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Abstract

This paper investigates the optimal ex-ante price mechanism design of selling a single indivisible object in a market that comprises one public risky buyer and one regular risky buyer under unlimited or limited liability where resale is allowed. First, we propose an endogenous liquidation rule requiring that the public buyer acquires the object in the liquidation stage. Next, we design an optimal bankruptcy transfer to prevent the buyer's strategic default. On the basis of this liquidation rule, the optimal ex-ante price mechanism is designed to achieve the seller's upper bound revenue under unlimited and limited liability when resale cannot be prohibited prior to the liquidation stage. Comparing the two mechanisms, the results illustrate that the effect on the seller's behavior and that revenue over the liability and information change in each case. In other words, (i) when faced with limited liability buyers, the regular buyer will obtain the object in the initial market, whereas they will become the loser under the unlimited liability case; (ii) the seller's expected revenue under unlimited liability is weakly higher than that under limited liability; and (iii) when faced with the risky buyer, the seller prefers the buyer's resale behavior and is averse to the speculator only under the limited liability case.

JEL classification:D82; G33
Keywords: Mechanism design Bankruptcy Endogenous bankruptcy recovery Resale

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1 Introduction

Consider the situation in which consumers purchase resources through advance reservation. Suppliers will introduce advance reservation while paying attention to the resale market and cancellation option , and if the cancellation option is available after the advance reservation, suppliers could encounter default when the worth is less than the demander's expected value. Therefore, suppliers that use advance reservation may prefer to prevent a resale that would, according to traditional perception, raise the default risk. On the other hand, the presence of post-reservation uncertainty enhances consumers who rely on the resale market. This is especially the case in the personal retail market through online shopping websites, which have prospered along with the development of the Internet. Once a consumer purchases a resource, its value will be observed after the details of the event information (e.g., available seats and participants) are revealed, and this resource will be resold if it is more profitable. Thus, suppliers may also benefit from the resale market. Examples of this include the sale of entertainment event tickets, luxury goods, etc.

Previous studies have investigated the mechanism design for applying resale or default that ensure ex-post efficiency. When resale cannot be prohibited, Zhang and Wang (2013) configure a direct optimal mechanism design and point out that regardless of the bargaining power in the resale market, the seller always benefits more from a buyer whose relevant information is common knowledge. Even if all the buyers have private information, a similar result is also described in other studies such as Zheng (2002). Haile (2003) introduces uncertainty in the model when faced with resale opportunities and concludes that if the information distribution is symmetric, the revenue equivalence may fail due to information conveyance and auction format choice in the secondary market. In those standard frameworks, the resale encourages an anticipatory bid from agents and may impose ex-post inefficiency when the ex-post value is lower than the bid. Once default is allowed, Board (2007) extend the work from DeMarzo et al. (2005) and propose a mechanism design using the endogenous bankruptcy recovery liquidation rule. The key to the existence of symmetry equilibrium in Board (2007) is that the public common shock makes bidders bid truthfully in the initial market. In the case where shocks are asymmetric, bidders may strategically misreport in order to obtain more profits during the liquidation stage. As a result, inefficient allocation arises due to both the information asymmetry and benefit of the liquidation stage.

These suggestions pose three questions to consider: (1) How to design an ex-ante price direct mechanism including resale and default that guarantees ex-post efficiency in the case of asymmetric information and uncertainty? (2) Does the seller reveal some information to buyers via a direct mechanism? If so, what information will be conveyed and how will the revealed information influence each player's belief? (3) Under this direct mechanism, can the seller benefit from resale even if the default is allowed?

As such, in this paper, we study a simple model that enables one risk-neutral seller to sell an individual object to two risk-neutral buyers while specifically incorporating resale and default opportunities. We assume that only the "regular" buyer has private information about all market participants, while the other is designated as "public". Once any buyer accepts the sales contract offered by the seller, the shock will be revealed, and then they can offer a take-it-or-leave-it resale contract. After the secondary market is closed, the winner can strategically declare bankruptcy.

