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Abstract

This study constructs an overlapping-generations model with endogenous fertility, mortality, and R&D activities. We demonstrate that the model explains the observed fertility dynamics of developed countries. When the level of per capita wage income is either low or high, an increase in such income raises the fertility rate. When the level of per capita wage income is in the middle, an increase in such income decreases the fertility rate. The model also predicts the observed relationship between population growth and innovative activity. At first, both the rates of population growth and technological progress increase, that is, there is a positive relationship. Thereafter, the rate of population growth decreases but the rate of technological progress increases, showing a negative relationship.

JEL classification: D91, J13, O10

Keywords: Fertility, Mortality, R&D

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1 Introduction

Over the past two centuries, many developed countries have experienced changes in fertility patterns. Galor (2005) identifies, what he calls, two distinct regimes *the post-Malthusian regime* and *the sustained growth regime.* In the post-Malthusian regime, technological progress showed a marked acceleration and the rising per capita income has a positive impact on fertility. That is, this period saw the positive Malthusian effect of per capita income on fertility.[1] In the sustained growth regime, although technological progress continues, the increase in per capita income has a negative impact on fertility.

Empirical studies in recent years have found that some developed countries are experiencing a reversal of the fertility pattern. For example, Myrskylä et al. (2009) analyze the cross-sectional relationship between total fertility rates (TFR) and the Human Development Index (HDI) in 1975 and 2005.[2] They look at 2005 and find that a higher HDI implies a higher fertility rate in sufficiently high HDI countries. Additionally, Luci-Greulich and Thévenon (2014) analyze the relationship between TFR and per capita GDP by using OECD countries from 1960 to 2007. They find results that are similar to those found by Myrskylä et al. (2009): a higher per capita GDP implies a higher fertility rate in several highly developed countries.[3]

In addition to fertility dynamics, Strulik et al. (2013) explore historical and empirical evidence to show that 'the erstwhile positive correlation between population growth and innovative activity turns negative during economic development.' They construct a unified growth model that incorporates R&D-based innovation to explain this "population-productivity reversal." The model proposes the following: At first, a high fertility rate promotes innovative activity; that is, the relation between population growth and innovative activity is positive. Subsequently, the emergence of mass education reduces the fertility rate. However, because the accumulation of human capital outweighs the effect of declining fertility, there is both R&D-based innovation and economic growth. Therefore,

[1] Galor (2005) refers the period before the post-Malthusian regime as *the Malthusian epoch.* In this epoch, although there is a positive relationship between per capita income and fertility, the average growth rate of per capita income is very low. However, our study does not focus on the Malthusian epoch and the source of take-off from the Malthusian epoch.

[2] HDI measures human development and is composed of per capita gross domestic product (GDP), life expectancy, and school enrolment.

[3] Hirazawa and Yakita (2017) also hint at a similar relationship between TFR and per capita GDP.

the relation between population growth and innovative activity becomes negative in the course of economic development.

The purpose of this study is to propose a simple model which explains this fertility dynamics of developed countries and provides an alternative mechanism to explain the population-productivity reversal. To examine the fertility dynamics, we construct a three-period-lived overlapping-generations (OLG) model with endogenous fertility and mortality. Furthermore, we integrate our model with the R&D-based growth model of Jones (1995). The two defining characteristics of our model are as follows: First, we consider two types of rearing costs: goods and time. When parents raise children, they must buy some final goods and sacrifice their time. Second, following Hirazawa and Yakita (2017), we assume that a young individual probably survives to old age and an increase in per capita wage income affects this probability positively. As discussed in Cutler et al. (2006), in developed countries, improved nutrition, vaccinations, and medical treatments are important factors in declining mortality during the course of economic development.[4] In our model, the only income earned during one's youth is wage income. Thus, a higher wage income can be taken as implying better nutrition. Furthermore, in our model, an increase in per capita wage income is induced by technological progress; that is, we can consider a low-income (high-income) economy as an economy with low (high) knowledge of health and technology.

Using this framework, we obtain the following results: When per capita wage income is low, an increase in such income leads to an increase in the number of children people have (this is the "income-effect"). When per capita income is at a middle level, an increase in income leads to a fall in fertility rates because the effect of an increasing probability of survival into old age comes into play. When per capita wage income increases, parents increase their savings to provide for consumption in their old age and decrease the number of children they have. When per capita wage income is high, the survival probability in old age reaches a sufficiently high level and increasing the survival probability in old age becomes smooth. Thus, when per capita wage income increases, the income-effect once

[4]Cutler et al. (2006) also emphasize the importance of public health. For example, Chakraborty (2004) incorporates this aspect by assuming that the probability of survival into old age depends on public health expenditure. However, for simplicity, we do not consider this aspect.

again comes into play; parents increase the number of children they have. In this study, an increase in per capita wage income is induced by technological progress. Hence, the economy experiences this fertility dynamics in the course of economic development.

With regard to the population-productivity reversal, our model makes the following predictions. At first, an increase in the fertility rate increases the size of the working-age population, which accelerates technological progress because the aggregate savings increase; that is, there is a positive relationship between population growth and innovative activity. Thereafter, an increase in the probability of survival into old age reduces the fertility rate and raises per capita savings. Although declining fertility rate has a negative effect on aggregate savings, the increase in per capita savings outweighs this negative effect. As a result, aggregate savings increase, which promotes technological progress and economic growth. Thus, our model predicts the population-productivity reversal as discussed in Strulik et al. (2013). In addition, our theoretical predictions are consistent with empirical findings (e.g., Bloom et al., 2003; Gehringer and Prettner, 2014).[5]

The remainder of this paper is organized as follows. Section 2 describes the relevant literature. Section 3 establishes the model. Section 4 investigates the relationship between fertility and per capita wage income. Section 5 examines the equilibrium dynamics of the economy, the sustainability of economic growth, and the balanced growth path (BGP). Section 6 investigates the model numerically. Section 7 concludes the paper.

2 Literature review

In this section, we review the relevant literature from the following two points of view:

2.1 Fertility dynamics

de la Croix and Licandro (2013) construct an OLG model with endogenous fertility, longevity, and education. The model explains the observed fertility dynamics from the Malthusian and post-Malthusian regimes to the sustained growth regime. Our study is different from theirs in the following respects. First, they do not propose the rebound of

[5] Bloom et al. (2003) find that an increase in life expectancy leads to higher savings rates. Gehringer and Prettner (2014) find a positive effect of rising longevity on total factor productivity.

the fertility rate. Second, they assume that longevity is extended by parents' investment in a child's physical development. Third, in their model, a demographic transition is induced by a Beckerian child quantity-quality trade-off. Finally, they do not consider R&D activities, or in other words, economic growth is not driven by R&D activities.

Hirazawa and Yakita (2017) attempt to explain the observed facts that an increase in per capita income raises both life expectancy and the fertility rate in several highly developed countries. They construct an OLG model with endogenous elderly labor supply and fertility. Their model proposes that rising longevity increases elderly labor supply, which increases per capita wage income in old age. Hence, young individuals decrease savings and raise their consumption and the number of children they have. Our study is different in the following respects. First, our model does not allow for elderly labor supply and our result is based on the notion of an increase in per capita wage income when people are young. Second, their model does not propose the initial increase in the fertility rate; that is, they do not consider the post-Malthusian regime. Third, in their model, economic growth is not driven by R&D activities.

Maruyama and Yamamoto (2010) integrate endogenous fertility into the Grossman and Helpman (1991) type of the variety expansion growth model and explain a new mechanism for fertility decline. In their model, variety expansion derived from innovation induces parents to raise consumption expenditure on differentiated goods and reduce the number of children they have. Hence, their result of fertility decline is derived differently from our mechanism and the fertility rate decreases monotonically in the course of economic development.

2.2 Population-productivity reversal and the effects of demographic change on economic growth

As mentioned in the Introduction, Strulik et al. (2013) propose a model that explains the population-productivity reversal. Our study is different in the following respects: First, in their model, a demographic transition is induced by the Beckerian child quantity-quality trade-off and the fertility rate decreases monotonically. Second, they do not consider endogenous mortality. Third, their mechanism of population-productivity reversal is different from ours.

This study is also related to studies showing that the growth rate can increase even under declining fertility in R&D-based growth models (e.g, Prettner and Trimborn, 2016; Hashimoto and Tabata, 2016). Prettner and Trimborn (2016) integrate the OLG model of Blanchard (1985) with the R&D-based growth model of Jones (1995) and examine the growth effects of demographic change during the transitional dynamics. In their model, demographic change is represented by an exogenous decrease in mortality rate and an exogenous reduction in fertility. They show that falling mortality increases savings, which outweighs the effect of declining fertility and promotes technological progress and economic growth in the short and medium run. Although their model describes the same mechanism as ours to show how these effects play out, they do not investigate the growth effect of demographic change under endogenous fertility and mortality.[6] Hashimoto and Tabata (2016) construct an OLG model with R&D-based growth and endogenous fertility and education. In their model, young individuals choose the number of children they have, the amount of education their children receive, and the amount of investment in their own education. They show that an increase in the probability of survival into old age reduces the fertility rate but raises young individuals' investment in their own education. When the probability of survival into old age is sufficiently low (high), the effect of human capital accumulation outweighs (falls short of) the effect of declining fertility; that is, an increase in the probability of survival into old age enhances (worsens) technological progress and economic growth. Their result is based on a Ben-Porath mechanism which our model does not consider.[7] In addition, they do not show the population-productivity reversal during the transitional dynamics.

