of private firms, and their R&D activities are inefficient. This means that, in the situation, the present value of the weighted sum of utility and aggregate profits of a mixed oligopoly is smaller than that of a pure oligopoly.,since this result is derived from the optimization problem of public firms.

Second, subsidies to public firms increase when they regard the representative household as important, at least when mixed oligopoly and pure oligopoly always exist. This paper focuses on steady state values of technology levels, a production cost over the economy, and subsidies in this situation. The reason for the result is that the public firm undervalues costs generated by its own activities because it is concerned about the household. This leads to lower prices of public firms relative to that of private firms and larger R&D investments. As a result, increases in subsidies occur. This brings about financial deficits that might be prevented by partial privatization or setting an upper limit on the subsidy.

The first result follows De Fraja (1991) and other related literature: privatization in the sufficiently competitive market improves social welfare. However, the present study is different from that literature in that no comparative statics were utilized to obtain this result. The literature shows that pure oligopoly is better that mixed oligopoly when the market is competitive. In the present study, the steady state is pure oligopoly as a result of public firms solving optimization problems, when the degree of differentiation is small. Since the small degree of differentiation implies that the market is competitive, this result corresponds to the previous results.

The second result follows Nishimori and Ogawa (2002) and Cato (2008). These studies show that investment by a public firm in a public monopoly is larger than that in a mixed oligopoly. Similarly, the present study shows that partial privatization has a negative effect on the amount of investment. In addition, this study shows that technology improvement in public firms has a negative effect on their mark-up.

The remainder of the paper is organized as follows. Section 2 sets up the model. We construct the model by solving the optimization problem of the representative household and firms, and obtaining equilibrium. The dynamics are presented in section 2 and comparative statics is presented in section 3. Section 4 concludes. Appendices discuss details of derivations of formulas .

2 Model

2.1 Household

Suppose that there is an infinitely lived representative household whose instantaneous utility at period t is

$$\ln C(t), \tag{1}$$

where $C(t)$ is an aggregate consumption index at period t. This index is composed of $m + n$ differentiated goods, which can be divided into two groups according to elasticities of substitution, which will be discussed later.

Let us denote one group as X and the other group as Y. We further denote $X(Y)$ group's consumption index as $X(t)$ $(Y(t))$. Suppose the elasticity of substitution between the two group as $\frac{1}{1-\gamma}$. Then, the aggregate consumption index is defined as

$$C(t) = [X(t)^{\gamma} + Y(t)^{\gamma}]^{\frac{1}{\gamma}}, \qquad \gamma \in [0, 1]. \tag{2}$$

The household can consume goods of the group X on interval $[0, m]$. Supposing elasticities of substitution between any two goods as constant $\frac{1}{1-\alpha}$, the consumption index of group X is defined as

$$X(t) = \left[\int_0^m x_i(t)^{\alpha} di\right]^{\frac{1}{\alpha}}, \qquad \alpha \in [0, 1], \tag{3}$$

where $x_i(t)$ is the consumption of firm i's product in group X. Similarly, the household can consume goods of group Y on interval $[0, n]$ and that of group Y is defined as

$$Y(t) = \left[\int_0^n y_j(t)^{\beta} dj\right]^{\frac{1}{\beta}}, \qquad \beta \in [0, 1], \tag{4}$$

where $y_j(t)$ is the consumption of firm j's product in the group Y.

The budget constraint of the household is as follows. By denoting a price index of $X(t)$ as $P_X(t)$ which is defined later, total expenditure to group X is

$$P_X(t)X(t) = \int_0^m p_{x_i}(t)x_i(t)di, \tag{5}$$

where $p_{x_i}(t)$ is a price of $x_i(t)$. In a similar way, by denoting a price index of group Y as $P_Y(t)$, total expenditure to the group Y is

$$P_Y(t)Y(t) = \int_0^n p_{y_j}(t)y_j(t)dj, \tag{6}$$

where $p_{y_j}(t)$ is a price of $y_j(t)$. Summing up, total expenditure of the household becomes

$$P_C(t)C(t) = P_X(t)X(t) + P_Y(t)Y(t), \tag{7}$$

where $P_C(t)$ is an aggregate price index defined later.

Since the household maximizes utility under these budget constraints, the maximization problem consists of two stages. In the first stage, the household maximizes (2) subject to (7) to obtain allocation between goods of X group and that of Y group as follows:

$$X(t) = \frac{P_X(t)^{\frac{1}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}} P_C(t)C(t), \tag{8}$$

and

$$Y(t) = \frac{P_Y(t)^{\frac{1}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}} P_C C(t). \tag{9}$$

Substituting these two demand functions into (2) and solving for $P_C(t)$, we can obtain the aggregate price index as

$$P_C(t) = (P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}})^{\frac{\gamma-1}{\gamma}}. \tag{10}$$

With this price index, (8) and (9) can be rewritten as

$$X(t) = \frac{P_X(t)^{\frac{1}{\gamma-1}}}{P_C(t)^{\frac{\gamma}{\gamma-1}}} P_C(t)C(t), \tag{11}$$

and

$$Y(t) = \frac{P_Y(t)^{\frac{1}{\gamma-1}}}{P_C(t)^{\frac{\gamma}{\gamma-1}}} P_C(t)C(t). \tag{12}$$

In the second step, we obtain quantities of consumption per firm. Maximizing (3) subject to (5) leads to

$$x_i(t) = \frac{p_{x_i}(t)^{\frac{1}{\alpha-1}}}{\int_0^m p_{x_{i'}}(t)^{\frac{\alpha}{\alpha-1}} di'} P_X(t)X(t). \tag{13}$$

Substituting this into (5) defines the price index of group X as

$$P_X(t) = \left[\int_0^m p_{x_i}(t)^{\frac{\alpha}{\alpha-1}} di\right]^{\frac{\alpha-1}{\alpha}}. \tag{14}$$

Combining (11), (13), and (14), we can obtain the demand function for the product of firm i as follows:

$$x_i(t) = \frac{p_{x_i}(t)^{\frac{1}{\alpha-1}}}{P_X(t)^{\frac{\alpha}{\alpha-1}-\frac{\gamma}{\gamma-1}} P_C(t)^{\frac{\gamma}{\gamma-1}}} P_C(t)C(t). \tag{15}$$

In the same manner, we can obtain the price index of group Y as follows:

$$P_Y(t) = \left[\int_0^n p_{y_j}(t)^{\frac{\beta}{\beta-1}} dj\right]^{\frac{\beta-1}{\beta}}. \tag{16}$$

and the demand function for the product of firm j as

$$y_j(t) = \frac{p_{y_j}(t)^{\frac{1}{\beta-1}}}{P_Y(t)^{\frac{\beta}{\beta-1}-\frac{\gamma}{\gamma-1}} P_C(t)^{\frac{\gamma}{\gamma-1}}} P_C(t)C(t). \tag{17}$$

We next turn to the intertemporal maximization problem of the representative household. We assume that the intertemporal utility function of the household is

$$U(t) = \int_t^\infty \mathrm{e}^{-\rho(s-t)} \ln C(s)\, ds, \tag{18}$$

where $\rho \in (0, 1)$ is a subjective discount rate. The household's intertemporal budget constraint is

$$\dot{A}(t) = r(t)A(t) + (1 - \tau(t))[w_P(t)L_P(t) + w_R(t)L_R(t)] - P_C(t)C(t), \tag{19}$$

where $r(t) \in (0, 1)$ is an interest rate and $\tau(t) \in [0, 1)$ is an income tax rate. The household supplies inelastic labour $\bar{L}$ and firms employ $L_P(t)$ units of labor for production at wage rate $w_P(t)$ and employ $L_R(t)$ units of labor for R&D investment at the wage rate $w_R(t)$. The household pays the government an income tax $\tau(t)[w_P(t)L_P(t) + w_R(t)L_R(t)]$ and divides after-tax income and asset income $rA(t)$ into saving and consumption. We first construct the current value Hamiltonian as follows:

$$H \equiv \ln C(t) + \nu(t)\left[r(t)A(t) + (1 - \tau(t))\left\{w_P(t)L_P(t) + w_R(t)L_R(t)\right\} - P_C(t)C(t)\right]. \tag{20}$$

From the necessary conditions of maximization, the following Euler equation obtains:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho - \frac{\dot{P_C}(t)}{P_C(t)}. \tag{21}$$

Normalizing $P_C(t)C(t)$ to one, we can rewrite above equation as

$$r(t) = \rho. \tag{22}$$

This means that the interest rate is constant over time.

2.2 Firms

This section discusses production behaviors of firms. Suppose that monopolistic competition prevails in the differentiated goods market and there are m firms belonging to group X and n firms belonging to group Y. Let firms in group X be public firms engaging in production and R&D investment. On the other hand, to keep matters simple, let firms in group Y be private firms who do not undertake R&D investment but produce goods.

Suppose that production and R&D investment require labor input. We specify their production and R&D technologies as follows. The public firm i produces a differentiated good with the following technology

$$x_i(t) = z_i(t)^{\varepsilon} l_{x_i}(t), \quad \varepsilon \in (0,1), \tag{23}$$

where $l_{x_i}(t)$ is labor input for the production of the public firm i and $z_i(t)$ indicates the technology level of the firm. The public firm i uses labor input $l_{R_i}(t)$ to improve its technology level $z_i(t)$ according to

$$\dot{z}_i(t) = l_{R_i}(t)^{\theta} - \delta z_i(t), \qquad \theta \in (0,1), \quad \delta \in (0,1), \tag{24}$$

where δ is an obsolescence rate of the technology level. This equation shows that technology becomes obsolete at the rate δ.