This paper studies sellers' revenue maximization problem on the basis of buyers' incentive compatibility and default behavior. In the presence of resale and limited liability, the seller will face buyers' private behaviors in three markets: an initial market, a resale market, and a liquidation market. After describing the general condition of every buyer in each market, we consider an optimal mechanism that maximizes the seller's expected revenue under unlimited liability and limited liability. In this analysis, the seller faces the problem of hidden information and hidden action because the regular buyer has private information and the winner's action in the resale market is unobservable by the seller. When the default is allowed, the regular winner also benefits from declaring bankruptcy because he has private information and there are no penalties in liquidation stage, i.e., the strategic default occurs. Moreover, the public winner will choose a strategic default on the basis of the design of the bankruptcy transfer.

To prevent the seller from strategic default, we propose endogenous liquidation recovery as "resale recovery" following the framework of Board (2007). In other words, the transaction will be forced to become invalid if any winner declares bankruptcy; however, the seller can resell an object only to the public buyer at the liquidation stage. With such a rule, bankruptcy transfer can be designed ignoring private information although a winner can strategically choose to default. This suggestion allows the seller to describe all liability in detail in an ex-ante direct revelation contract. In addition, the results on optimal bankruptcy transfer design indicate that neither strategic default nor welfare loss arise at the liquidation stage.

In Section 4, an optimal ex-ante price mechanism is designed on the basis of an analysis of buyers' resale behaviors. We first find that regardless of the level of the buyer's liability, the seller can extract more profit allowing for resale while not revealing any information to buyers. It is no surprise that both parties benefit from resale regardless of the liability: the resale can either reduce the welfare loss due to uncertainty or decrease the probability of default. In our private information environment, the regular buyer can benefit more from misreporting during resale. Regarding information disclosure, the seller never reveals anything to the winner in order to separate the information in the initial market, where a pooled recommended resale price offer from the winner is the best response for the seller. From there, a direct mechanism is designed only from information reports.

When liability is unlimited, the seller will allocate the object to the public buyer in order to extract the entire surplus in the resale market through the optimal mechanism; In other words, the risk of uncertainty is translated to the public winner. As far as limited liability exists, the public winner will get some information rent by offering higher resale price strategically because either the ex-post efficiency holds or screening resale behavior is costly. The seller will reduce the surplus gain from the resale market in order to decrease the information rent paid. Therefore, we address an important issue that has hitherto been neglected in the literature: the effect of liability and information change on sellers' behaviors and revenue in each case. If post-resale defaulting is allowed, the seller prefers to set an ex-ante price to protect at the least one buyer capable of staying solvent. Through analyzing the information distribution, we highlight the result that the seller prefers, and has the ability to set, the ex-ante price in advance to prohibit resale when the resale market becomes certain. Intuitively, in the optimal mechanism under unlimited liability, the seller's expected revenue is weakly higher than under limited liability. Thus, in some situations, the seller can prevent the resale between buyers according to "resale recovery" and the ex-post allocation will also achieve the Myerson allocation. These foregoing results illustrate that "resale recovery" and the optimal ex-ante mechanism under limited liability can both effectively prevent the public speculator from emerging, also ensuring that when facing limited liability buyers, the seller's revenue does not decrease in some cases. These phenomena have important implications for sellers' preferences for resale or speculators in real markets.

Finally, we analyze the relationship between the optimal ex-ante price mechanism and information structure and show that our results are robust in more general situations, but the price may not be unique; for example, some properties of the hazard rate cannot hold. On the other hand, if the information parameter is relevant, the seller can still achieve maximum revenue through the optimal ex-ante price mechanism but never needs to be concerned with the information correlation.