3 Model

We consider a three-period overlapping-generations model following Diamond (1965). An individual lives for three periods–childhood, youth, an old age. In childhood, they do not make any economic decision. In youth, every individual supplies labor to the market and

[6]Prettner (2013) integrates the OLG model of Blanchard (1985) with the R&D-based growth models of Romer (1990) and Jones (1995) and investigates the effects of demographic change on long-run growth. However, he does not take into account the transitional dynamics.

[7]The Ben-Porath mechanism works as follows: rising life expectancy prolongs active working lives, which has a positive impact on investment in human capital.

earns labor income. Young individuals also raise children. From youth to old age, an individual dies with a probability $1 - \lambda_t \in [0, 1)$. That is, the probability of an individual surviving to the next period is $\lambda_t \in (0, 1]$. In old age, individuals retire, consume their savings, and have no bequest motive.

3.1 Individuals

Each young individual at period t maximizes the following utility:

$$U_t = \log c_t^y + \beta\lambda_t \log c_{t+1}^o + \gamma \log n_t, \tag{1}$$

where c_t^y is their consumption when young, c_{t+1}^o is their consumption when they are old, n_t is the number of children each young individual has, β is the subjective discount rate, and γ is the weight on utility derived from the number of children an individual has.[8] We assume that each young individual has one unit of time endowment. Raising children involves two kinds of costs: *time* and *goods*. When each young individual raises n_t units of children, he or she has to incur δn_t units of final goods and ρn_t units of time. Let w_t be the wage rate. The disposable working income becomes $(1 - \rho n_t)w_t$. Additionally, we normalize the price of final goods to one. Thus, the budget constraint for each young individual can be expressed as,

$$c_t^y + s_t + \delta n_t = (1 - \rho n_t)w_t, \quad \delta > 0, \;\; \rho \in (0, 1), \tag{2}$$

where s_t represents the savings in youth. The disposable working income $(1 - \rho n_t)w_t$ is divided among consumption, savings, and child-rearing expenses.

As in Yaari (1965) and Blanchard (1985), we assume that there are actuarially fair insurance companies and the annuities market is perfectly competitive. At the end of the youth period, each individual deposits his or her savings with an insurance company. The company invests them and pays the return on that investment to the surviving insured old individuals. Owing to perfect competition, the rate of return on the annuities becomes $(1 + r_{t+1})/\lambda_t$, where r_{t+1} is the interest rate in period $t + 1$. Hence, the consumption of

[8] Following Galor (2005) and Strulik et al. (2013), we assume that n_t is the number of surviving children because we do not want to consider child mortality, for the sake of analytical simplicity.

the surviving old individuals becomes

$$c^o_{t+1} = \frac{1 + r_{t+1}}{\lambda_t} s_t. \tag{3}$$

By using (1), (2), and (3), the intertemporal utility maximization problem can be stated as,

$$\max_{s_t, n_t} \quad \log\Big[(1 - \rho n_t)w_t - \delta n_t - s_t\Big] + \beta\lambda_t \log\left(\frac{1 + r_{t+1}}{\lambda_t} s_t\right) + \gamma \log n_t.$$

Solving this maximization problem, we obtain

$$s_t = \frac{\beta\lambda_t w_t}{1 + \beta\lambda_t + \gamma}, \tag{4a}$$

$$n_t = \frac{\gamma w_t}{(1 + \beta\lambda_t + \gamma)(\rho w_t + \delta)}. \tag{4b}$$

3.2 Final goods sector

Following Romer (1990) and Jones (1995), we consider three production sectors: a final goods sector, an intermediate goods sector, and an R&D sector. We assume that the final goods market is perfectly competitive. The production technology of the final good is given by

$$Y_t = L^{1-\alpha}_{Y,t} \int_0^{A_t} x^{\alpha}_{j,t} dj, \qquad 0 < \alpha < 1, \tag{5}$$

where Y_t, $L_{Y,t}$, A_t, and $x_{j,t}$ represent the output level, labor input, the variety of intermediate goods, and the input of the jth intermediate good, respectively. Because markets are perfectly competitive, factor prices equal their marginal products:

$$w_t = (1 - \alpha)L^{-\alpha}_{Y,t} \int_0^{A_t} x^{\alpha}_{j,t} dj = (1 - \alpha)\frac{Y_t}{L_{Y,t}}, \tag{6a}$$

$$q_{j,t} = \alpha L^{1-\alpha}_{Y,t} x^{\alpha-1}_{j,t}, \tag{6b}$$

where $q_{j,t}$ is the price of the jth intermediate good. From (6b), the demand function for intermediate good j is given as,

$$x_{j,t} = \left(\frac{\alpha}{q_{j,t}}\right)^{\frac{1}{1-\alpha}} L_{Y,t}. \tag{7}$$

3.3 Intermediate goods sector

We make the following assumptions about the intermediate goods sector: Each differentiated intermediate good is produced by a single firm because an intermediate good is infinitely protected by a patent. That is, the intermediate goods market is monopolistically competitive. Firm entering the intermediate goods market must acquire new blueprints. They issue shares to raise funds.

We further assume that one unit of labor input produces one unit of a differentiated intermediate good. Therefore, a firm manufacturing an intermediate good j (firm j) maximizes its own profit, $\pi_{j,t} = q_{j,t}x_{j,t} - w_t x_{j,t}$, subject to the demand function (7). Solving this maximization problem, we obtain the following price charged by firm j:

$$q_{j,t} = q_t = \frac{1}{\alpha} w_t. \tag{8}$$

Thus, all intermediate goods are priced equally. From (7) and (8), the output and monopoly profits of firm j are given by,

$$x_{j,t} = x_t = \left(\frac{\alpha^2}{w_t}\right)^{\frac{1}{1-\alpha}} L_{Y,t}, \tag{9a}$$

$$\pi_{j,t} = \pi_t = \frac{1-\alpha}{\alpha} w_t x_t. \tag{9b}$$

Because the market value of new blueprints is equal to the sum of the discounted present value of the profit flow after period t, we can express it as,

$$v_t = \sum_{\tau=t+1}^{\infty} \frac{\pi_\tau}{\prod_{u=t+1}^{\tau}(1+r_u)},$$

where v_t is the market values of new blueprints. Using this, we obtain the following no-arbitrage condition:

$$r_{t+1}v_t = \pi_{t+1} + v_{t+1} - v_t.$$

The return on stock purchase is equal to the sum of dividend π_{t+1} and capital gains or losses $v_{t+1} - v_t$.

3.4 R&D sector

Next, we consider the technology involved in developing a new intermediate good. R&D activities require labor inputs and the R&D sector is also perfectly competitive. New blueprints developed by R&D activities are sold to intermediate good firms. We assume the following R&D technologies:

$$A_{t+1} - A_t = \theta_t L_{A,t}, \tag{10}$$

where θ_t and $L_{A,t}$ represent the productivity of R&D and the amount of labor devoted to R&D. $A_{t+1} - A_t$ measures new intermediate goods. Following Romer (1990), Grossman and Helpman (1991), and Jones (1995), we assume the productivity of R&D depends on existing knowledge produced through previous R&D activities. We assume the following productivity for R&D:

$$\theta_t = A_t^{\phi}. \tag{11}$$

Thus, A_t also represents the stock of technological knowledge. As discussed in Jones (1995), we consider a parameter range where $0 < \phi < 1$. The profit of R&D firms is given by $\pi_t^A = v_t(A_{t+1} - A_t) - w_t L_{A,t} = (v_t \theta_t - w_t) L_{A,t}$. R&D activities are perfectly competitive. Hence, when R&D is undertaken, the following equality holds:

$$v_t = \frac{w_t}{A_t^{\phi}}. \tag{12}$$

On the other hand, when $v_t \theta_t < w_t$, R&D is not conducted because $\pi_t^A < 0$.

3.5 Market clearing condition and dynamic system

Labor is used for production of final and intermediate goods and for R&D activities. As mentioned above, the labor supply of each individual is $1 - \rho n_t$. Let N_t be the population size of young individuals at period t. The labor market clearing condition becomes

$$L_{Y,t} + A_t x_t + L_{A,t} = (1 - \rho n_t) N_t. \tag{13}$$

The aggregate savings of young individuals in period t must be used for investment in R&D $(A_{t+1} - A_t)v_t$ or for the purchase of existing stocks $v_t A_t$. Hence, the asset market clearing condition is given by,

$$(A_{t+1} - A_t)v_t + A_t v_t = A_{t+1} v_t = s_t N_t. \tag{14}$$

Finally, we consider the equilibrium condition for the final goods market. The final goods are used for consumption and for raising children. The final goods-market clearing condition is as follows:

$$Y_t = c_t^y N_t + c_t^o \lambda_{t-1} N_{t-1} + \delta n_t N_t.$$

Note, because the probability of survival of old individuals in period t is λ_{t-1}, the population size of old individuals in period t becomes $\lambda_{t-1} N_{t-1}$.

Next, we characterize the dynamic system of this economy. Substituting (9a) into (5) yields

$$Y_t = \left(\frac{\alpha^2}{w_t} \right)^{\frac{\alpha}{1-\alpha}} A_t L_{Y,t}. \tag{15}$$

Using (6a) and (15), we obtain,

$$w_t = \hat{\alpha} A_t^{1-\alpha}, \tag{16}$$

where $\hat{\alpha} \equiv \alpha^{2\alpha}(1-\alpha)^{1-\alpha}$. Let us define the growth rate of A_t as $g_{A,t} \equiv (A_{t+1} - A_t)/A_t$. (4a), (12), and (14) yield

$$1 + g_{A,t} = \frac{s_t N_t}{w_t} A_t^{\phi-1} = \frac{\beta \lambda_t}{1 + \beta \lambda_t + \gamma} N_t A_t^{\phi-1}. \tag{17}$$

Following Hirazawa and Yakita (2017), we assume that the probability of survival rate λ_t depends on the wage rate w_t, that is, $\lambda_t = \lambda(w_t)$. (16) shows there is a one-to-one relationship between the wage rate w_t and the stock of technological knowledge A_t. Therefore, we can say that the probability of the survival rate λ_t depends on the stock of technological knowledge.