We next consider the private firm j. The private firm j produces a differentiated good with the following technology

$$y_j(t) = l_{y_j}(t), \tag{25}$$

where $l_{y_j}(t)$ is labor input for production of the private firm j.

From (23) and (24), the profit of the public firm i becomes

$$\pi_{x_i}(t) = (p_{x_i}(t) - w_P(t) z_i(t)^{-\epsilon}) x_i(t) - w_R(t) l_{R_i}(t) + s_i(t), \tag{26}$$

where $s_i(t)$ is a subsidy to the public firm i. The government subsidies $s_i(t)$ satisfy the public firm's budget loss because of taxation on the household. Similarly, (25) gives the profit of the private firm j as follows:

$$\pi_{y_j}(t) = (p_{y_j}(t) - w_P(t)) y_j(t). \tag{27}$$

We now proceed to these firms' optimization problems. The objective function of the private firm j is

$$v(t) = \int_t^{\infty} \quad \mathrm{e}^{-r(s-t)} \quad \pi_{y_j}(s) \quad ds. \tag{28}$$

By using (17) and (27), we can obtain the price of the private firm j as

$$p_{y_j}(t) = \frac{w_P(t)}{\beta}. \tag{29}$$

We have assumed that private firms take price indices $P_X(t)$, $P_Y(t)$ and $P_C(t)$ as given.

Unlike private firms, the public firm i considers the household's utility and other firms' profits. Hence, the objective function of the public firm i is defined as follows:

$$\int_t^{\infty} \mathrm{e}^{-\rho(s-t)} \left[\psi \ln C(s) + (1-\psi) \left[\int_0^m \pi_{x_i}(s) di + \int_0^n \pi_{y_j}(s) dj \right] \right] ds, \tag{30}$$

where $\psi \in [0, 1]$. The public firm i put weight ψ on the household's utility and weight $1 - \psi$ on the sum of all firms' profits. We assume that the discount rate of the public firm i is ρ, since the owner of the public firm i is the government, which takes care of the welfare of the household. We construct the current value Hamiltonian of the public firm i as follows:

$$\begin{aligned} H \equiv \psi \ln C(t) + (1-\psi) \left[\int_0^{i-1} \pi_{x_{i'}}(t) di' + \pi_{x_i}(t) + \int_{i+1}^m \pi_{x_{i'}}(t) di' + \int_0^n \pi_{y_j}(t) dj \right] \\ + \mu_i(t) \left[l_{R_i}^{\theta}(t) - \delta z_i(t) \right], \end{aligned} \tag{31}$$

where $\mu_i(t)$ is the co-state variable of the technology level of the public firm i.

Before proceeding to the first order condition of the maximization problem, we consider derivatives of demand functions. When determining its behavior, the public firm i considers the direct effect of one unit of increase in $p_{x_i}(t)$ on its profit and the indirect effect of that on other agents' benefits through changes in the price indices. This indirect effect is generally ignored in the case of differential monopoly. However, it is hard to ignore this effect in mixed differential monopoly, since the scale of public firms are basically large in comparison with private firms. This is the reason we differentiate $P_C(t)$ with respect to $p_{x_i}(t)$ when deriving the first order conditions.

Using (26) and (27), the first order conditions of maximization are derived as

$$\begin{aligned} \frac{\partial H}{\partial p_{x_i}(t)} = 0 \Leftrightarrow \psi \frac{\frac{\partial C(t)}{\partial p_{x_i}(t)}}{C(t)} + (1-\psi) \Bigg[\int_0^{i-1} \left(p_{x_{i'}}(t) - w_P(t) z_{i'}(t)^{-\varepsilon} \right) \frac{\partial x_{i'}(t)}{\partial p_{x_i}(t)} di' \\ + x_i(t) + \left(p_{x_i}(t) - w_P(t) z_i(t)^{-\varepsilon} \right) \frac{\partial x_i(t)}{\partial p_{x_i}(t)} + \int_{i+1}^m \left(p_{x_{i'}}(t) - w_P(t) z_{i'}(t)^{-\varepsilon} \right) \frac{\partial x_{i'}(t)}{\partial p_{x_i}(t)} di' \\ + \int_0^n \left(p_{y_j}(t) - w_P(t) \right) \frac{\partial y_j(t)}{\partial p_{x_i}(t)} dj \Bigg] = 0, \end{aligned} \tag{32}$$

$$\frac{\partial H}{\partial l_{R_i}(t)} = 0 \Leftrightarrow (1-\psi) w_R(t) = \theta \mu_i(t) l_{R_i}(t)^{\theta - 1}, \tag{33}$$

$$\frac{\partial H}{\partial z_i(t)} = \rho \mu_i(t) - \dot{\mu}_i(t) \Leftrightarrow \dot{\mu}_i(t) = (\rho - \delta) \mu_i(t) - \varepsilon (1-\psi) w_P(t) z_i(t)^{-\varepsilon - 1} x_i(t). \tag{34}$$

The derivations of these derivatives in the equation (32) are explained in Appendix A.

To obtain the price of public firm i requires several steps. Taking derivatives of (13) and (17) and substituting them into (32), we obtain

$$\psi x_i(t) = (1-\psi) \left\{ 1 - \frac{1}{1-\alpha} \left(1 - \frac{w_P(t) z_i(t)^{-\varepsilon}}{p_{x_i}(t)} \right) + \frac{\gamma}{1-\gamma} \Gamma(t) \right\} x_i(t), \tag{35}$$

where $\Gamma(t) \equiv \int_0^m (p_{x_i}(t) - w_P(t) z_i(t)^{-\varepsilon}) x_i(t) di + \int_0^n (p_{y_j}(t) - w_P(t)) y_j(t) dj$.

The left-hand side of the above equation represents the weighted marginal benefit of the household. This is the increase in the household benefit caused by one additional unit of p_{x_i}. Similarly, the right-hand side of the above equation represents the weighted marginal profit over all firms. This is the change in the total profit over all firms caused by one additional unit of p_{x_i}. The price p_{x_i}(t) is determined by the equality between the weighted marginal benefit and weighted aggregate marginal profits. Furthermore, weighted aggregate marginal profits has two components. The component $1 - \frac{1}{1-\alpha}(1 - \frac{w_P(t)z(t)^{-\varepsilon}}{p_x(t)})$ means the direct effect on the profit of the public firm i by one additional unit of $p_{x_i}(t)$. The other component $\frac{\gamma}{1-\gamma}\Gamma(t)$ means the indirect effect on the profit through the change in the consumption by that amount.

Here, we check that $\Gamma(t)$ represents aggregate profits of all firm except for public firms' R&D expenditures. Rearranging $\Gamma(t)$ provides

$$\Gamma(t) = \left[\int_0^m p_{x_i}(t)x_i(t)di + \int_0^n p_{y_j}(t)y_j(t)dj\right] - w_P(t)\left[\int_0^m z_i(t)^{-\varepsilon}x_i(t)di + \int_0^n y_j(t)dj\right]. \quad (36)$$

Substituting (5) and (6) into the above equation yields

$$\Gamma(t) = \left[P_X(t)X(t) + P_Y(t)Y(t)\right] - w_P(t)\left[\int_0^m l_{x_i}(t)di + \int_0^n l_{y_j}(t)dj\right]. \quad (37)$$

By using (7)、(23) and (25), we can rewrite this equation as follows:

$$\Gamma(t) = P_C(t)C(t) - w_P(t)L_P(t). \quad (38)$$

Defining $w_P(t)L_P(t)$ as aggregate production cost of all firms, that is, $C_P(t)$ and remembering that total expenditure is one, we can obtain

$$\Gamma(t) = 1 - C_P(t). \quad (39)$$

Let us turn back to the first order condition with respect to $p_{x_i}(t)$. Substituting (39) into (35) results in

$$\psi x_i(t) = (1-\psi)\left\{1 - \frac{1}{1-\alpha}\left(1 - \frac{w_P(t)z_i(t)^{-\varepsilon}}{p_{x_i}(t)}\right) + \frac{\gamma}{1-\gamma}(1 - C_P(t))\right\} x_i(t). \quad (40)$$

For simplicity of notation, setting $(1-\alpha)\frac{\gamma}{1-\gamma}$ as Q and setting $(1-\alpha)\frac{1-2\psi}{1-\psi}$ as R, the price of the public firm i can be expressed as

$$p_{x_i}(t) = \frac{w_P(t)z_i(t)^{-\varepsilon}}{QC_P(t) + 1 - Q - R}. \quad (41)$$

The mark-up $\frac{1}{QC_P(t)+1-Q-R}$ is an increasing function of $C_P(t)$.

2.3 Equilibrium

In this section, we examine equilibrium conditions of the goods market, the labor market and a capital market. The aggregate production cost over firms will play an important role in these equilibrium conditions.