Related Literature

This work is associated with the framework of mechanism design with resale, which encompasses studies by Zhang and Wang (2013), Zheng (2002), Haile (2003), Hafalir and Krishna (2008), Calzolari and Pavan (2006), Garratt and Tröger (2006), Garratt et al. (2009), and Groes and Tranaes (1999). These studies have tended to indicate the optimal mechanism design with resale by using auction theory, which shows that for optimal mechanism, the seller prefers the first-price auction (FPA) over the second-price auction (SPA).

Zhang and Wang (2013) consider a mechanism design with a bargaining resale market. They focus on the bargaining game in a resale market that will change the seller's revenue between Myerson's revenue and efficiency revenue. They demonstrate that in optimal mechanism, the public buyer always obtains the object regardless of their bargaining power in the resale market. Our result for unlimited liability is closely related to this framework and extends also to more general cases, such as the uncertainty of the resale market; in other words, their work describes a special case, whereas we obtain more general results. Our paper extends the work of Haile (2003), which contains uncertain bidders who have only one dimension of information in the first stage.

Furthermore, the work presented herein is related to Board (2007) and Saral (2009), both of which concern the optimal auction design to maximize the seller's revenue. Board (2007) discusses the seller's preference over the choice of auction with differing forms of limited liability and designs the optimal auction format for the seller to achieve higher revenue. In his work, an endogenous liquidation rule is configured in a way the seller can resell the object over the loser when ex-post auction bankruptcy occurs, the outcome being that the seller prefers the SPA under this liquidation rule. Saral (2009) analyzes the bidder's behaviors and the seller's revenue in the SPA with limited liability. His results show that the winner's resale option increases the seller's revenue and affects the bidding strategy. Thus, when using the SPA, the seller may prefer the speculator. For further details on the topic of optimal auction design with limited liability, see DeMarzo et al. (2005) and Gorbenko and Malenko (2011).

The analysis of strategic default behavior in this paper most closely relates to that in Alary and Gollier (2004) and Rhodes-Kropf and Viswanathan (2000), both of which study the relationship between strategic default behavior and the liquidation rule (penalties payment). For pertinent studies concerning renegotiation, monitoring, and bankruptcy, see Livshits and Kovrijnykh (2015) and Martimort et al. (2015).

Throughout this paper, by analyzing the surplus gain from each winner, we explain the seller's preference for speculators and resale. Cui et al. (2014) propose also a similar argument but on the basis of a stronger assumption that the speculator faces a lower resale cost. The idea of the resale market in this paper is derived also from the financial intermediaries problem (see Park (2000) and Allen and Gale (2004) who show that financial intermediaries increase social welfare in incomplete markets).

2 Model

Consider an economy where a seller ("she") provides one indivisible object to two risk-neutral buyers ("he") facing the initial market and the secondary (resale) market. The buyer whose information is commonly known to the market participants is defined as a public buyer, namely, P. The regular agent, R, has private information. Thereafter, the economy will be regarded as the model whereby the seller only faces one public buyer and one regular buyer. A sales contract will be concluded between the seller and the buyer (now the "winner") in the initial market. Transactions, i.e., the object of trade and payments, will be then completed at the end of the secondary market. In addition, resale is permitted between the winner and loser and is modeled as a monopoly problem; in other words, the winner will provide a resale contract to maximize their expected payoff. We assume that the seller cannot reject the offered sale contract; therefore, the winner is able to resell the object although they will receive it after the resale contract is offered.