Because $A_{t+1} > A_t$ ($A_{t+1} = A_t$) corresponds to $g_{A,t} > 0$ ($g_{A,t} = 0$) and the dynamics of the young individual population size becomes $N_{t+1} = n_t N_t$, equations (4b), (16), and (17) characterize the dynamic system with respect to A_t and N_t.[9] Note that both A_t and N_t in period t are predetermined variables.

[9]In this study, $A_{t+1} < A_t$ does not hold because there is no product obsolescence.

4 Relationship between fertility and per capita wage income

In this section, we examine the relationship between n_t and w_t. First, following Hirazawa and Yakita (2017), we specify $\lambda(w_t)$ as follows:

$$\lambda(w_t) = \frac{\nu}{1 + \chi e^{-\psi w_t}}, \tag{18}$$

where $\nu \in (0, 1]$ and χ, $\psi > 0$. Under this functional form, we have

$$\lambda'(w_t) = \frac{\nu \psi \chi e^{-\psi w_t}}{(1 + \chi e^{-\psi w_t})^2} > 0, \tag{19a}$$

$$\lambda''(w_t) = \frac{\nu \psi^2 \chi e^{-\psi w_t}}{(1 + \chi e^{-\psi w_t})^3} \left(\chi e^{-\psi w_t} - 1\right). \tag{19b}$$

$$\lim_{w_t \to 0} \lambda(w_t) \equiv \underline{\lambda} = \frac{\nu}{1 + \chi}, \tag{19c}$$

$$\lim_{w_t \to +\infty} \lambda(w_t) \equiv \overline{\lambda} = \nu. \tag{19d}$$

From (19b), we obtain the following properties of $\lambda(w_t)$. If $0 < \chi \leq 1$, $\lambda''(w_t) < 0$ holds. That is, $\lambda(w_t)$ is concave for any w_t. If $\chi > 1$, $\lambda''(w_t) \gtreqless 0$ holds for any $w_t \lesseqgtr w_\lambda$, where $w_\lambda \equiv \frac{1}{\psi} \log \chi$. When w_t is either low or high, an increase in w_t raises λ_t but this increase is not large. On the other hand, when w_t is at a middle level, an increase in w_t increases λ_t substantially.

Next, using (4b), we differentiate n_t with respect to w_t:

$$\frac{\partial n_t}{\partial w_t} = \frac{\gamma \delta}{[1 + \beta \lambda(w_t) + \gamma] (\rho w_t + \delta)^2} - \frac{\gamma \beta \lambda'(w_t) w_t}{[1 + \beta \lambda(w_t) + \gamma]^2 (\rho w_t + \delta)}. \tag{20}$$

The first term shows that an increase in per capita wage income increases the fertility rate. Because we normalize the price of final goods to one, an increase in the wage rate decreases the relative price of final goods, which reduces the child-rearing costs by final goods and raises the number of children. We call this the "income-effect." The second term shows that a rise in per capita wage income raises the probability of surviving into old age, which decreases the fertility rate and increases the savings meant for consumption in old age. As shown in Appendix A, we clarify which one is larger, and thus state the following proposition:

Proposition 1.

1. *If* $\chi > \max\left\{\frac{\psi\delta-2\rho}{\psi\delta+2\rho}, \tilde{\chi}\right\}$*, an increase in* w_t *raises* n_t *for any* $0 < w_t < \bar{w}_1$ *or* $w_t > \bar{w}_2$*, while an increase in* w_t *reduces* n_t *for any* $\bar{w}_1 < w_t < \bar{w}_2$*, where* $\tilde{\chi}$*,* $\bar{w}_1$*, and* $\bar{w}_2$ *are defined in Appendix A.*

2. *If* $0 < \chi < \frac{\psi\delta-2\rho}{\psi\delta+2\rho}$ *or* $0 < \chi < \tilde{\chi}$*, an increase in* w_t *raises* n_t *for all* w_t*.*

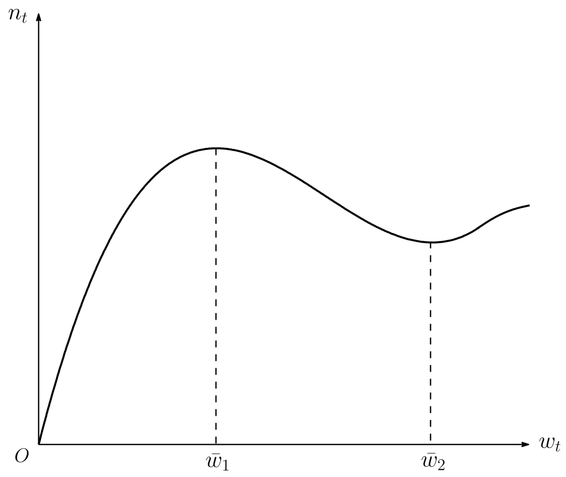

Figure 1: Relationship between n_t and w_t.

In addition, by using (4b) and (19d), n_t converges to the following value:

$$\lim_{w_t \to \infty} n_t \equiv n^* = \frac{\gamma}{(1+\beta\nu+\gamma)\rho}.$$

Figure 1 illustrates the relationship between n_t and w_t, corresponding to the first case of Proposition 1. As discussed above, an increase in per capita wage income has opposite effects on fertility. When per capita wage income is low, the income-effect is relatively large. That is, rising per capita wage income has a positive impact on the number of children. When per capita wage income is at a middle level, the effect of an increasing probability of survival into old age comes into play. Thus, when per capita wage income increases, parents increase their savings for consumption in old age and decrease consumption in their youth and the number of children they have. When per capita wage income is high, the probability of survival into old age becomes smooth. Thus, when per capita wage income increases, the income-effect is higher, or in other words, parents once again increase the number of children they have.

At the end of this section, we mention the probability of survival rate $\lambda(w_t)$. Intuitively, it seems that the first case of Proposition 1 does not hold if $\lambda(w_t)$ is a concave function. However, in Section 6, we provide a numerical example and show that this case holds even under a concave function $\lambda(w_t)$. Moreover, our result of the fertility rebound is derived from the following: the survival probability in old age attains a sufficiently high level and increasing the survival probability in old age becomes smooth. To show the empirical plausibility, Figure 2 presents the ratio of survivors at different age points in high-income countries from 1950 to 2100. In addition, Table 1 shows the average rate of increase for the data presented in Figure 2. In the first decade of 2000, the ratio of survivors at ages 50, 60, and 70 is sufficiently high, and the average rate of increase begins to decline from 2015 to 2100.

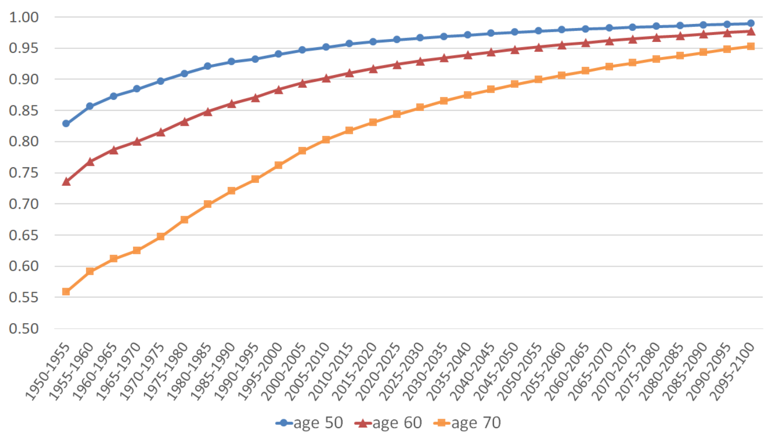

Figure 2: Ratio of survivors at each age in high-income countries.

Source: World Population Prospects: The 2017 Revision (United Nations).

Note: *World Population Prospects* presents the number of survivors at different age points assuming that there are 100,000 people at birth. We calculate the ratio of survivors, using this data.

	age 50	age 60	age 70
1950-1985	0.0164	0.0227	0.0370
1985-2015	0.0057	0.0106	0.0240
2015-2100	0.0019	0.0040	0.0086

Table 1: Average rise rate of the data presented in Figure 2

5 Equilibrium path

5.1 Phase diagram

We consider a phase diagram in the (A_t, N_t) plane. For our purpose, we impose $\chi > \frac{\psi\delta-2\rho}{\psi\delta+2\rho}$ and $\chi > \tilde{\chi}$, which corresponds to the first case of Proposition 1. To investigate whether $A_{t+1} > A_t$ or $A_{t+1} = A_t$ at each point of the (A_t, N_t) plane, we set $g_{A,t} = 0$ in (17) as follows:

$$N_t = \frac{1+\beta\lambda(w_t)+\gamma}{\beta\lambda(w_t)} A_t^{1-\phi} \equiv \Gamma(A_t). \tag{21}$$

We have $A_{t+1} > A_t$ above this locus and $A_{t+1} = A_t$ below this locus. Hereafter, we refer to the region where $A_{t+1} > A_t$ as *R&D region* and the region where $A_{t+1} = A_t$ as *no R&D region*. Using (16), we differentiate $\Gamma(A_t)$ with respect to A_t as follows:

$$\begin{aligned}\Gamma'(A_t) &= -\frac{(1+\gamma)\lambda'(w_t)}{\beta\left[\lambda(w_t)\right]^2} A_t^{1-\phi}\frac{\partial w_t}{\partial A_t} + \frac{1+\beta\lambda(w_t)+\gamma}{\beta\lambda(w_t)}(1-\phi)A_t^{-\phi}, \\ &= \frac{A_t^{-\phi}}{\beta\lambda(w_t)}\left\{-(1-\alpha)(1+\gamma)\frac{\lambda'(w_t)w_t}{\lambda(w_t)} + (1-\phi)\left[1+\beta\lambda(w_t)+\gamma\right]\right\}. \end{aligned} \tag{22}$$