2.3.1 Goods Market Equilibrium

Let us derive the aggregate production cost over $m + n$ firms to obtain the goods market equilibrium condition. Because of symmetry across public firms and across private firms, combining (10), (14), (15), (16), (17), (29), and (41) yields

$$x(t) = \frac{1}{m}\frac{QC_P(t) + 1 - Q - R}{w_P(t)z(t)^{-\varepsilon}}\frac{P_X(t)^{\frac{\gamma}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}}, \tag{42}$$

and

$$y(t) = \frac{1}{n}\frac{\beta}{w_P(t)}\frac{P_Y(t)^{\frac{\gamma}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}}. \tag{43}$$

It is obvious that these output levels are functions of $w_P(t)$, $z(t)$, $C_P(t)$, $P_X(t)$, and $P_Y(t)$.

From (23) and (25), production of these differentiated goods requires the following labor inputs:

$$l_x(t) = \frac{1}{m}\frac{QC_P(t) + 1 - Q - R}{w_P(t)}\frac{P_X(t)^{\frac{\gamma}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}}, \tag{44}$$

and

$$l_y(t) = \frac{1}{n}\frac{\beta}{w_P(t)}\frac{P_Y(t)^{\frac{\gamma}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}}. \tag{45}$$

Because $L_P(t) = ml_x(t) + nl_y(t)$ and $C_P(t) = w_P(t)L_P(t)$, (44) and (45) give the aggregate production cost as

$$C_P(t) = \frac{(QC_P(t) + 1 - Q - R)P_X(t)^{\frac{\gamma}{\gamma-1}} + \beta P_Y(t)^{\frac{\gamma}{\gamma-1}}}{P_X(t)^{\frac{\gamma}{\gamma-1}} + P_Y(t)^{\frac{\gamma}{\gamma-1}}}. \tag{46}$$

Rearranging this equation, we obtain

$$\frac{P_X(t)}{P_Y(t)} = \left(\frac{C_P(t) - \beta}{(1 - Q)(1 - C_P(t)) - R}\right)^{\frac{\gamma-1}{\gamma}}. \tag{47}$$

By using this ratio, we can rewrite (42) and (43) as follows:

$$x(t) = \frac{1}{m}\frac{1}{w_P(t)z(t)^{-\varepsilon}}\frac{(QC_P(t) + 1 - Q - R)(C_P(t) - \beta)}{QC_P(t) + 1 - Q - R - \beta}, \tag{48}$$

and

$$y(t) = \frac{1}{n}\frac{1}{w_P(t)}\frac{((1-Q)(1-C_P(t))-R}{(QC_P(t)+1-Q-R-\beta)\,\beta^{-1}}. \tag{49}$$

2.3.2 Labor market equilibrium

Suppose that labor can freely move between production and R&D activities, the labor market equilibrium condition becomes

$$w_P(t) = w_R(t) \equiv w(t), \tag{50}$$

and

$$L_P(t) + L_R(t) = \bar{L}. \tag{51}$$

Utilizing $C_P(t) = w(t)L_P(t)$ and $L_R(t) = ml_R(t)$, (51) can be rewritten as

$$\frac{C_P(t)}{w(t)} + ml_R(t) = \bar{L}. \tag{52}$$

From (33) and (52), we can obtain the equilibrium wage rate as

$$C_P(t) = w(t)\left[\bar{L} - m\left(\frac{\theta\mu(t)}{(1-\psi)w(t)}\right)^{\frac{1}{1-\theta}}\right]. \tag{53}$$

This implicitly defines $w(t)$ as a function of the aggregate production function cost and the co-state variable. We denote the equilibrium wage rate as $w(C_P(t), \mu(t))$. In addition, (52) and (53) give the equilibrium labor input for R&D activities as

$$l_R(t) = \left[\frac{\theta\mu(t)}{(1-\psi)w(C_P(t), \mu(t))}\right]^{\frac{1}{1-\theta}}. \tag{54}$$

2.3.3 Capital market equilibrium

Private firms pay their profit to the household as dividends. From (27), (28), (29), and (49), we can obtain the non-arbitrage condition as

$$\dot{v}(t) = rv(t) - \frac{1-\beta}{n}\frac{(1-Q)(1-C_P(t))-R}{QC_P(t)+1-Q-R-\beta}, \tag{55}$$

where $v(t)$ is a stock price of the private firm.

2.3.4 Constraints for the aggregate production cost over all firms

Let us discuss conditions guaranteeing the existence of firms. To prove concavity of objective functions of public firms, we must check that the second order condition of (31) is negative. This imposes the following constraint on the aggregate production cost over firms:

$$C_P(t) > \beta, \tag{56}$$

when $m > \frac{1-\gamma}{\gamma}$. Its proof is given in Appendix B.

The equilibrium profits of private firms give the free entry condition of private firms as

$$C_P(t) \leq \frac{1-Q-R}{1-Q}. \tag{57}$$

Accordingly, (56) and (57) conclude that the mixed oligopoly exists if the aggregate production cost over firms satisfies the following inequality.

$$\beta < C_P(t) \leq \frac{1-Q-R}{1-Q}. \tag{58}$$

Moreover, it is easy to check that the market is a pure oligopoly when $C_P(t)$ is β. [1] Now, we have completed the discussion of equilibrium conditions.

3 Dynamics

3.0.1 The structure of dynamics

We now examine the dynamics of the system. Substituting (52) into (24) yields

$$\dot{z}(t) = \left[\frac{1}{m}\left(\bar{L} - \frac{C_P(t)}{w(C_P(t), \mu(t))}\right)\right]^{\theta} - \delta z(t), \tag{59}$$

Substituting (48) into (34) yields the following:

$$\rho = \delta + \frac{\dot{\mu}(t)}{\mu(t)} + \frac{\varepsilon(1-\psi)}{m}\frac{1}{z(t)}\frac{(QC_P(t)+1-Q-R)(C_P(t)-\beta)}{QC_P(t)+1-Q-R-\beta}\frac{1}{\mu(t)}. \tag{60}$$

The left-hand side of this equation is the cost incurred by giving up a unit of consumption for investment and the right-hand side represents the return on R&D investment. The right-hand side is composed for three terms: compensation for technological obsolescence, the rate of the marginal value change of the technology, and cost-reducing effects of R&D investment.

We next explore the relationship between $z(t)$ and $C_P(t)$. Equilibrium price indices can be derived by (14), (16), (29) and (41). Substituting these indices into (47) and solving this for $z(t)$ gives

$$z(t) = \phi^{\frac{1}{\varepsilon}}\left[\frac{C_P(t)-\beta}{(1-Q)(1-C_P(t))-R}\right]^{\frac{1-\gamma}{\varepsilon\gamma}}(QC_P(t)+1-Q-R)^{-\frac{1}{\varepsilon}}, \tag{61}$$

[1] See Appendix. B

where $\phi \equiv \left(\frac{m^{\frac{\alpha-1}{\alpha}}}{n^{\frac{\beta-1}{\beta}}}\beta\right)$. This equation indicates that $C_P(t)$ is a function of $z(t)$. Let us denote the function as $C_P(z(t))$. Equation (61) reveals that the differential equations (59) and (60) have two endogenous variables $z(t)$ and $\mu(t)$.

Consequently, we can summarize the structure of dynamics as follows. The dynamics is comprised of three differential equations (55), (59) and (60) and two equations (53) and (61). These equations involve three endogenous variables $z(t)$, $\mu(t)$, and $\nu(t)$.

3.0.2 Steady states

Before proceeding to the phase diagram analysis, we solve for the steady states value of the dynamics. At the steady state, $\dot{z}(t) = \dot{\mu}(t) = 0$ must be satisfied. Thus, the following holds at the steady state.

$$z^* = \frac{1}{\delta}\left(\frac{\bar{L}}{m}\right)^{\theta}\left[\frac{\varepsilon\theta\delta(QC_P^* + 1 - Q - R)(C_P^* - \beta)}{(\rho-\delta)(QC_P^* + 1 - Q - R - \beta) + \varepsilon\theta\delta(QC_P^* + 1 - Q - R)(C_P^* - \beta)}\right]^{\theta}. \tag{62}$$

Using (61), we can obtain the following equation for the steady state value of C_P^*,

$$\begin{aligned}
&\phi^{\frac{1}{\varepsilon}}\frac{(C_P^* - \beta)^{\frac{1-\gamma}{\varepsilon\gamma}-\theta}}{\left\{(1-Q)(1-C_P^*) - R\right\}^{\frac{1-\gamma}{\varepsilon\gamma}}(QC_P^* + 1 - Q - R)^{\frac{1}{\varepsilon}+\theta}} \\
=\ &\frac{1}{\delta}\left(\frac{\bar{L}}{m}\right)^{\delta}\left[\frac{\rho-\delta}{\varepsilon\theta\delta}(QC_P^* + 1 - Q - R - \beta)C_P^* + (QC_P^* + 1 - Q - R)(C_P^* - \beta)\right]^{-\theta}.
\end{aligned} \tag{63}$$

The existence of the steady state can be proved by graphical investigation of (63) in $\beta \le \frac{1-Q-R}{1-Q}$. Differentiating the right-hand side of (63) with respect to C_P^* shows that it is a downward sloping curve and takes non-negative values in the domain of C_P. In the same way, we can find that the left-hand side of (63) is an increasing function of C_P if $\gamma < \frac{1}{1+\varepsilon\theta}$ and U-shaped curve if $\gamma \ge \frac{1}{1+\varepsilon\theta}$. These results provide figure 1(a) which proves the existence of C_P^* in the case of $\gamma < \frac{1}{1+\varepsilon\theta}$ and figure 1(b) which shows the case of $\gamma \ge \frac{1}{1+\varepsilon\theta}$.