Each buyer $i \in \{P, R\}$ has type (signal) $\theta_i \in [\underline{\theta}, \bar{\theta}]$ supported by an independently identical distribution (i.i.d) function $F_i(\theta)$ and density $f_i(\theta)$, and a shock $s_i \in [\underline{s}, \bar{s}]$, which is independent of θ_i, supported with i.i.d $G_i(s)$ and density $g_i(s)$. Assume that those distributions are common knowledge. Notably the public buyer's information will commonly be observed by all the market participants; thus, the public buyer's type is common knowledge for all the player while his shock will be revealed by the regular buyer in resale market and by the seller only in liquidation stage. Denote the post using a valuation function as $\nu : (\theta_i, s_i) \mapsto R_+$, which is twice differentiable, increasing over both the parameters θ, s, and uniformly bound. After the shock s_i is revealed, the winner i will choose to use the object or propose a take-it-or-leave-it resale offer to the loser. The loser can choose to accept or reject this offer. If any resale price $P_i \in R_+$ is provided by the winner, the resale will occur if and only if $\nu(\theta_j, s_j) \geq P_i$. In addition, we assume the monotonicity to be as follows:

Assumption 1.

$$\nu(\theta_i, s_i) - \frac{\partial \nu(\theta_i, s_i)}{\partial \theta_i} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} > 0$$

increases in θ_i for every s_i.

Note that we do not need to discuss the effect of information disclosure on buyer beliefs if the inequality holds.

Taking into account the winner's resale and default behavior, the seller will design a mechanism to solve her revenue maximization problem. The timing of the unlimited liability game proceeds as follows:

(1) type θ_i is naturally observed for each buyer;

(2) the seller provides a sales contract to a buyer on the basis of the information report;

(3) shock s_i is revealed by each buyer after the trade occurs;

(4) the secondary (resale) market opens, the winner proposes a take-it-or-leave-it resale offer into the resale market, and the loser chooses to accept or not accept it;

(5) after the secondary market closes, the winner receives the object, and then makes the payment while reselling it when the resale offer is accepted.

The timing of limited liability game is the same as the unlimited liability game in periods (1)-(4). In period (5), instead, all the players will enter the liquidation stage when the winner receives the object but declares bankruptcy; otherwise, the winner will do the same as in an unlimited liability game. It is notable that once a sales contract is signed, the seller can only participate again in the liquidation stage.

In the case of unlimited liability, shocks are treated as uncertainty as the seller cannot commit it during the whole game. However, once the signals are revealed, the emergence of shock can be predicted. Therefore, those signals allow the seller to convey the resale price in every situation by designing a direct mechanism. On the other hand, when any winner declares bankruptcy, the seller can observe the public buyer's ex-post using valuation after participation in the liquidation stage. Thus, the seller can provide the winner direct mechanisms, including the bankruptcy transfer, based only on the public buyer's valuation.

In this game, we assume that the winner will never receive the object in the initial market but has resale opportunity in the secondary market; thus, the initial market also can be called reservation market. All the transactions will be completed after secondary market is closed. Therefore, the ex ante contract can be considered a promised contract, and the default will become the winner's strategy. Once the resale contract is accepted, any buyer will never benefit by breaking it strategically. Because it will reduce each buyer's ex post payoff, but never affect the acceptance of contract in the initial market.

On the basis of this game setting, we focus on perfect Bayesian equilibria in a three-stage game, initial market stage, resale market stage and liquidation stage, using backward induction, in which the unlimited liability case is characterized as a two-stage game without the liquidation stage. Accordingly, to derive the individual rationality and incentive compatibility (IR_i and IC_i) for each buyer in each stage, a direct mechanism will be designed to maximize the seller's expected revenue directly where the public buyer's type is given as $\theta_p^* \in [\underline{\theta}, \bar{\theta}]$. For simplification, a feasible ex-ante price mechanism is defined as follows:

Definition 1. *A mechanism, as quadruple parameters (ρ_i, Q_i, P_i, B_i), is designed by the seller to maximize her revenue, which depends on the buyers' signal reports.*