The first term of (22) implies that an increase in A_t increases the probability of surviving $\lambda(w_t)$, from (16) and (19a). This increases the per capita savings, and thus, the no R&D region shrinks. The second-term of (22) implies that an increase in A_t reduces the marginal productivity of the R&D sector derived from the knowledge spillover, which enlarges the no R&D region. If the first-term's effect is larger, $\Gamma'(A_t) < 0$ holds. On the other hand, if the second-term's effect is larger, $\Gamma'(A_t) > 0$ holds. As shown in Appendix B, we clarify which one is larger. If $0 < \chi < \hat{\chi}$, we obtain

$$\Gamma'(A_t) > 0 \quad for\ all\ A_t,$$

where $\hat{\chi}$ is as defined in Appendix B. We depict this case in the left-side panel of Figure 3. The shaded area represents the no R&D region. In contrast, if $\chi > \hat{\chi}$, we obtain

$$\Gamma'(A_t) > 0 \quad when \quad 0 < A_t < A_{\Gamma,1},\, A_t > A_{\Gamma,2},$$
$$\Gamma'(A_t) < 0 \quad when \quad A_{\Gamma,1} < A_t < A_{\Gamma,2},$$

where $A_{\Gamma,1}$ and $A_{\Gamma,2}$ are defined as $\Gamma'(A_{\Gamma,1}) = 0$ and $\Gamma'(A_{\Gamma,2}) = 0$. We depict this case in the right-side panel of Figure 3. As discussed below, the properties of the phase diagram remain the same in both cases.

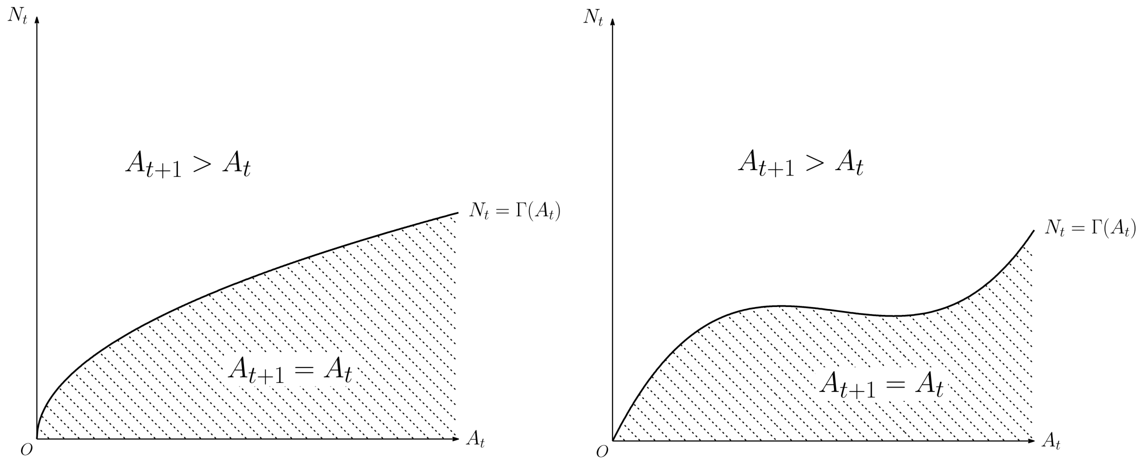

Figure 3: $A_{t+1} > A_t$ and $A_{t+1} = A_t$ regions on the (A_t, N_t) plane.

Next, we examine whether $N_{t+1} > N_t$ or $N_{t+1} < N_t$ at each point of the (A_t, N_t) plane. As discussed below, if n^* is less than 1, the population size of young individuals continues to decrease in the long run, and as a result, the economy definitely falls into the no R&D region in the long run. To ensure sustainable growth, we impose the following assumption:

Assumption 1. $n^* = \dfrac{\gamma}{(1 + \beta\nu + \gamma)\rho} > 1.$

We show in Appendix C the following lemma:

Lemma 1. *Suppose that* $\chi > \max\left\{\frac{\psi\delta-2\rho}{\psi\delta+2\rho}, \tilde{\chi}\right\}$ *and Assumption 1 holds.*

1. *If* $\gamma\bar{w}_2 > [1+\beta\lambda(\bar{w}_2)+\gamma](\rho\bar{w}_2+\delta)$*, the following holds.* $N_{t+1} > N_t$ *holds when* $w_t > \hat{w}_1$*, while* $N_{t+1} < N_t$ *holds when* $0 < w_t < \hat{w}_1$*, where* $\hat{w}_1$ *is defined as shown in Appendix C.*

2. *If* $\gamma\bar{w}_1 > [1+\beta\lambda(\bar{w}_1)+\gamma](\rho\bar{w}_1+\delta)$ *and* $\gamma\bar{w}_2 < [1+\beta\lambda(\bar{w}_2)+\gamma](\rho\bar{w}_2+\delta)$*, the following holds.* $N_{t+1} > N_t$ *holds when* $\hat{w}_1 < w_t < \hat{w}_2$ *or* $w_t > \hat{w}_3$*, while* $N_{t+1} < N_t$ *holds when* $0 < w_t < \hat{w}_1$ *or* $\hat{w}_2 < w_t < \hat{w}_3$*, where* $\hat{w}_2$ *and* $\hat{w}_3$ *are as defined in Appendix C.*

Figure 4 uses the results obtained so far to illustrate the relationship between n_t and w_t and phase diagrams on the (A_t, N_t) plane. The first (second) case of Lemma 1 corresponds to the left-side (right-side) panels in Figure 4. Here, we define $\bar{A}_j$ and $\hat{A}_j$ as $\bar{w}_j = \hat{\alpha}\bar{A}_j^{1-\alpha}$ and $\hat{w}_j = \hat{\alpha}\hat{A}_j^{1-\alpha}$ $(j = 1, 2, 3)$. As shown in Appendix C, the definition of $\hat{w}_j$ $(j = 1, 2, 3)$ implies that $\hat{A}_1 < \bar{A}_1$ and $\hat{A}_2 < \bar{A}_2 < \hat{A}_3$. Since A_t and N_t are predetermined variables at time t, the initial state of the economy is given by point (A_0, N_0) on the (A_t, N_t) plane. If the economy is in the no R&D region (the lower side of the $N_t = \Gamma(A_t)$ locus), A_t is constant, and as a result, the fertility rate n_t is also constant from (4b), (16), and (19a). In this case, when $N_{t+1} > N_t$, the economy can reach the R&D region in a finite time period. On the other hand, when $N_{t+1} < N_t$, the economy falls into a trap. For our research purposes, we assume that the initial stock of technological knowledge is given by $A_0 \in (\hat{A}_1, \bar{A}_1)$ in the discussions that follow. We describe the transition dynamics in the R&D region (the upper side of $N_t = \Gamma(A_t)$ locus).

First, we consider the first case of Lemma 1. Under $A_0 \in (\hat{A}_1, \bar{A}_1)$, both A_t and N_t continue to increase, that is, the economy is on a sustainable growth path. With regard to the fertility dynamics, the fertility rate decreases with time under $\bar{A}_1 < A_t < \bar{A}_2$, and increases with time under $\hat{A}_1 < A_t < \bar{A}_1$ or $A_t > \bar{A}_2$.

Next, we consider the second case of Lemma 1. If N_0 is sufficiently small, as represented by Q_1 in the lower-right panel of Figure 4, R&D is undertaken and both the population size of young individuals and the fertility rate increase, at first. After exceeding the vertical line $\bar{A}_1$, the fertility rate begins to fall. After exceeding the vertical line