Figure 1(a) depicts the case of $\gamma < \frac{1}{1+\varepsilon\theta}$. In this case, there is one intersection in the interval, $\beta < C_P^* < \frac{1-Q-R}{1-Q}$. However, this graph implies that there are two steady states. It is clear that the left-hand side of (61) and the right-hand side of (62) are zero when C_P is β. It follows that β is also the steady state value of C_P. Hence the number of steady points is the number of intersections on the figures plus one.

Similar arguments can be applied to the case of $\gamma \ge \frac{1}{1+\varepsilon\theta}$. Figure 1(b) shows that the number of steady points for this case can be zero, one or two, depending on parameters. When $\bar{L}$ is sufficiently large, there exits two steady states.

Using implicit function theorem, we can find that the $\dot{z}_i(t) = 0$ locus goes through the origin and has a positive slope. Similarly, we can find characters of the $\dot{\mu}_i(t) = 0$ locus. If $\gamma < \frac{1}{1+\varepsilon}$, the $\dot{\mu}_i(t) = 0$ locus has a negative slope and asymptotically approaches the horizontal and vertical axes, respectively. If $\gamma \ge \frac{1}{1+\varepsilon}$, the $\dot{\mu}_i(t) = 0$ locus is an inverted U-shape and goes through the origin and asymptotically approaches the horizontal axis. The characters are explained in Appendix E.

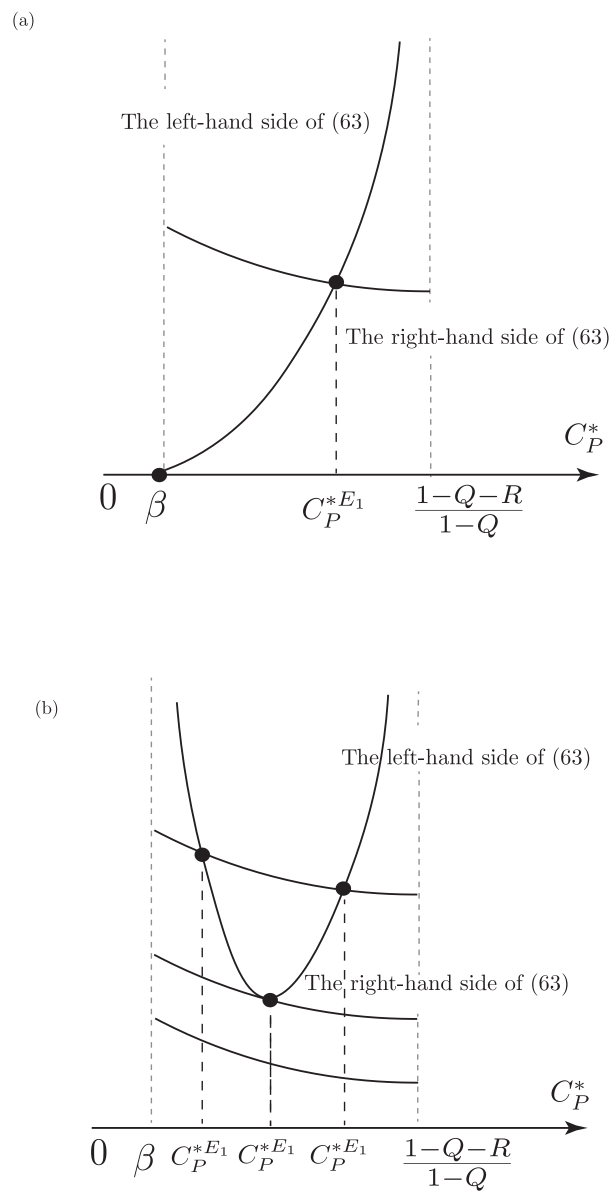

Figure 1: (a) The number of steady state in $\gamma < \frac{1}{1+\varepsilon\theta}$
(b) The number of steady state in $\gamma \geq \frac{1}{1+\varepsilon\theta}$

Figure 2(a) shows the phrase diagram of $\gamma < \frac{1}{1+\varepsilon}$. There are two steady points which are E_0, the origin and E_1, an intersection of two loci. The value of z^* is zero at E_0 which means that public firms do not exist in this steady point. Then the market is a pure oligopoly market, where only private firms operate. Even if this steady point has no transition path, once a sufficient upward jump of μ occurred, public firms would begin to operate and the economy would reach steady point E_1. At the steady point E_1, z^* is no longer zero, that is, C_P^* is greater than β. This implies that the market is a mixed oligopoly in E_1.

The phase diagram of $\frac{1}{1+\varepsilon} \leq \gamma < \frac{1}{1+\varepsilon\theta}$ is represented as figure 2(b). This figure is also the phase diagram of one intersection case on figure 1(b). If there are two intersection points on figure 1(b), the two loci have three intersections on the phase diagram. Figure 2(c) shows this situation. A similar argument as before can be applied to these cases. The steady points E_0 of these two figures exhibit the pure oligopoly and the steady points E_1 of these figures exhibit the mixed oligopoly, respectively. [2]

However, the steady state E_1 disappears if there is no intersection on the figure 1(b). When two loci have no intersection on figure 1(b), the phase diagram is shown as figure 2(d). It is obvious that there is only one steady point E_0 corresponding to the pure oligopoly. This implies that mixed oligopoly does not always exist when $\gamma \geq \frac{1}{1+\varepsilon\theta}$.

Hence, we conclude that the steady state differs from γ as follows.

Proposition 1 *In this economy, the market can be either a mixed oligopoly or a pure oligopoly, depending on the initial variables $z(0)$ and $\mu(0)$. However, there is a set of parameters which make the market never be the mixed oligopoly market when $\gamma \leq \frac{1}{1+\varepsilon\theta}$.*

As mentioned before, γ is the parameter referring to the elasticity of substitution between $X(t)$ and $Y(t)$. It is obvious that the larger γ indicates the more competitive market. Moreover, larger ε and θ imply that one unit of labor input has a bigger effect on cost reduction. Therefore, the market is more competitive and R&D activities are more productive when parameters γ, ε, and θ satisfy $\gamma \geq \frac{1}{1+\varepsilon\theta}$.

4 Welfare

In the previous section, we have clarified that there always exists the steady point of the mixed oligopoly when $\gamma < \frac{1}{1+\varepsilon\theta}$. We now proceed to the steady state effects of increasing ψ under the condition $\gamma < \frac{1}{1+\varepsilon\theta}$.

By (63), we define

$$\begin{aligned}
\Phi(C_P^{*E_1}, \psi) \quad \equiv \quad & \phi^{\frac{1}{\varepsilon}} \frac{(C_P^{*E_1} - \beta)^{\frac{1-\gamma}{\varepsilon\gamma} - \theta}}{\{(1-Q)(1-C_P^{*E_1}) - R\}^{\frac{1-\gamma}{\varepsilon\gamma}}} (QC_P^{*E_1} + 1 - Q - R)^{-\frac{1}{\varepsilon} - \theta} \\
- \quad & \frac{1}{\delta}\left(\frac{\bar{L}}{m}\right)^{\delta} \left[\frac{\rho - \delta}{\varepsilon\theta\delta}(QC_P^{*E_1} + 1 - Q - R - \beta)C_P^{*E_1}\right. \\
+ \quad & \left.(QC_P^{*E_1} + 1 - Q - R)(C_P^{*E_1} - \beta)\right]^{-\theta}, \qquad (64)
\end{aligned}$$

[2] We suppose that midpoint of three intersections is unstable point.