In general, the direct mechanism allows the seller to convey the information as much as she wants, such as the allocation rule $\rho_i(\theta_p^*, \theta_r)$, the monetary transfers $Q_i(\theta_p^*, \theta_r)$, the recommended resale price $P_i(\theta_p^*, \theta_r)$, and the bankruptcy transfer B_i. Similarly, the mechanism, as triple parameters (ρ_i, Q_i, P_i), will be offered to the buyers with unlimited liability. As mentioned above, the seller is able to recommend a resale price depending on the information report and shock prediction, but neither the resale action nor the resale price choice can be committed. Assume that the buyers report the information (θ_p^*, θ_r) truthfully; in that case, the seller can forecast the winner's resale behavior against all available states of nature as $\{(\theta_p^*, \theta_r, s_p, s_r)\}_{(s_p, s_r) \in [\underline{s}, \bar{s}]^2}$. In equilibrium, the seller believes that the winner will act with the incentive-compatible recommended resale price in each state of nature. Thus, the recommended resale price can be specified in an ex-ante direct mechanism.

In the limited liability game, the winner can commonly declare bankruptcy regardless of his ex-post payoff. If the winner i declares bankruptcy, he should pay the bankruptcy transfers B_i during the liquidation stage. Additionally, we do not allow any bankruptcy transfers greater than the social welfare during the liquidation stage. Accordingly, if the winner stays solvent, their payment is equal to an ex-ante price Q_i. The definition of the liquidation rule and the design of bankruptcy transfers will be described in detail in Section 3.

Thus, the seller's optimization problem involves the transfer of two different cases, one equivalent to the ex-ante price when the winner stays solvent and the other from liquidation. In Section 3, we will describe the liquidation rule and design the optimal bankruptcy transfers that will facilitate the characterization of the seller's optimization problem. In Section 4, we analyze the resale behavior of each winner and design the optimal mechanism under unlimited and limited liability.

3 Liquidation rule

When a bankruptcy is considered in an economy, the winner will declare bankruptcy if they can get a higher payoff in an insolvent rather than solvent state. In other words, strategic default arises in this economy. In this section, we will analyze the strategic default behavior of each winner in the case of asymmetric information and the optimal bankruptcy transfer B_i will be designed.

Because the seller faces limited liability buyers, we will then address the liquidation rule of "resale recovery", which is used in this paper as follows:

(1) if any winner declares bankruptcy, the object will be compulsorily transacted to the public buyer;

(2) otherwise, the seller will receive the promised transfers as Q_i.

This "resale recovery" rule is different from that employed by Board (2007). As in part (1), we assume that the resale contract, if accepted, will be annulled if any winner declares bankruptcy. Thus, the public buyer will receive the object in the liquidation stage whether or not he is winner. In other words, the object will be resold if the winner is a regular buyer; on the other hand, the public winner will hold the object compulsorily. According to the above definition, this liquidation recovery is close to an endogenous recovery i.e., the seller has the ability to set bankruptcy transfers for each bankrupt without disclosing the regular buyer's information. Because the public buyer's information is common knowledge during the liquidation stage, the seller can extract the entire surplus, but less than the social welfare, in the liquidation stage. Therefore, we have the following proposition:

Proposition 1. *The bankruptcy transfer is optimal if and only if* $B_i \equiv \{\nu(\theta_p^*, s_p) | s_p \in [\underline{s}, \bar{s}]\}$.

The bankruptcy transfer can be specified in an ex-ante contract after revealing the public buyer's signal even if the seller never receives any information report from the regular buyer. This is because the public buyer's ex-post valuation in each state can be forecasted on the basis of the commonly known signal θ_p^* and, in the liquidation stage, it will be revealed by the seller. Using such common knowledge, the seller will be able to specify all the liability in each liquidation state. Therefore, the seller can offer an ex-ante contract that conveys the bankruptcy payment but is never greater than the public buyer's ex-post valuation.