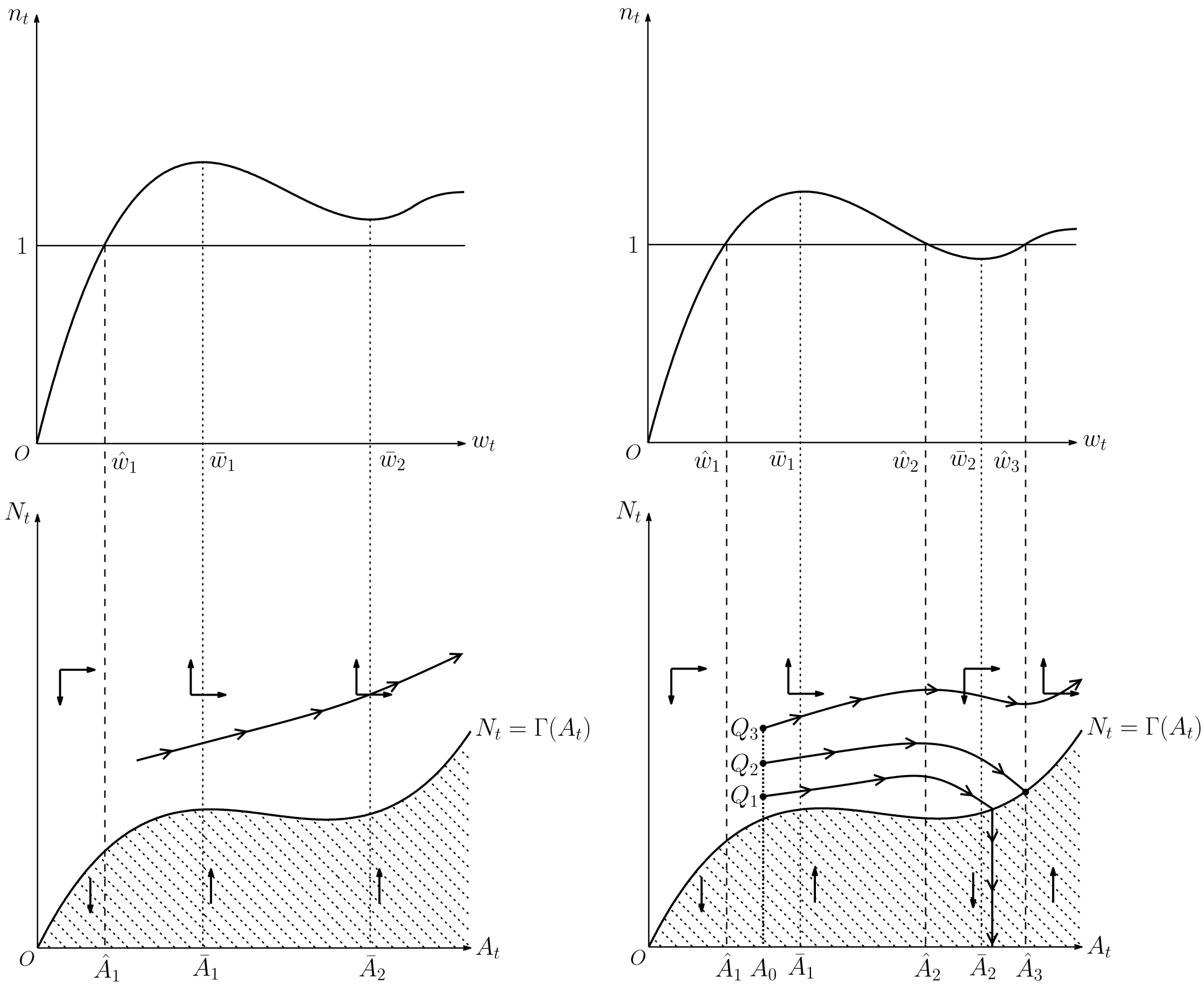

Figure 4: Relationship between n_t and w_t and phase diagram on the (A_t, N_t) plane.

$\hat{A}_2$, the fertility rate drops to below 1; the population size of young individuals decreases with time, and as a result, the economy falls into the no R&D region. If N_0 is a specific value as represented by Q_2, the economy follows a path that is similar to that of the initial point Q_1. In this case, the economy converges to the point $(\hat{A}_3, \hat{N}_3)$, where $\hat{N}_3 \equiv \Gamma(\hat{A}_3)$. At this point, the fertility rate is 1 and since there is no R&D, the economy remains at this point. If N_0 is sufficiently large as represented by Q_3, the economy initially follows a path similar to the one detailed above. However, after exceeding the vertical line $\bar{A}_2$, the fertility rate increases with time. Hence, the economy goes beyond the vertical line $\hat{A}_3$ and once again, the population size of young individuals increases. That is, the economy attains a sustainable growth path.[10] In summary, we can state that the path converging to the point $(\hat{A}_3, \hat{N}_3)$ is a *threshold curve* that sustains economic growth. If (A_0, N_0) falls into the lower side of this arm, the economy falls into a trap in the long run. These results are summarized in the following lemma:

Lemma 2. *Suppose that* $\chi > \max\left\{\frac{\psi\delta-2\rho}{\psi\delta+2\rho}, \tilde{\chi}\right\}$*,* $A_0 \in (\hat{A}_1, \bar{A}_1)$*, and Assumption 1 holds.*

1. *If* $\gamma\bar{w}_2 > [1 + \beta\lambda(\bar{w}_2) + \gamma](\rho\bar{w}_2 + \delta)$*, the economy can be on a sustainable growth path.*

2. *If* $\gamma\bar{w}_1 > [1 + \beta\lambda(\bar{w}_1) + \gamma](\rho\bar{w}_1 + \delta)$ *and* $\gamma\bar{w}_2 < [1 + \beta\lambda(\bar{w}_2) + \gamma](\rho\bar{w}_2 + \delta)$*, there is a threshold curve that sustains economic growth. This curve is an inverted-U on the* (A_t, N_t) *plane.*

When the economy is in the region where $A_t \in (\hat{A}_1, \bar{A}_1)$ and $N_t > \Gamma(A_t)$, technological progress is triggered by R&D activities. As mentioned earlier, both per capita wage income and the fertility rate increase with time. Therefore, this region corresponds to the post-Malthusian regime. After exceeding the vertical line $\bar{A}_1$, the economy enters the sustained growth regime; that is, the economy experiences declining fertility although per capita wage income continues to rise. As in Lemma 2, a reduction in the fertility rate may lead the economy into a trap. However, if the initial population size of young individuals is sufficiently large, sustainable growth can be attained. Along a sustainable

[10] If we do not impose Assumption 1, $N_{t+1} < N_t$ holds for all $A_t > \hat{A}_2$. In this case, the economy definitely falls into the no R&D region. Thereafter, A_t and n_t become constant and n_t falls to below 1. Since $N_{t+1} < N_t$, the economy cannot get out of the trap.

growth path, if the economy exceeds the vertical line $\bar{A}_2$, the fertility rate rebounds, and the economy converges to the BGP . In summary, we can state the following proposition:

Proposition 2. *Suppose that* $\chi > \max\left\{\frac{\psi\delta-2\rho}{\psi\delta+2\rho}, \tilde{\chi}\right\}$*,* $A_0 \in (\hat{A}_1, \bar{A}_1)$*, and Assumption 1 hold. If* N_0 *is sufficiently large, the economy can be on a sustainable growth path. Along the sustainable growth path, the fertility rate increases at first and then starts to fall. Subsequently, the fertility rate rebounds and converges to* n^**.*

5.2 Growth rates

So far we have discussed the fertility dynamics and the sustainability of economic growth; however, some important questions remain in our model. In this subsection, we discuss the growth rate of A_t during the transitional dynamics, the value of GDP, and the growth rates for each economic value along the BGP.

5.2.1 Growth rate of A_t

We first consider the growth rate of A_t. (17) leads to the following remark:

Remark 1. *During the transitional dynamics, a lower* A_t *or a higher* N_t *or* λ_t *implies a higher* $g_{A,t}$*.*

This can be intuitively explained as follows. A rising A_t reduces the effect of knowledge spillover and the marginal productivity of the R&D sector (see (11)). By using (4a), we obtain

$$\frac{s_t}{w_t} = \frac{\beta\lambda_t}{1+\beta\lambda_t+\gamma}. \tag{23}$$

A higher λ_t implies a higher per capita savings relative to wage income. From (17) and (23), rising N_t or λ_t increases aggregate savings, which raises the supply of funds in the asset market (see (14)). These two effects have a positive impact on investment in R&D. Moreover, from individuals' optimal decisions as discussed in Section 3, young individuals allocate their wage income to consumption when young, and to savings for consumption in old age and for raising children. Thus, we obtain the following relationship between the fertility rate and the growth rate of the stock of technological knowledge.

Remark 2. *When the economy is in transitional dynamics, one sees the following static and dynamic effects. Let us consider any period t:*

1. *Increasing (decreasing) n_t has a negative (positive) impact on s_t, which can reduce (raise) aggregate savings at period t and $g_{A,t}$.*

2. *Increasing (decreasing) n_t has a positive (negative) impact on N_{t+1}, which can raise (reduce) aggregate savings at period $t+1$ and $g_{A,t+1}$.*

Because analytical investigation of the evolution of $g_{A,t}$ is too complicated, we explore this through a numerical example (see Section 6).

5.2.2 GDP and BGP

In our model, the value of GDP is not equivalent to that of Y_t. The correct value of GDP is as given below:

$$GDP = Y_t + v_t(A_{t+1} - A_t).$$

As mentioned above, when R&D is undertaken, $\pi_t^A = v_t(A_{t+1} - A_t) - w_t L_{A,t} = 0$ holds. By using this and the final goods-market clearing condition, we obtain[11]

$$GDP = Y_t + w_t L_{A,t} \equiv Z_t.$$

The calculation of Y_t is shown in Appendix D. In this study, we consider three generations (children, young individuals, and old individuals). To calculate the per capita GDP, we define the population size of the economy at time t as

$$M_t \equiv n_t N_t + N_t + \lambda_{t-1} N_{t-1}.$$

[11] Gross domestic income (GDI) is calculated by

$$GDI = (1 - \rho n_t) w_t N_t + \pi_t A_t.$$

From (9b), (13), and (D.2), we obtain

$$GDI = w_t \left(L_{Y,t} + A_t x_t + L_{A,t}\right) + \frac{1-\alpha}{\alpha} w_t A_t x_t = \frac{1}{1-\alpha} w_t L_{Y,t} + w_t L_{A,t}.$$

Because (6a) implies $w_t L_{Y,t} = (1-\alpha) Y_t$, GDI is as given below:

$$GDI = Y_t + w_t L_{A,t}.$$

Thus, we confirm that the value of GDI is equivalent to that of GDP.

Hence, we define per capita GDP as $z_t \equiv \frac{Z_t}{M_t}$.

If the economy is on a sustainable growth path, A_t and N_t grow forever. We define BGP as that state of the economy in which the growth rates and the employment share of each sector is constant. Let us define the growth rate of a variable X_t as $g_{X,t} \equiv (X_{t+1} - X_t)/X_t$. As mentioned above, the fertility rate converges to n^* in the long run. As shown in Appendix E, we can derive the growth rates along the BGP as follows:

$$
\begin{aligned}
1 + g_M^* &= n^*, \\
1 + g_A^* &= (n^*)^{\frac{1}{1-\phi}}, \\
1 + g_Y^* = 1 + g_Z^* &= (n^*)^{\frac{1-\alpha}{1-\phi}+1}, \\
1 + g_z^* &= (n^*)^{\frac{1-\alpha}{1-\phi}}.
\end{aligned}
$$

As in Jones (1995), the growth rates along the BGP are determined by the fertility rate.