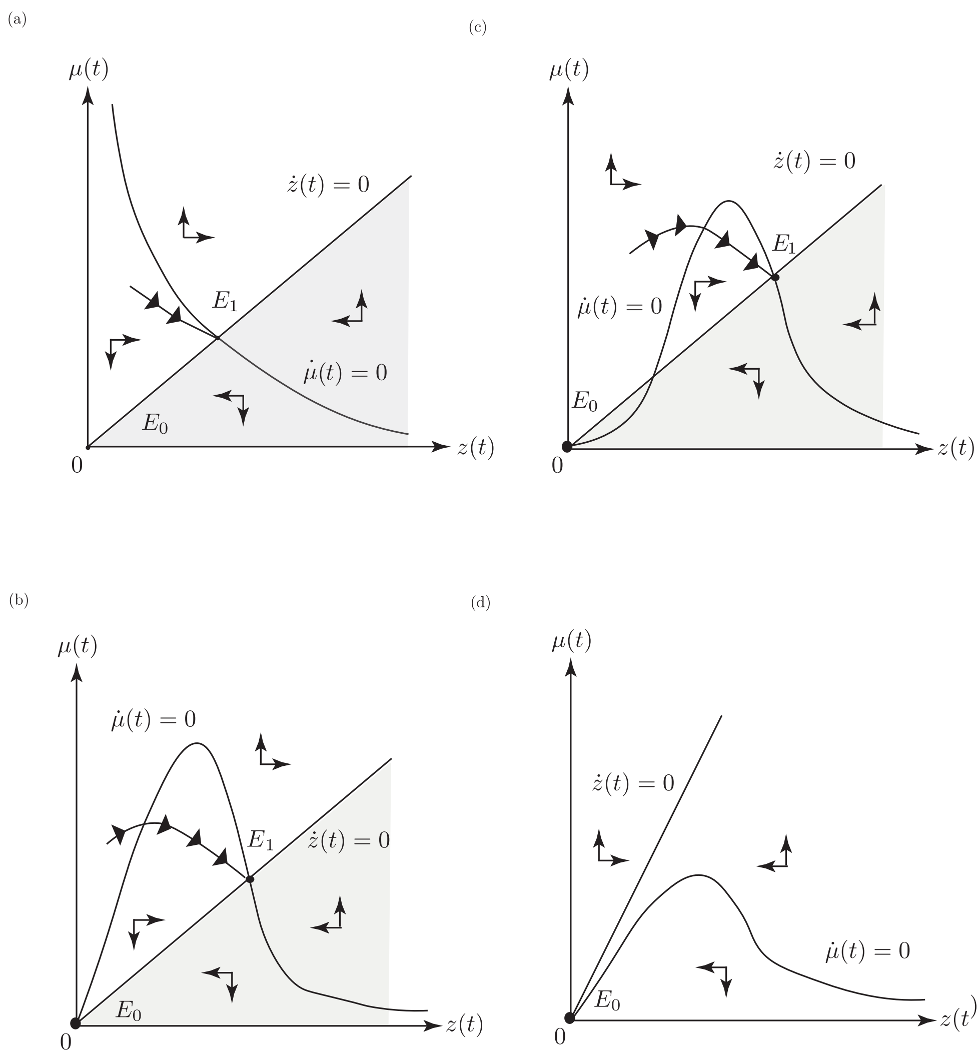

Figure 2: (a) The phase diagram in $\gamma < \frac{1}{1+\varepsilon}$
(b) The phase diagram in $\frac{1}{1+\varepsilon} \leq \gamma < \frac{1}{1+\varepsilon\theta}$ or in $\gamma \geq \frac{1}{1+\varepsilon\theta}$ with one intersection case on figure 1(b)
(c) The phase diagram in $\gamma \geq \frac{1}{1+\varepsilon\theta}$ with two intersections case on figure 1(b)
(d) The phase diagram in $\gamma \geq \frac{1}{1+\varepsilon\theta}$ with no intersection case on figure 1(b)

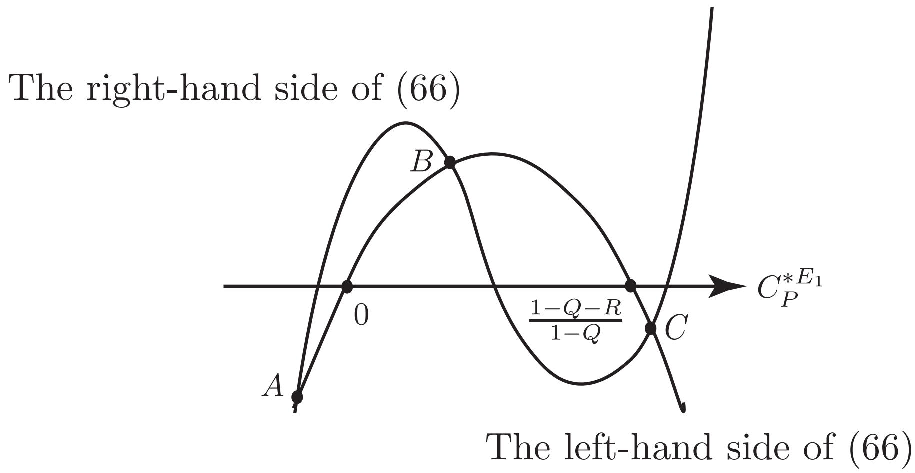

Figure 3: The sign of $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial R}$

where $C_P^{*E_1}$ refers to the sate value of aggregate production cost at steady state E_1. Using the implicit function theorem to the above equation yields

$$\frac{dC_P^{*E_1}}{d\psi} = -\frac{\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial\psi}}{\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial C_P^{*E_1}}}. \tag{65}$$

We can see at once that the molecule $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial\psi}$ can be rewritten as $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial R}\frac{\partial R}{\partial\psi}$ by the chain rule and that the sign of $\frac{\partial R}{\partial\psi}$ is negative by the definition of R. The task is now to find the sign of $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial R}$. Differentiating (64) with respect to R, we can find that the sign is negative if $C_P^{*E_1}$ satisfies the following inequality.

$$\begin{aligned}
&\beta\gamma\frac{\rho-\delta}{\delta}\{(1-Q)(1-C_P^{*E_1})-R\} \quad C_P^{*E_1} \\
&> \Big[(1-\gamma)(QC_P^{*E_1}+1-Q-R) \quad + \quad \gamma\{(1-Q)(1-C_P^{*E_1})-R\}\Big] \\
&\Big[\{(\frac{\rho-\delta}{\delta\varepsilon\theta}+1)C_P^{*E_1}-\beta\}(QC_P^{*E_1} \quad + \quad 1-Q-R)-\frac{\rho-\delta}{\delta\varepsilon\theta}\beta C_P^{*E_1}\Big].
\end{aligned} \tag{66}$$

To check that $C_P^{*E_1}$ satisfies the inequality, we draw the left-hand side and the right-hand side of (66). The left-hand side has a negative coefficient of $C_P^{*E_1 2}$ and is zero when $C_P^{*E_1}$ are 0 or $\frac{1-Q-R}{1-Q}$. The right-hand side of (66) has a positive coefficient of $C_P^{*E_1 3}$ and three icon points: two of them are smaller than β and one of them is larger than $\frac{1-Q-R}{1-Q}$. Depicting the graphs of these equations, we can find that these two curvatures have three intersections, A, B, and C as shown in figure 3. Since A is smaller than 0 and C is larger than $\frac{1-Q-R}{1-Q}$, (66) is not satisfied when $C_P^{*E_1}$ is between 0 and B and is not satisfied when $C_P^{*E_1}$ is between B and $\frac{1-Q-R}{1-Q}$. Substituting β into (66) shows that

(66) is obtained only if $(1-Q)(1-\beta)-R+(1-\gamma-\gamma\varepsilon\theta)\beta>0$. Hence, B is smaller than β when $(1-Q)(1-\beta)-R+(1-\gamma-\gamma\varepsilon\theta)\beta>0$. This condition is satisfied at least when $\gamma<\frac{1}{1+\varepsilon\theta}$, since we set R less than $(1-Q)(1-\beta)$. In conclusion, (66) is satisfied only if $\gamma<\frac{1}{1+\varepsilon\theta}$.

Now, we turn to the denominator of (65). Figure 1(b) indicates the sign of $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial C_P^{*E_1}}$ is positive. Because of (63) and (64), we can find that the difference of gradients of each curve at the steady point E_1 on figure 1(b) correspond to the sign of the denominator. Therefore, $\frac{\partial\Phi(C_P^{*E_1},\psi)}{\partial C_P^{*E_1}}$ has positive sign when $\gamma<\frac{1}{1+\varepsilon\theta}$.

These results reveal that increases in ψ have positive effects on $C_P^{*E_1}$ when $\gamma<\frac{1}{1+\varepsilon\theta}$. Moreover, Appendix C shows that increase in $C_P(t)$ has a positive effects on $z(t)$. Hence, we establish the following proposition.

Proposition 2 *Increase in ψ has positive effect on z^{*1} and $C_P^{*E_1}$ under the condition $\gamma<\frac{1}{1+\varepsilon\theta}$.*

This result mainly comes from the fact that public firms undervalue their costs of production and R&D investment compared to the household's utility as ψ is increasing. This underestimation of cost leads to larger R&D investments and lower prices relative to private firms' prices. Since the negative effect on $C_P^{*E_1}$ by technology improvement is smaller than the positive effect on $C_P^{*E_1}$ due to cheaper prices of the public firms, an increase of ψ has a positive effect on aggregate production cost.

Last, we discuss the subsidy to public firms. Substituting (41), (48) and (54) into (26) gives

$$\pi_x^*=\frac{1}{m}\left[\frac{(-QC_P^{*E_1}+Q+R)(C_P^{*E_1}-\beta)}{QC_P^{*E_1}+1-Q-R-\beta}-\frac{m(\delta z^*)^{-\theta}}{\bar{L}-m(\delta z^*)^{-\theta}}C_P^{*E_1}\right]+s^*. \tag{67}$$

The government sets the subsidy to compensate the public firms' budget losses as follows:

$$(sum\,of\,subsidies)=\frac{m(\delta z^*)^{-\theta}}{\bar{L}-m(\delta z^*)^{-\theta}}C_P^{*E_1}-\frac{(-QC_P^{*E_1}+Q+R)(C_P^{*E_1}-\beta)}{QC_P^{*E_1}+1-Q-R-\beta}. \tag{68}$$

which is derived by (67) It is obvious that this equation is an increasing function in $C_P^{*E_1}$. Considering proposition 2 and this result gives following proposition.

Proposition 3 *An increase in ψ leads to larger subsidies if $\gamma>\frac{1}{1+\varepsilon\theta}$*

That is, an increase in ψ leads to an increase in the technology level and the aggregate production cost, and then, the amount of subsidy increases. The public firm that assigns more importance the household's utility is more aggressive in investing and sets lower prices compared to private firms. Consequently, the existence of such public firms leads to financial pressure.