To clarify the importance and practicability of this liquidation rule and the bankruptcy transfers, we will explain the following aspects. Under the "resale recovery" and optimal bankruptcy transfers defined in Proposition 1, the strategic default chosen by any winner will reduce their profit. Assume that the regular buyer is the initial market winner; in that case, in accordance with the definition, the regular winner will compulsorily resell the object when they declare bankruptcy, and thus, the net payoff of regular bankrupt equals to zero. As a result, there is no incentive to motivate the regular buyer to reject the contract offered by the seller because his ex-post payoff is indifferent to whether he declares bankruptcy or becomes a loser. When the public buyer is the winner on the initial market, the result is more complex. The seller cannot commit with the resale behavior of the public buyer and thus a problem of hidden action, private resale behavior, arises. However, due to the liquidation, the resale offer will be mandatorily rejected. Thus, this rule prevents the public winner from choosing the strategic default; in other words, the public winner has no incentive to resell the object but declare bankruptcy. Furthermore, because the public bankrupt's utility equals zero in the liquidation stage, the optimal bankruptcy transfers B_i also effectively prevent any solvent public winner from choosing the strategic default. Therefore, the strategic default has never been a weakly dominated strategy for each winner and welfare loss does not arise during the liquidation stage. Then, the strategy of declaring bankruptcy will never affect the signal report in the initial market even with no penalties; hence, a direct mechanism as in Definition 1 can be designed without another information report.

Intuitively, this result shows that the asymmetric information problem will not appear in the liquidation stage. Therefore, the seller needs to observe neither any buyer's shock nor the resale behavior. Through "resale recovery", the seller can set an ex-ante price directly depending on $theta_i$. In this way, on the basis of "resale recovery" and Proposition 1, we can design an ex-ante price direct mechanism to maximize the seller's expected revenue, subject to the individual rationality and incentive compatibility constraints of each buyer in the initial market.

4 Mechanism design with unlimited and limited liability

In this section, the optimal ex-ante price mechanism will be designed according to the behavior of each buyer under unlimited and limited liability, which is useful to investigate the effect of changes in buyer's liability on the seller's behavior and revenue. In subsection 4.1, on the basis of each buyer's incentive compatibility and individual rationality in the unlimited liability case, a directly optimal ex-ante price mechanism will be designed to achieve the upper bound of the seller's revenue. When facing the limited liability buyer, the optimal ex-ante price mechanism can still be uniquely designed according to the "resale recovery" even if the resale cannot be prohibited. These results will be described in subsection 4.2.

4.1 Ex-ante price mechanism design with unlimited liability

In the unlimited liability case, the ex-ante price mechanism contains parameters ρ_i, Q_i, and P_i. In addition, the resale price will maximize the winner's surplus in the secondary market, and this resale behavior, which cannot be observed by the seller, affects not only the allocation rule but also the ex-ante price. Using backward intuition, the directly optimal mechanism will maximize the seller's surplus in the two-stage game, which depends on each buyer's equilibrium path in two markets. In addition, because the regular buyer has private information, we have an assumption as follows:

Assumption 2. *For every interval* $\Theta_r \in [\underline{\theta}, \bar{\theta}]$,

$$E_{\Theta_r} \frac{1 - H(\nu|\theta_r)}{h(\nu|\theta_r)}$$

is increasing in ν.

Note that $H(\nu|\theta_i)$ is denoted as a conditional probability distribution function and, therefore, $h(\nu|\theta_i)$ is the density function. Assumption 2 also derives from the framework of Myerson (1981, 1983) and is useful for calculating the unique equilibrium path for each buyer. Based on assumptions 1 and 2, this equilibrium path will be separately discussed for two situations, i.e., each winner sets the resale price following the seller's recommendation if it satisfies incentive compatibility.