6 Numerical example

In this section, we analyze the model by using numerical examples. We set the maximum value of λ_t to 1 (that is, $\overline{\lambda} = \nu = 1$) and the minimum value of λ_t to 0.2 (that is, $\underline{\lambda} = \nu/(1+\chi) = 0.2$ leads to $\chi = 4$). The time cost of raising children is assumed to be $\rho = 0.08$, which lies between the value of Hirazawa and Yakita (2017) and Strulik et al. (2013). To ensure sustained economic growth, we assume $n^* = 1.2$. We set $\beta = 0.5$ and adjust γ to satisfy $n^* = 1.2$. According to Alvarez-Pelaez and Groth (2005), a plausible range of monopoly markup in US industry is the interval [1.05, 1.40]. To achieve this, we set $\alpha = 0.72$. We let $\phi = 0.9$ and obtain $g_z^* = 0.67$. This implies that the annual growth rate of per capita GDP is about 1.5%, assuming one period to be 35 years. We choose the remaining parameter values as $\delta = 0.05$, $\psi = 0.1$, $\chi = 4$ and set the initial values of A_t and N_t as $A_{-1} = 500$ and $N_{-1} = 50$. Figure 5 presents the evolution of the survival probability, fertility, the growth rate of the stock of technological knowledge, and the growth rate of per capita GDP from period $t = 0$ to $t = 10$.

The upper left- and upper right-side panels in Figure 5 show the evolution of the survival probability, λ_t, and the evolution of fertility, n_t, respectively. The numerical result is consistent with our theoretical result in Sections 4 and 5. That is, the fertility rate first rises and then begins to fall. Subsequently, the fertility rate rebounds and converges to the BGP.

The lower left-side panel in Figure 5 shows that $g_{A,t}$ first rises and then falls, converging to the BGP. Moreover, we notice that the evolution of $g_{z,t}$ is similar to that of $g_{A,t}$, from the lower right-side panel in Figure 5. As mentioned in the Introduction, our model predicts the population-productivity reversal. Initially, the rising fertility rate increases the population size of young individuals. This enlarges aggregate savings and the growth rate of the stock technological knowledge. Later, the increasing probability of survival into old age decreases the fertility rate and raises per capita savings. As mentioned in Remarks 1 and 2, the growth rate of the stock of technological knowledge has the following effects: (i) A decrease in n_t has a negative effect on $g_{A,t+1}$, (ii) an increase in λ_t has a positive effect on $g_{A,t}$, and (iii) an increase in A_t has a negative effect on $g_{A,t}$. Our results show that the positive effect (ii) outweighs the negative effects (i) and (iii); that is, the growth rate of the stock technological knowledge increases with time.

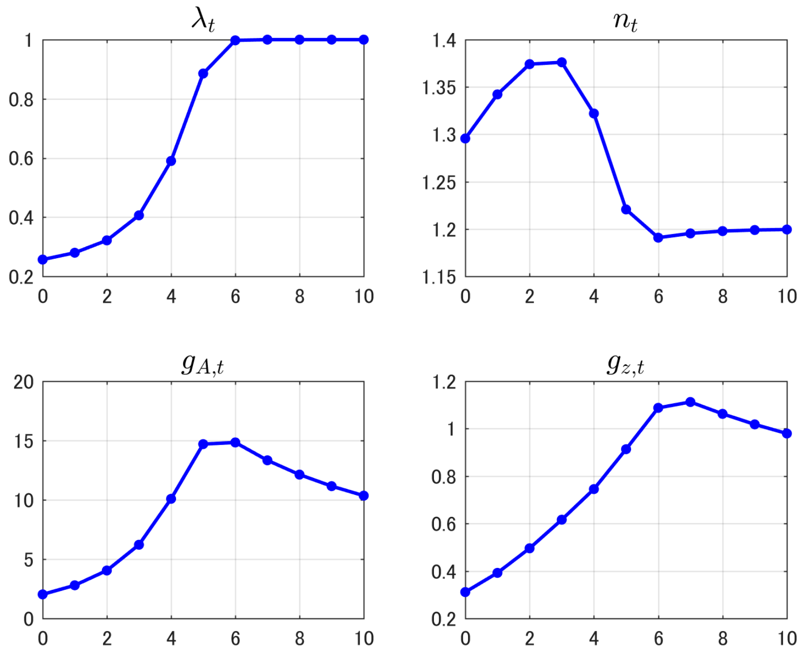

Figure 5: The evolution of λ_t, n_t, $g_{A,t}$, and $g_{z,t}$ from period $t = 0$ to $t = 10$.

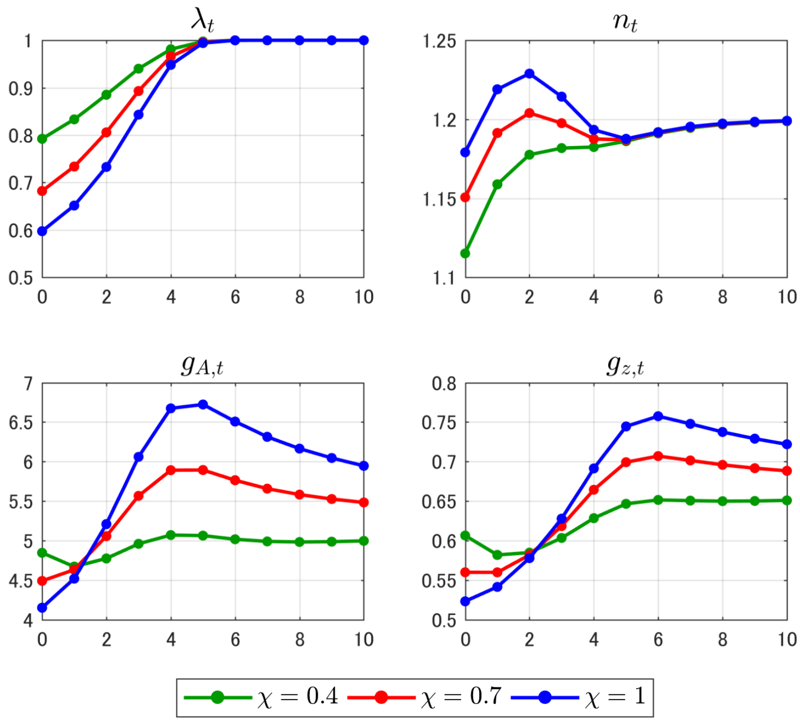

Figure 6: The evolution of λ_t, n_t, $g_{A,t}$, and $g_{z,t}$ at each value of χ from period $t = 0$ to $t = 10$.

This suggests the population-productivity reversal. Subsequently, the negative effect (iii) becomes sufficiently large, and the growth rate of the stock of technological knowledge begins to decline and converge to the BGP.

Next, we show that our main results hold even if $\lambda(w_t)$ is a concave function. The calculation of $\tilde{\chi}$ and $\frac{\psi\delta-2\rho}{\psi\delta+2\rho}$ using the same parameter values as in Figure 5 yields the value of $\tilde{\chi} \approx 0.405$ and $\frac{\psi\delta-2\rho}{\psi\delta+2\rho} \approx -0.939$. As discussed in Section 4, the condition of concavity of $\lambda(w_t)$ is $0 < \chi \leq 1$. Thus, if $0.405 < \chi \leq 1$, the first case of Proposition 1 holds. To confirm this result, we present Figure 6, which shows the evolution of λ_t, n_t, $g_{A,t}$, and $g_{z,t}$ for each value of χ from period $t = 0$ to $t = 10$. The green line represents the case where $\chi = 0.4$, and the red and blue lines show the cases where $\chi = 0.7$ and $\chi = 1$. As we can see from Figure 6, the cases where $\chi = 0.7$ and $\chi = 1$ correspond to the first case of Proposition 1, whereas the case where $\chi = 0.4$ corresponds to the second case of Proposition 1. In addition, the cases where $\chi = 0.7$ and $\chi = 1$ show that the population-productivity reversal holds under the concave function $\lambda(w_t)$.

7 Conclusion

In this study, we construct an overlapping-generations model with endogenous fertility, mortality, and R&D activities. Our model demonstrates the observed fertility dynamics from the post-Malthusian regime through the sustained growth regime to the rebound of fertility rate. Furthermore, our model depicts another mechanism for the population-productivity reversal.

There are several interesting directions for future research. First, we do not consider human capital investment, even though human capital investment is an important factor in fertility and economic growth. Most existing studies examine various important roles of the Beckerian child quantity-quality trade-off and the Ben-Porath mechanism. Moreover, in recent years, many developed countries' TFR has fallen to below the so-called replacement level (2.1 children per women) even after the fertility rebound. Our model predicts that the economy will definitely fall into a trap if the long-run fertility rate is lower than the replacement level. However, as pointed out by Strulik et al. (2013), human capital investment can ensure economic growth if human capital growth exceeds the decline in population. Future research could examine the implication of human capital in our model. Second, we do not calibrate the model. As shown in Figure 5, our numerical result does not match the actual data. In future research, incorporating human or physical capital could help match the results to actual data as in Strulik et al. (2013). Third, Luci-Greulich and Thévenon (2013) state that family policies (paid leave, childcare services, and financial transfers) of developed countries have a positive impact on fertility; that is, family policies can contribute to the recent fertility rebound. Thus, investigating the effects of child-rearing policies offers another area for further research. Fourth, as mentioned in footnote 8, we do not consider child mortality. According to Cutler et al. (2006), the child mortality rate declined significantly before 1960. Incorporating child mortality could provide interesting insights into fertility dynamics. Fifth, in life-cycle models, an expansion of public pension is crucial to individual decisions on fertility and savings. If there is a pay-as-you-go pension system, individual savings may decrease, which works against the population-productivity reversal. Future research should examine the effects of public pension in our model.