5 Conclusion

This paper proposes a dynamic general equilibrium model of the mixed oligopoly with public firms' R&D activities. We formulate public firms' intertemporal optimization problem.

We find that the steady state of the mixed oligopoly may not exist if public firms manufacture similar products to those of private firms and R&D activities are efficient. This result follows De Fraja (1991) and other related literatures. However, we obtain this result without comaparative statistics.

We show that investments increase as public firms regard the household as important. This follows Nishimori and Ogawa (2002) and Cato (2008). However, we also show that this leads to lower prices of public firms relative to that of private firms and larger subsidies to public firms.

In this paper, we ignore R&D activities of private firms. The model with both kinds of firms' R&D activities would yield very different results. It is important to examine how the results change.

6 Appendix

A Derivatives of demand functions

Let us obtain derivatives in the first order condition (32). Differentiating $p_{x_i}(t)^{\frac{1}{1-\alpha}}$ in the numerator of (15) and $P_C(t)$ in its denominator provides the derivative of the public firm i's demand function as

$$\begin{aligned}\frac{\partial x_i(t)}{\partial p_{x_i}(t)} &= \frac{1}{P_X(t)^{\frac{\alpha}{\alpha-1}-\frac{\gamma}{\gamma-1}}P_C(t)^{\frac{\gamma}{\gamma-1}}}\frac{\partial p_{x_i}(t)^{\frac{1}{\alpha-1}}}{\partial p_{x_i}(t)} + \frac{p_{x_i}(t)^{\frac{1}{\alpha-1}}}{P_X(t)^{\frac{\alpha}{\alpha-1}-\frac{\gamma}{\gamma-1}}}\frac{\partial P_C(t)^{-\frac{\gamma}{\gamma-1}}}{\partial p_{x_i}(t)}, \\ &= \left(-\frac{1}{1-\alpha}p_{x_i}(t)^{-1} + \frac{\gamma}{1-\gamma}\frac{\partial P_C(t)}{\partial p_{x_i}(t)}P_C(t)^{-1}\right)x_i(t). \end{aligned} \tag{69}$$

Differentiating $P_C(t)$ in (15) obtains the derivatives of other public firms' demand functions as

$$\begin{aligned}\frac{\partial x_{i'}(t)}{\partial p_{x_i}(t)} &= \frac{p_{x_{i'}}(t)^{\frac{1}{\alpha-1}}}{P_X(t)^{\frac{\alpha}{\alpha-1}-\frac{\gamma}{\gamma-1}}}\frac{\partial P_C(t)^{-\frac{\gamma}{\gamma-1}}}{\partial p_{x_i}(t)}, \\ &= \frac{\gamma}{1-\gamma}\frac{\partial P_C(t)}{\partial p_{x_i}(t)}P_C(t)^{-1}x_{i'}(t). \end{aligned} \tag{70}$$

Differentiating private firm j's demand function (17), we obtain

$$\begin{aligned}\frac{\partial y_j(t)}{\partial p_{x_i}(t)} &= \frac{p_{y_j}(t)^{\frac{1}{\beta-1}}}{P_Y(t)^{\frac{\beta}{\beta-1}-\frac{\gamma}{\gamma-1}}}\frac{\partial P_C(t)^{-\frac{\gamma}{\gamma-1}}}{\partial p_{x_i}(t)}, \\ &= \frac{\gamma}{1-\gamma}\frac{\partial P_C(t)}{\partial p_{x_i}(t)}P_C(t)^{-1}y_j(t). \end{aligned} \tag{71}$$

The partial derivative of $P_C(t)$ with respect to $p_{x_i}(t)$ becomes

$$\frac{\partial P_C(t)}{\partial p_{x_i}(t)} = x_i(t)P_C(t), \tag{72}$$

from (10) and (14).

The derivatives of the household's utility function with respect to $p_{x_i}(t)$ in (32) can be derived as follows. Totally differentiating $P_C(t)C(t) = 1$ by $p_{x_i}(t)$, we obtain

$$\frac{\partial P_C(t)}{\partial p_{x_i}(t)} C(t) + P_C(t) \frac{\partial C(t)}{\partial p_{x_i}(t)} = 0. \tag{73}$$

Combining this and (72) yields

$$\frac{\partial C(t)}{\partial p_{x_i}(t)} C(t)^{-1} = -x_i(t). \tag{74}$$

Substituting these derivatives into (32) yields

$$\frac{\partial H}{\partial p_{x_i}(t)} = \left[-\psi + (1-\psi) \left\{ 1 - \frac{1}{1-\alpha} \left(1 - \frac{w_P(t) z_i(t)^{-\varepsilon}}{p_{x_i}(t)} \right) + \frac{\gamma}{1-\gamma} \Gamma(t) \right\} \right] x_i(t). \tag{75}$$

B Second order conditions

In this appendix, we derive second order conditions to check concavity. Using (1), (2), and (3), the household's first order derivative with respect to $x_i(t)$ is

$$\frac{\partial \ln C(t)}{\partial x_i(t)} = \frac{1}{C(t)} \frac{\partial C(t)}{\partial X(t)} \frac{\partial X(t)}{\partial x_i(t)} = C(t)^{-\gamma} X(t)^{\gamma-\alpha} x_i(t)^{\alpha-1}, \tag{76}$$

and then the second order derivative becomes

$$\frac{\partial^2 \ln C(t)}{\partial x_i(t)^2} = -\frac{\partial \ln C(t)}{\partial x_i(t)} \left[\gamma \frac{\partial C(t)}{\partial x_i(t)} C(t)^{-1} + (\alpha - \gamma) \frac{\partial X(t)}{\partial x_i(t)} X(t)^{-1} + (1-\alpha) x_i(t)^{-1} \right]. \tag{77}$$

Hence, the following condition is sufficient for concavity.

$$\alpha > \gamma. \tag{78}$$

Similar arguments can be applied to the case of $y_j(t)$. We can obtain the condition

$$\beta > \gamma, \tag{79}$$

to ensure that the second order condition of $y_j(t)$ is a concave function.

We next turn to public firm i. Its first order condition has been derived in Appendix A. With the envelope theorem, the second order condition of $p_{x_i}(t)$ becomes

$$\frac{\partial^2 H}{\partial p_{x_i}(t)^2} = -\frac{1-\psi}{1-\alpha} \left[\frac{w_P(t) z_i(t)^{-\varepsilon}}{p_{x_i}(t)^2} - Q \frac{\partial \Gamma(t)}{\partial p_{x_i}(t)}) \right] x_i(t). \tag{80}$$

Using $\Gamma(t)$, (15), (17), (23) and (25), the partial of $\Gamma(t)$ with respect to $p_{x_i}(t)$ in the above equation can be derived. Substituting this into (80) can be rewritten as

$$\frac{\partial^2 H}{\partial p_{x_i}(t)^2} = -\frac{1-\psi}{1-\alpha}\left[\frac{w_P(t)z_i(t)^{-\varepsilon}}{p_{x_i}(t)^2} - \frac{Q}{1-\alpha}\left(\frac{w_P(t)z_i(t)^{-\varepsilon}}{p_{x_i}(t)} - QC_P(t)\right)x_i(t)\right]x_i(t). \tag{81}$$

Substituting (41) and (48) and $Q \equiv (1-\alpha)\frac{\gamma}{1-\gamma}$ into (81), we can obtain

$$\begin{aligned}\frac{\partial^2 H}{\partial p_{x_i}(t)^2} &= -\frac{1-\psi}{1-\alpha}\left[\left(\frac{1}{p_{x_i}(t)} - \frac{\gamma}{1-\gamma}x_i(t)\right)\right.\\ &+ \left.\frac{1}{m}\frac{Q^2}{1-\alpha}\frac{1}{w_P(t)z(t)^{-\varepsilon}}\frac{C_P(t)(C_P(t)-\beta)}{QC_P(t)+1-Q-R-\beta}\right](QC_P(t)+1-Q-R)x_i(t). \quad (82)\end{aligned}$$

The task is now to find conditions which make (82) negative. Assuming $x_i(t)$ is a positive value, we must show the signs of terms in the parentheses. It follows that

$$1-Q>0, \tag{83}$$

because of (78). This inequality and the fact that the maximum value of R is $1-\alpha$ brings

$$1-Q-R>0. \tag{84}$$

These conditions show that the mark-up of public firms is non-negative as follows:

$$QC_P(t)+1-Q-R \geq 0. \tag{85}$$

It is natural to assume the mark-up of the private firm i is larger than that of the public firm. Thus, (29) and (41) give

$$QC_P(t)+1-Q-R-\beta \geq 0. \tag{86}$$

Consequently, the parentheses in (82) would be negative only if

$$C_P(t) > \beta. \tag{87}$$

under the conditions (84), (85), (86), and $m > \frac{1-\gamma}{\gamma}$ [3].

C The relation of $z(t)$ and $C_P(t)$

Differentiating (61) with respect to $C_P(t)$ provides

$$\frac{(1-Q)(1-\beta)-R}{(1-Q)(1-C_P(t))-R} \geq \left(\frac{\gamma}{1-\gamma}\right)\frac{Q(C_P(t)-\beta)}{QC_P(t)+1-Q-R}. \tag{88}$$

[3] This condition can be derived by (41), (42), (47), and (57).