First, consider the case where the regular buyer is the initial market winner and offers a take-it-or-leave-it resale contract to the loser at price P_r. Because the public buyer's information is common knowledge, the optimal resale price P_r must satisfy the following:

$$P_r^*(s_p) = \begin{cases} \nu(\theta_p^*, s_p) & \nu(\theta_r, s_r) < \nu(\theta_p^*, s_p) \\ > \nu(\theta_p^*, s_p) & \text{otherwise} \end{cases}. \tag{1}$$

It can be easily verified that this recommended price satisfies the public buyer's incentive compatibility constraints. In addition, suppose that the regular winner reports $\theta_r^{'}$ instead of the true signal θ_r. Then, the offer price is unchanged because it depends only on common knowledge for the regular buyer. Therefore, this price is incentive-compatible in the resale market, and the seller will recommend the resale price equal to $P_r^*(s_p)$. Furthermore, although the seller cannot observe the shock of each buyer, she will believe that the buyer will accept the recommended resale price if it satisfies the incentive compatibility constraint in the secondary market. Thus, we obtain the following result:

Lemma 1. *The resale price offer* $P_r^*(s_p)$ *is incentive-compatible if and only if it satisfies:*

$$P_r^*(s_p) = \begin{cases} \nu(\theta_p^*, s_p) & \nu(\theta_r, s_r) < \nu(\theta_p^*, s_p) \\ > \nu(\theta_p^*, s_p) & \textit{otherwise} \end{cases}.$$

Furthermore, the seller will recommend price $P_r^*(s_p)$ *when she receives the signal report as* θ_r*, and the regular winner will follow this recommendation.*

Clearly, IRR_r is an individual rationality constraint and ICR_r is the incentive compatibility constraint for the regular loser in the resale market. Furthermore, because the public winner follows the recommendation and the seller cannot observe the regular loser's shock, the ICR_r only contains the truthful type report. The final equation,

$\sigma(\theta_r, \theta_r^{'})$, shows that the resale occurs if and only if the regular buyer's ex-post valuation is greater than the resale price. According to the ICR_r, the public winner has no incentive to reject the recommended price and set up an alternative resale price on the basis of the information report from the regular loser in the resale market. Furthermore, the ICR_r shows that this recommended price must be the global maximum. In that case, the optimization problem in the resale market can be rewritten as follows:

$$\max_{P_p(s_p)} E_{\theta_r}(\nu(\theta_p^*, s_p)H(P_p(s_p)|\theta_r) + P_p(s_p)[1 - H(P_p(s_p)|\theta_r)])$$

Solving this optimization problem, we have the next result:

Lemma 2. *The resale price offer $P_p^*(s_p)$ that satisfies the following:*

$$\begin{aligned}
&\int_{\max\{\underline{s},\underline{s}_r^*\}}^{\min\{\bar{s},\bar{s}_r^*\}} \left(\frac{[\nu(\theta_r^*|\xi) - \nu(\theta_p^*, s_p)]f_r(\theta_r^*|\xi)}{\frac{\partial \nu(\theta_r^*|\xi)}{\partial \theta_r^*}} - (1 - F_r(\theta_r^*|\xi)\right)dG_r(\xi) \\
= \;& 1 - G_r(\min\{\bar{s}, \bar{s}_r^*\})
\end{aligned}$$

is incentive-compatible, is the global maximum, and is increasing in s_p where $\nu(\theta_r^*|\xi) = P_p^*(s_p)$. Furthermore, the seller will recommend a price $P_p^*(s_p)$ regardless of the information report she receives and the public winner will follow this recommendation.

Proof. Appendix □

Note that the resale occurs if and only if the regular loser's valuation is greater than the resale price $P_p^*(s_p)$. As a result, the public winner has no incentive to reject the recommended price. If $\min\{\bar{s}, \bar{s}_r^*\} = \bar{s}_r^*$, the resale price will balance two parts: the information rent and the utility of the lowest type regular loser. In other words, if resale price increases, the information rent will increase while the lowest type regular loser's utility will decrease.