Appendix

A Proof of Proposition 1

To investigate the sign of $\frac{\partial n_t}{\partial w_t}$, we rearrange (20) as follows:

$$\frac{\partial n_t}{\partial w_t} = \gamma \frac{\Phi(w_t)}{\left[1+\beta\lambda(w_t)+\gamma\right]^2 (\rho w_t+\delta)^2},$$

where $\Phi(w_t) \equiv \delta[1+\beta\lambda(w_t)+\gamma] - \beta\lambda'(w_t)w_t(\rho w_t+\delta)$. The sign of $\frac{\partial n_t}{\partial w_t}$ is determined by that of $\Phi(w_t)$, that is, $\Phi(w_t) \gtreqless 0$ implies $\frac{\partial n_t}{\partial w_t} \gtreqless 0$. Differentiating $\Phi(w_t)$ with respect to w_t and using (19a) and (19b) yield

$$\begin{aligned}
\Phi'(w_t) &= -\beta w_t\Big[2\rho\lambda'(w_t) + \lambda''(w_t)(\rho w_t+\delta)\Big], \\
&= -\frac{\beta\nu\psi\chi w_t e^{-\psi w_t}}{\left(1+\chi e^{-\psi w_t}\right)^3}\left\{2\rho - \psi(\rho w_t+\delta) + \chi\big[2\rho+\psi(\rho w_t+\delta)\big]e^{-\psi w_t}\right\}, \\
&\equiv -\frac{\beta\nu\psi\chi w_t e^{-\psi w_t}}{\left(1+\chi e^{-\psi w_t}\right)^3}\Omega(w_t).
\end{aligned}$$

Here, $\Omega(w_t)$ satisfies

$$\begin{aligned}
\Omega(0) &= 2\rho(1+\chi) + \psi\delta(\chi-1), \\
\Omega'(w_t) &= -\psi\rho - \psi\chi\big[\rho+\psi(\rho w_t+\delta)\big]e^{-\psi w_t} < 0.
\end{aligned}$$

If $\Omega(0) < 0$, we have $\Phi'(w_t) > 0$ for all w_t. On the other hand, if $\Omega(0) > 0$, we have $\Phi'(w_t) \lesseqgtr 0$ for any $w_t \lesseqgtr \tilde{w}$, where $\tilde{w}$ is defined as $\Phi'(\tilde{w}) = 0$. Here, the condition of $\Omega(0) > 0$ is as follows:

$$\chi > \frac{\psi\delta - 2\rho}{\psi\delta + 2\rho}.$$

These results imply that the relationship between $\Phi(w_t)$ and w_t has a U-shape and $\tilde{w}$ minimizes the value of $\Phi(w_t)$.

We then substitute (18) and (19a) into $\Phi(w_t)$ as follows:

$$\Phi(w_t) = \delta(1+\gamma) + \frac{\delta\beta\nu}{1+\chi e^{-\psi w_t}} - \frac{\beta\nu\psi w_t(\rho w_t+\delta)\chi e^{-\psi w_t}}{\left(1+\chi e^{-\psi w_t}\right)^2}.$$

Using this, we obtain

$$\lim_{w_t \to 0} \Phi(w_t) = \delta(1+\gamma) + \frac{\delta\beta\nu}{1+\chi} > 0,$$
$$\lim_{w_t \to \infty} \Phi(w_t) = \delta(1+\gamma) + \delta\beta\nu > 0.$$

In addition, if $\Phi(\tilde{w}) < 0$, we obtain the following relationship:

$$\Phi(w_t) > 0 \quad when \quad 0 < w_t < \bar{w}_1 \quad or \quad w_t > \bar{w}_2,$$
$$\Phi(w_t) < 0 \quad when \quad \bar{w}_1 < w_t < \bar{w}_2,$$

where $\bar{w}_1$ and $\bar{w}_2$ are defined as $\Phi(\bar{w}_1) = 0$ and $\Phi(\bar{w}_2) = 0$ hold. In contrast, if $\Phi(\tilde{w}) > 0$, we have $\Phi(w_t) > 0$ for all w_t.

Next, we examine the condition of $\Phi(\tilde{w}) < 0$. Because $\tilde{w}$ satisfies $\Phi'(\tilde{w}) = 0$, we obtain

$$2\rho - \psi(\rho\tilde{w}+\delta) + \chi\big[2\rho + \psi(\rho\tilde{w}+\delta)\big]e^{-\psi\tilde{w}} = 0,$$
$$\Leftrightarrow 1 + \chi e^{-\psi\tilde{w}} = \frac{2\psi(\rho\tilde{w}+\delta)}{2\rho + \psi(\rho\tilde{w}+\delta)}. \tag{A.1}$$

An investigation of (A.1) gives us the following properties:

$$\frac{d\tilde{w}}{d\chi} = \frac{[2\rho + \psi(\rho\tilde{w}+\delta)]e^{-\psi\tilde{w}}}{\psi\rho + \psi\chi[\rho + \psi(\rho\tilde{w}+\delta)]e^{-\psi\tilde{w}}} > 0, \tag{A.2}$$
$$\tilde{w} \to \infty \quad when \quad \chi \to \infty. \tag{A.3}$$

Using (A.1), we rearrange the condition of $\Phi(\tilde{w}) < 0$ as given below:

$$\begin{aligned}
&\Phi(\tilde{w}) < 0,\\
\Leftrightarrow\ &\delta(1+\gamma) + \frac{\delta\beta\nu}{1+\chi e^{-\psi\tilde{w}}} - \frac{\beta\nu\psi\tilde{w}(\rho\tilde{w}+\delta)\chi e^{-\psi\tilde{w}}}{(1+\chi e^{-\psi\tilde{w}})^2} < 0,\\
\Leftrightarrow\ &\delta(1+\gamma) + \delta\beta\nu\frac{2\rho + \psi(\rho\tilde{w}+\delta)}{2\psi(\rho\tilde{w}+\delta)}\\
&\qquad - \beta\nu\psi\tilde{w}(\rho\tilde{w}+\delta)\frac{\psi(\rho\tilde{w}+\delta) - 2\rho}{2\rho + \psi(\rho\tilde{w}+\delta)}\left[\frac{2\rho + \psi(\rho\tilde{w}+\delta)}{2\psi(\rho\tilde{w}+\delta)}\right]^2 < 0,\\
\Leftrightarrow\ &4\delta(1+\gamma)\psi(\rho\tilde{w}+\delta) + 2\delta\beta\nu[2\rho + \psi(\rho\tilde{w}+\delta)] < \beta\nu\tilde{w}\left\{[\psi(\rho\tilde{w}+\delta)]^2 - 4\rho^2\right\},\\
\Leftrightarrow\ &\psi\rho\left[\frac{2(1+\gamma)}{\beta\nu} + 1\right] + \left[\frac{2(1+\gamma)\psi\delta}{\beta\nu} + (2\rho + \psi\delta)\right]\frac{1}{\tilde{w}} < \frac{\psi^2(\rho\tilde{w}+\delta)^2 - 4\rho^2}{2\delta}.
\end{aligned} \tag{A.4}$$

Let us define the left- and right-hand sides of (A.4) as $\eta_L(\tilde{w})$ and $\eta_R(\tilde{w})$. $\eta_L(\tilde{w})$ is decreasing in $\tilde{w}$ and $\lim_{\tilde{w}\to 0}\eta_L(\tilde{w}) = \infty$ and $\lim_{\tilde{w}\to\infty}\eta_L(\tilde{w}) = \psi\rho\left[\frac{2(1+\gamma)}{\beta\nu} + 1\right]$ hold. On the other hand, $\eta_R(\tilde{w})$ is increasing in $\tilde{w}$ and $\lim_{\tilde{w}\to 0}\eta_R(\tilde{w}) = \frac{\psi^2\delta^2-4\rho^2}{2\delta}$ and $\lim_{\tilde{w}\to\infty}\eta_R(\tilde{w}) = \infty$ hold. From (A.2) and (A.3), we find that there is a unique $\tilde{\chi}$ that satisfies $\eta_L(\tilde{w}) = \eta_R(\tilde{w})$. Hence, $\chi > \tilde{\chi}$ implies $\eta_L(\tilde{w}) < \eta_R(\tilde{w})$. In contrast, if $0 < \chi < \tilde{\chi}$, $\eta_L(\tilde{w}) > \eta_R(\tilde{w})$ holds (that is, $\Phi(\tilde{w}) > 0$ holds).