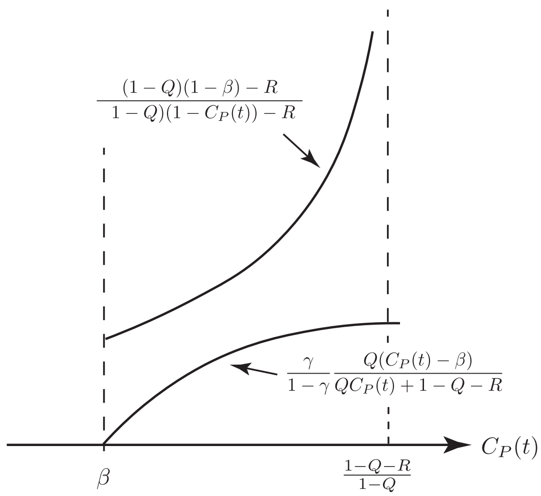

Figure 4: $\frac{\partial z(t)}{\partial C_P(t)} > 0$

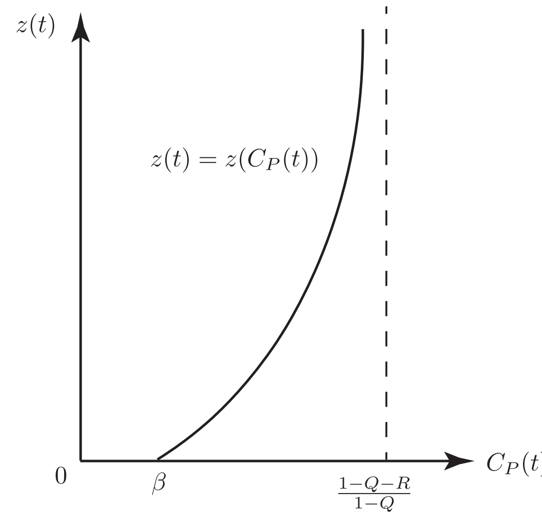

Figure 5: $z(t) = z(C_P(t))$

We will prove that this inequality is established under $\beta \leq C_P(t) < \frac{1-Q-R}{1-Q}$ by depicting a graph of these equations. Both sides of (88) have positive slopes in the constraint. Introducing β into $C_P(t)$, we can obtain that the left hand-side is 1 and the right hand side is 0. Hence, the left hand-side is larger than the right hand-side when $C_P(t) = \beta$. Introducing $\frac{1-Q-R}{1-Q}$ into $C_P(t)$, we can obtain that the left hand-side is infinity but the right hand-side is finite in the neighborhood of $\frac{1-Q-R}{1-Q}$. These results enable us to depict both sides of (88) as figure 4 and then the graph of (61) can be represented as figure 5. [4] Since the graphs intersect the horizontal line at β and increase to infinity in the domain, it follows that an increase in $z(t)$ leads to an increase in $C_P(t)$.

D $C_P(t)$ of a pure oligopoly

We prove that aggregate production cost over all firms of the pure oligopoly where only private firms exist is β. Since there are no public firms, (2) and (4) can be rewritten as

$$C(t) = Y(t) = \left[\int_0^n y_j(t)^\beta dj\right]^{\frac{1}{\beta}}. \tag{89}$$

Lack of R&D activities provides the intertemporal budget constraint as

$$\dot{A}(t) = rA(t) + w_P(t)\bar{L} - P_Y(t)Y(t), \tag{90}$$

by (7) and (19). Therefore, a demand function for private firm j is

$$y_j(t) = \frac{p_{y_j}(t)^{\frac{1}{\beta-1}}}{\int_0^\infty p_{y_j}(t)^{\frac{\beta}{\beta-1}} dj.}. \tag{91}$$

[4]For simplicity, we ignore that two graphs never cross under $\beta \leq C_P(t) < \frac{1-Q-R}{1-Q}$ on figure 4.

From (27) and (91), we can obtain

$$p_{y_j}(t) = \frac{w_P(t)}{\beta}. \tag{92}$$

Assuming a symmetric equilibrium , we obtain the following goods market equilibrium condition.

$$Y(t) = \frac{w_P(t)}{\beta}, \tag{93}$$

from (89), (91), and (92). The labor market equilibrium condition becomes

$$C_P(t) = w_P(t)ny(t) = \beta, \tag{94}$$

from（25 and (93). Hence, we complete the derivation.

E Preparation for phase diagrams

In this appendix, we discuss loci of (59) and (60). Setting $\dot{z}(t) = 0$, (59) can be rewritten as

$$\left[\frac{1}{m}\left(\bar{L} - \frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))}\right)\right]^{\theta} - \delta z(t) = 0. \tag{95}$$

Here, we define

$$\Theta(z(t), \mu(t)) \equiv \left[\frac{1}{m}\left(\bar{L} - \frac{C_P(t)}{w(C_P(t), \mu(t))}\right)\right]^{\theta} - \delta z(t), \tag{96}$$

and apply the implicit function theorem to this equation as follows:

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}z(t)} = -\frac{\frac{\partial\Theta(z(t),\mu(t))}{\partial z(t)}}{\frac{\partial\Theta(z(t),\mu(t))}{\partial \mu(t)}}. \tag{97}$$

Differentiating (96) with respect to $z(t)$ yields

$$\begin{aligned}\frac{\partial\Theta(z(t), \mu(t))}{\partial z(t)} &= -\theta\left[\frac{1}{m}\left(\bar{L} - \frac{C_P(z(t))}{w(C_P(t), \mu(t))}\right)\right]^{\theta-1} \\ &\frac{\left(w(C_P(z(t)), \mu(t)) - C_P(z(t))\frac{\partial w(C_P(z(t)),\mu(t))}{\partial C_P(z(t))}\right)\frac{\mathrm{d}C_P(z(t))}{\mathrm{d}z(t)}}{w(C_P(z(t)), \mu(t))^2} - \delta,\end{aligned} \tag{98}$$

where $\frac{1}{m}\left(\bar{L} - \frac{C_P(t)}{w(t)}\right)$ equals $l_R(t)$. The derivative of (53) with respect to $w(t)$ and (54) become

$$\begin{aligned}\frac{\partial C_P(z(t))}{\partial w(C_P(z(t)), \mu(t))} &- \frac{C_P(z(t))}{w(C_P(z(t), \mu(t))}, \\ &= \left(\bar{L} + \frac{\theta}{1-\theta}ml_R(t)\right) - \left(\bar{L} - ml_R(t)\right), \\ &= \frac{1}{1-\theta}ml_R(t) \geq 0.\end{aligned} \tag{99}$$

Rearranging this inequality leads to $w(C_P(z(t)), \mu(t)) - C_P(z(t))\frac{\partial w(C_P(z(t)),\mu(t))}{\partial C_P(z(t))} > 0$. Hence, (98) and (99) show

$$\frac{\partial \Theta(z(t), \mu(t))}{\partial z(t)} \leq 0. \tag{100}$$

Turing to the denominator of (97), differentiating (96) with respect to $\mu(t)$ leads to

$$\begin{aligned}\frac{\partial \Theta(z(t), \mu(t))}{\partial \mu(t)} &= \frac{\theta}{m}\left[\frac{1}{m}\left(\bar{L} - \frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))}\right)\right]^{\theta-1} \\ &\quad \frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))^2}\frac{\partial w(C_P(z(t)), \mu(t))}{\partial \mu(t)}.\end{aligned} \tag{101}$$

It is obvious that $\bar{L} - \frac{C_P(z(t))}{w(C_P(z(t)),\mu(t))}$ is positive from (52). We must examine the partial of $w(C_p(z(t)), \mu(t))$ with respect to $\mu(t)$. Solving (53) for $\mu(t)$ gives

$$\mu(t) = \frac{(1-\psi)}{\theta m} w(C_P(z(t)), \mu(t))\left[\bar{L} - \frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))}\right]^{1-\theta}. \tag{102}$$

The derivative of (102) with respect to $w(C_P(z(t)), \mu(t))$ is

$$\begin{aligned}\frac{\partial \mu(t)}{\partial w(C_P(z(t)), \mu(t))} &= \frac{(1-\psi)}{m\theta}\left[\bar{L} - \frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))}\right]^{-\theta} \\ &\quad \left[\bar{L} - \theta\frac{C_P(z(t))}{w(C_P(z(t)), \mu(t))}\right].\end{aligned} \tag{103}$$

Using (54), it is easy to see that (103) is non-negative. Therefore, the following inequality is established.

$$\frac{\partial \Theta(z(t), \mu(t))}{\partial \mu(t)} \geq 0. \tag{104}$$

From (100) and (104), the locus of $z(t)$ has positive slope. Incidentally, (59) and (61) show that the locus goes through the point $(z(t), \mu(t)) = (0, 0)$.

We now turn to the locus of $\mu(t)$ which is defined as

$$(\rho - \delta)\mu(t) - \frac{\varepsilon(1-\psi)}{m}\frac{1}{z(t)}\frac{(QC_P(z(t)) + 1 - Q - R)(C_P(z(t)) - \beta)}{QC_P(z(t)) + 1 - Q - R - \beta} = 0. \tag{105}$$

from (60). Denoting the left-hand side of (105) as

$$\begin{aligned}\Omega(z(t), \mu(t)) &\equiv \\ (\rho - \delta)\mu(t) &\quad \frac{\varepsilon(1-\psi)}{m} \quad \frac{1}{z(t)}\frac{(QC_P(z(t)) + 1 - Q - R)(C_P(z(t)) - \beta)}{QC_P(z(t)) + 1 - Q - R - \beta},\end{aligned} \tag{106}$$

and applying the similar method as in the case of $z(t)$, we can obtain

$$\frac{\partial \mu(t)}{\partial z(t)} = -\frac{\frac{\partial \Omega(z(t),\mu(t))}{\partial z(t)}}{\frac{\partial \Omega(z(t),\mu(t))}{\partial \mu(t)}}. \tag{107}$$

The derivative of (106) with respect to $z(t)$ shows that $C_P(t)$ satisfies

$$\frac{\mathrm{d}\ln z(t)}{\mathrm{d}C_P(z(t))} > \frac{\mathrm{d}}{\mathrm{d}C_P(z(t))}\left[\ln \frac{(QC_P(z(t)) + 1 - Q - R)(C_P(z(t)) - \beta)}{QC_P(z(t)) + 1 - Q - R - \beta}\right], \tag{108}$$

if the condition, $\frac{\partial \Omega(z(t),\mu(t))}{\partial z(t)} > 0$ holds. Let us calculate both sides of the above inequality, respectively. Here, (61) provides the left-hand side of this inequality as

$$\frac{1}{\varepsilon}\left[\left(\frac{1-\gamma}{\gamma}\right)\frac{(1-Q)(1-\beta)-R}{\{(1-Q)(1-C_P(z(t)))-R\}(C_P(z(t))-\beta)} - \frac{Q}{QC_P(z(t))+1-Q-R}\right], \tag{109}$$

and that of the right-hand side as

$$\frac{Q}{QC_P(z(t)) + 1 - Q - R} + \frac{(1-Q)(1-\beta) - R}{(QC_P(z(t)) + 1 - Q - R - \beta)(C_P(z(t)) - \beta)}. \tag{110}$$

Therefore, by rearranging these, we obtain

$$\begin{aligned}\{(1-Q)(1-\beta)-R\}\,(QC_P(z(t))+1-Q-R)\Big[(1-\gamma)(QC_P(z(t))\\ +1-Q-R-\beta)-\gamma\varepsilon\,\{(1-Q)(1-C_P(z(t))-R\}\Big] > \gamma(1+\varepsilon)Q\\ (C_P(z(t))-\beta)(QC_P(z(t))+1-Q-R-\beta)\\ \{(1-Q)(1-C_P(z(t)))-R\}\,.\end{aligned} \tag{111}$$

Let us define the left-hand side of (111) as the function $f(C_P)$ and the right-hand side as the function $g(C_P)$. The function f can be depicted as a graph on figure 6, since its coefficient of $C_P(t)^2$ is positive and $-\frac{1-Q-R}{Q}$ and $\frac{-(1-\gamma-\varepsilon\gamma)(1-Q-R)+(1-\gamma)\beta}{(1-\gamma-\varepsilon\gamma)Q+\varepsilon\gamma}$ satisfies $f(C_P) = 0$. Similarly, the function g can be depicted as a graph on figure 7. The coefficient of $C_P(t)^3$ is negative and g equals zero when $C_P(t)$ is $-\frac{1-Q-R-\beta}{Q}$, β, or $\frac{1-Q-R}{1-Q}$.

Let us compare five values of C_P that make $f = 0$ or $g = 0$. It is obvious that

$$-\frac{1-Q-R}{Q} < -\frac{1-Q-R-\beta}{Q}. \tag{112}$$

Since (58) provides $(1-Q)(1-\beta) - R > 0$, we can get

$$-\frac{1-Q-R-\beta}{Q} < \beta < \frac{1-Q-R}{1-Q}, \tag{113}$$

and

$$-\frac{1-Q-R-\beta}{Q} < \frac{-(1-\gamma-\gamma\varepsilon)(1-Q-R)+(1-\gamma)\beta}{(1-\gamma-\gamma\varepsilon)Q+\gamma\varepsilon} < \frac{1-Q-R}{1-Q}. \tag{114}$$

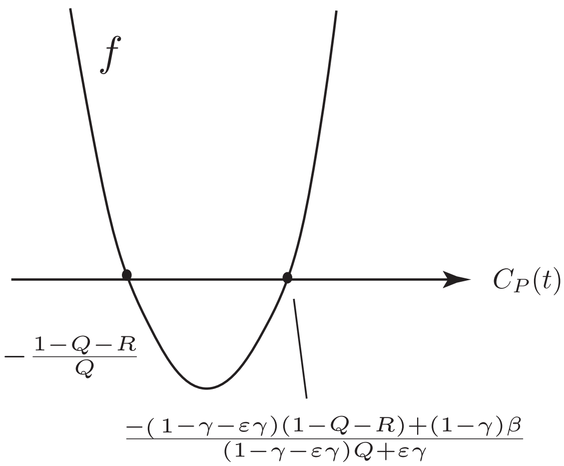

Figure 6: Function f

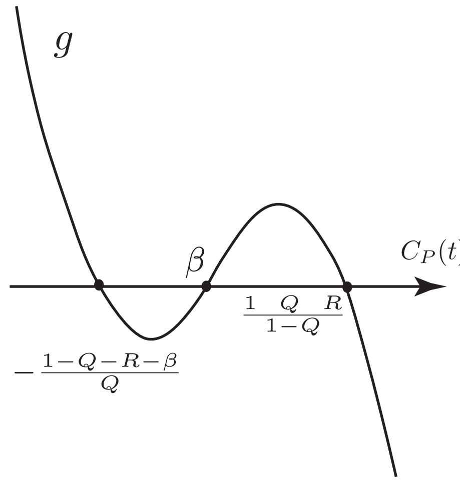

Figure 7: Function g

Moreover, we can get

$$\beta > \frac{-(1-\gamma-\gamma\varepsilon)(1-Q-R)+(1-\gamma)\beta}{(1-\gamma-\gamma\varepsilon)Q+\gamma\varepsilon} \qquad if \quad \gamma < \frac{1}{1+\varepsilon}, \tag{115}$$

$$\beta \leq \frac{-(1-\gamma-\gamma\varepsilon)(1-Q-R)+(1-\gamma)\beta}{(1-\gamma-\gamma\varepsilon)Q+\gamma\varepsilon} \qquad if \quad \gamma \geq \frac{1}{1+\varepsilon}. \tag{116}$$

With this result, figure 6 and figure 7 show that $f > g$ under the condition $\gamma > \frac{1}{1+\varepsilon}$, that is,

$$\frac{\partial\Omega(z(t),\mu(t))}{\partial z(t)} > 0, \tag{117}$$

if $\gamma \geq \frac{1}{1+\varepsilon}$. Now, we turn to the case of $\gamma \geq \frac{1}{1+\varepsilon}$. Assuming $\hat{C_P}$ as the variable satisfies $f = g$ in $\beta \leq C_p(t) \leq \frac{1-Q-R}{1-Q}$, figure 6 and figure 7 shows that $f < g$ if $\beta \leq C_P(t) \leq \hat{C_P}$ and $f > g$ if $\hat{C_P} < C_P(t) < \frac{1-Q-R}{1-Q}$. [5] This implies that

$$\frac{\partial\Omega(z(t),\mu(t))}{\partial z(t)} \begin{cases} < 0 & if \beta < C_P(t) < \hat{C}_P, \\ > 0 & if \hat{C_P} \leq C_P(t) < \frac{1-Q-R}{1-Q}. \end{cases} \tag{118}$$

By the way, it is obvious that

$$\frac{\partial\Omega(\mu(t),z(t))}{\partial\mu(t)} = \rho - \delta > 0. \tag{119}$$

Therefore, (107), (117) and (119) give the following results. If $\gamma < \frac{1}{1+\varepsilon}$,

[5] We eliminate the case that $function\ f$ and $function\ g$ have more than three intersections.

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}z(t)} < 0. \tag{120}$$

If $\gamma \geq \frac{1}{1+\varepsilon}$,

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}z(t)} \begin{cases} > 0 & if \quad \beta < C_P(t) < \hat{C}_P, \\ \leq 0 & if \quad \hat{C}_P \leq C_P(t) \leq \frac{1-Q-R}{1-Q}. \end{cases} \tag{121}$$

Having discussed the slope of the locus of $\mu(t)$, we now turn to its intercept. Introducing (61) into $\Omega(\mu(t), z(t)) = 0$ and setting $\mu(t)$ as 0 yield

$$C_P(z(t)) = \begin{cases} \frac{1-Q-R}{1-Q} & if \gamma < \frac{1}{1+\varepsilon}, \\ \beta, \frac{1-Q-R}{1-Q}, & if \gamma \geq \frac{1}{1+\varepsilon}. \end{cases} \tag{122}$$

In conclusion, (120), (121) and (122) show that [6]

- If $\gamma < \frac{1}{1+\varepsilon}$, the $\dot{\mu}(t) = 0$ locus has a negative slope and asymptotically approaches the horizontal and vertical axes.
- If $\gamma \geq \frac{1}{1+\varepsilon}$, the $\dot{\mu}(t) = 0$ locus is an inverted-U shape going through the origin and asymptotically approaching the horizontal axis.

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