B Properties of $\Gamma(A_t)$

Using (18), (19a), and (22), we obtain

$$\begin{aligned}\Gamma'(A_t) &= \frac{A_t^{-\phi}}{\beta\nu}\Big\{(1-\phi)(1+\gamma+\beta\nu) + \big[(1-\phi)-(1-\alpha)(1+\gamma)\psi w_t\big]\chi e^{-\psi w_t}\Big\},\\ &\equiv \frac{A_t^{-\phi}}{\beta\nu}\Theta(w_t).\end{aligned}$$

Here, $\Theta(w_t)$ satisfies

$$\begin{aligned}\Theta(0) &= (1-\phi)(1+\gamma+\beta\nu) + (1-\phi)\chi > 0,\\ \Theta'(w_t) &= -\big[(1-\alpha)(1+\gamma)+1-\phi-(1-\alpha)(1+\gamma)\psi w_t\big]\psi\chi e^{-\psi w_t}.\end{aligned}$$

Hence, $\Theta'(w_t) \lesseqqgtr 0$ when $w_t \lesseqqgtr w_\Theta$, where $w_\Theta \equiv \frac{(1-\alpha)(1+\gamma)+1-\phi}{(1-\alpha)(1+\gamma)\psi}$. Using these results, we can define the following two cases: When $\Theta(w_\Theta) > 0$, we obtain

$$\Theta(w_t) > 0 \quad for\ all\ w_t.$$

On the other hand, when $\Theta(w_\Theta) < 0$, we obtain

$$\begin{aligned}&\Theta(w_t) > 0 \quad when \quad 0 < w_t < w_{\Gamma,1},\ w_t > w_{\Gamma,2},\\ &\Theta(w_t) < 0 \quad when \quad w_{\Gamma,1} < w_t < w_{\Gamma,2},\end{aligned}$$

where $w_{\Gamma,1}$ and $w_{\Gamma,2}$ are defined as $\Theta(w_{\Gamma,1}) = 0$ and $\Theta(w_{\Gamma,2}) = 0$. Because the sign of $\Gamma'(A_t)$ is the same as that of $\Theta(w_t)$ and w_t is determined by A_t, we can state the properties of $\Gamma(A_t)$ as in Subsection 5.1.

We then investigate the condition $\Theta(w_\Theta) < 0$. Rearranging $\Theta(w_\Theta) < 0$ yields

$$\begin{aligned}&\Theta(w_\Theta) < 0,\\ \Leftrightarrow\ &(1-\phi)(1+\gamma+\beta\nu) - (1-\alpha)(1+\gamma)\chi e^{-\psi w_\Theta} < 0.\end{aligned}$$

This implies that $\Theta(w_\Theta) < 0$ holds when $\chi > \hat{\chi}$, where $\hat{\chi}$ is defined as follows:

$$\hat{\chi} \equiv \frac{(1-\phi)(1+\gamma+\beta\nu)}{(1-\alpha)(1+\gamma)e^{-\psi w_\Theta}}.$$

In contrast, we obtain $\Theta(w_\Theta) > 0$ when $0 < \chi < \hat{\chi}$.

C Proof of Lemma 1

From Proposition 1, n_t has a local maximum (minimum) value at $w_t = \bar{w}_1$ ($w_t = \bar{w}_2$). Because $N_{t+1} \gtreqless N_t$ is equivalent to $n_t \gtreqless 1$, we examine the following two cases:[12] First, if $n_t > 1$ at $w_t = \bar{w}_2$, we obtain

$$N_{t+1} \gtreqless N_t \quad \textit{when} \quad w_t \gtreqless \hat{w}_1,$$

where $\hat{w}_1$ is defined as $n_t|_{w_t=\hat{w}_1} = 1$. This case corresponds to the panels on the left side of Figure 4. We next consider the other case. From Assumption 1, if $n_t > 1$ at $w_t = \bar{w}_1$ and $n_t < 1$ at $w_t = \bar{w}_2$, we obtain

$$\begin{aligned} &N_{t+1} > N_t \quad \textit{when} \quad \hat{w}_1 < w_t < \hat{w}_2 \;\; \textit{or} \;\; w_t > \hat{w}_3, \\ &N_{t+1} < N_t \quad \textit{when} \quad 0 < w_t < \hat{w}_1 \;\; \textit{or} \;\; \hat{w}_2 < w_t < \hat{w}_3, \end{aligned}$$

where $\hat{w}_2$ and $\hat{w}_3$ are defined as $n_t|_{w_t=\hat{w}_j} = 1$ $(j = 2, 3)$ and $\hat{w}_1 < \hat{w}_2 < \hat{w}_3$. This case corresponds to the panels on the right side of Figure 4. Using (4b), we obtain the following relationship:

$$n_t \gtreqless 1 \quad \Leftrightarrow \quad \gamma w_t \gtreqless [1 + \beta\lambda(w_t) + \gamma]\,(\rho w_t + \delta).$$

Hence, the case where $n_t > 1$ at $w_t = \bar{w}_2$ corresponds to $\gamma\bar{w}_2 > [1 + \beta\lambda(\bar{w}_2) + \gamma]\,(\rho\bar{w}_2 + \delta)$. On the other hand, the case where $n_t > 1$ at $w_t = \bar{w}_1$ and $n_t < 1$ at $w_t = \bar{w}_2$ corresponds to $\gamma\bar{w}_1 > [1 + \beta\lambda(\bar{w}_1) + \gamma]\,(\rho\bar{w}_1 + \delta)$ and $\gamma\bar{w}_2 < [1 + \beta\lambda(\bar{w}_2) + \gamma]\,(\rho\bar{w}_2 + \delta)$, respectively.

[12] We rule out the case where $n_t < 1$ at $w_t = \bar{w}_1$ because this runs contrary to the observed data.

D Calculation of Y_t

Substituting (16) into (15), we obtain

$$Y_t = \left(\frac{\alpha^2}{1-\alpha}\right)^{\alpha} A_t^{1-\alpha} L_{Y,t}, \tag{D.1}$$

We consider the labor market clearing condition to derive $L_{Y,t}$. By using (9a) and (16), the labor input into production of intermediate goods becomes

$$A_t x_t = \frac{\alpha^2}{1-\alpha} L_{Y,t}. \tag{D.2}$$

From (13) and (D.2), we obtain

$$L_{Y,t} = \frac{1-\alpha}{1-\alpha+\alpha^2}\Big[(1-\rho n_t)\, N_t - L_{A,t}\Big] \quad if \quad L_{A,t} > 0. \tag{D.3}$$

Furthermore, (10), (11), and (17) yield

$$L_{A,t} = \frac{\beta\lambda(w_t)N_t}{1+\beta\lambda(w_t)+\gamma} - A_t^{1-\phi}. \tag{D.4}$$

By using (4b), (16), (D.1), (D.3), and (D.4), we can calculate the output of final goods Y_t.

E Derivation of the growth rates along the BGP

E.1 Derivation of g_M^*

Because $N_{t+1} = n^* N_t$ and $\lambda_t = \bar{\lambda}$ hold along the BGP, the growth rate of population along the BGP is given as,:

$$1 + g_M^* = \frac{n^* N_{t+1} + N_{t+1} + \bar{\lambda} N_t}{n^* N_t + N_t + \bar{\lambda} N_{t-1}} = \frac{(n^*)^2 + n^* + \bar{\lambda}}{n^* + 1 + \frac{\bar{\lambda}}{n^*}} = n^*.$$

E.2 Derivation of g_A^*

By using (D.4) and (19d), we obtain the employment share of R&D along the BGP as follows:

$$\frac{L_{A,t}}{N_t} = \frac{\beta\bar{\lambda}}{1+\beta\bar{\lambda}+\gamma} - \frac{A_t^{1-\phi}}{N_t}.$$

Because the employment share of R&D is constant, the term $A_t^{1-\phi}/N_t$ also becomes constant along the BGP. Thus, the growth rate of A_t along the BGP is as follows:

$$(1+g_A^*)^{1-\phi} = \left(\frac{A_{t+1}}{A_t}\right)^{1-\phi} = \frac{N_{t+1}}{N_t} = n^*.$$

E.3 Derivation of g_Y^*

From (D.1), the growth rate of Y_t is determined by those of A_t and $L_{Y,t}$. (D.3) implies

$$\frac{L_{Y,t}}{N_t} = \frac{1-\alpha}{1-\alpha+\alpha^2}\left(1-\rho n^* - \frac{L_{A,t}}{N_t}\right).$$

Because the employment share of final goods production becomes a constant, the growth rate of $L_{Y,t}$ along the BGP is equal to n^*. Therefore, the growth rate of Y_t along the BGP is given by

$$1+g_Y^* = (1+g_A^*)^{1-\alpha} n^* = (n^*)^{\frac{1-\alpha}{1-\phi}+1}.$$

E.4 Derivation of g_Z^* and g_z^*

From the definition of Z_t, the growth rate of Z_t is as follows:

$$\begin{aligned} 1+g_{Z,t} = \frac{Z_{t+1}}{Z_t} &= \frac{Y_{t+1}+w_{t+1}L_{A,t+1}}{Y_t+w_tL_{A,t}} = \frac{w_{t+1}}{w_t}\frac{\frac{1}{1-\alpha}L_{Y,t+1}+L_{A,t+1}}{\frac{1}{1-\alpha}L_{Y,t}+L_{A,t}}, \\ &= \frac{w_{t+1}}{w_t}\frac{\frac{1}{1-\alpha}\frac{L_{Y,t+1}}{N_{t+1}}+\frac{L_{A,t+1}}{N_{t+1}}}{\frac{1}{1-\alpha}\frac{L_{Y,t}}{N_t}+\frac{L_{A,t}}{N_t}}\frac{N_{t+1}}{N_t}. \end{aligned}$$

Along the BGP, $\frac{L_{Y,t}}{N_t}$ and $\frac{L_{A,t}}{N_t}$ become a constant. These results and (16) imply

$$1+g_Z^* = (1+g_A^*)^{1-\alpha} n^* = (n^*)^{\frac{1-\alpha}{1-\phi}+1}.$$

Because $z_t = \frac{Z_t}{M_t}$, the growth rate of z_t along the BGP is given by

$$1+g_z^* = \frac{z_{t+1}}{z_t} = \frac{Z_{t+1}}{Z_t}\frac{M_t}{M_{t+1}} = \frac{1+g_Z^*}{1+g_M^*} = (n^*)^{\frac{1-\alpha}{1-\phi}}.$$

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