On the basis of the resale behavior and price $P_p^*(s_p)$, $P_r^*(s_p)$ of each winner on the equilibrium path, the seller will choose ρ_i, Q_i to maximize her expected revenue subject to the individual rationality IR_i and incentive compatibility IC_i for all buyers $i = p$, r in the initial market. Because the public buyer has no private information, he does not have an incentive compatibility constraint. Suppose that the public buyer is the winner, if the seller receives information report as θ_r, the individual rationality constraint of the public winner can be written as follows:

$$E_{s_p}(\nu(\theta_p^*, s_p)H(P_p(s_p)|\theta_r) + P_p(s_p)[1 - H(P_p(s_p)|\theta_r)]) - Q_p(\theta_p^*, \theta_r) \geq 0 \quad IRI_p$$

As Lemma 1 shows, the public loser's individual rationality constraint in the resale market is as follows:

$$\nu(\theta_p^*, s_p) - P_r(s_p) \geq 0 \quad IRR_p$$

Next, for the regular buyer, suppose he reports the type $\theta_r^{'}$ instead of θ_r. Then, the ex-post payoff of the regular winner is as follows:

$$E_{s_p}[\int_{\underline{s}}^{s_r} \nu(\theta_r^*, s_p)dG_r(\xi) + \int_{s_r}^{\bar{s}} \nu(\theta_r, \xi)dG_r(\xi)] - Q_r(\theta_r^{'})$$

where s_r satisfies $\nu(\theta_r, s_r) = \nu(\theta_p^*, s_p)$. For the regular loser, his payoff is as follows:

$$E_{s_r}[\nu(\theta_r, s_r) - P_p(\theta_r^{'}|s_p)]\sigma(\theta_r, \theta_r^{'}|s_p)$$

Furthermore, the information report will affect the allocation rule, i.e., the regular buyer's expected payoff is calculated as follows:

$$\begin{aligned} U_r(\theta_r, \theta_r^{'}) &= \rho_p(\theta_r^{'}) E_{s_r}[\nu(\theta_r, s_r) - P_p(\theta_r^{'}|s_p)]\sigma(\theta_r, \theta_r^{'}|s_p) \\ &+ \rho_r(\theta_r^{'})[E_{s_p}\int_{\underline{s}}^{s_r} \nu(\theta_r^*, s_p) dG_r(\xi) + \int_{s_r}^{\bar{s}} \nu(\theta_r, \xi) dG_r(\xi) - Q_r(\theta_r^{'})] \end{aligned}$$

Thus, the incentive compatibility constraint of the regular buyer in the initial market can be written as follows:

$$U_r(\theta_r, \theta_r) \geq U_r(\theta_r, \theta_r^{'}) \quad IC_r$$

Moreover, in each market, the individual rationality constraint of the regular buyer can be simplified as follows:

$$\begin{aligned} E_{s_p}[\int_{\underline{s}}^{s_r} \nu(\theta_r^*, s_p) dG_r(\xi) + \int_{s_r}^{\bar{s}} \nu(\theta_r, \xi) dG_r(\xi)] - Q_r(\theta_r) \geq 0 \quad & IRI_r \\ \nu(\theta_r, s_r) - P_p(\theta_r|s_p) \geq 0 \quad & IRR_r \end{aligned}$$

Therefore, the seller's optimization problem is subject to individual rationality, incentive compatibility, and the allocation rule of each buyer in each market, which is clarified by the following equation :

$$\begin{aligned} \max_{(\rho_p, \rho_r), (Q_p, Q_r)} \quad & E_{\theta_r}[\rho_p(\theta_r) Q_p(\theta_r) + \rho_r(\theta_r) Q_r(\theta_r)] \\ \text{subject to} \quad & IRI_p,\ IRI_r,\ IRR_p,\ IRR_r,\ IC_r, \\ & \rho_p(\theta_r) + \rho_r(\theta_r) \leq 1 \end{aligned}$$

Generally, if there is an upper bound of expected revenue for all feasible mechanisms, intuitively, an optimal mechanism exists to maximize the seller's expected revenue. By this approach, the optimal allocation rule can be directly determined and the optimal ex-ante price can be derived according to the optimal allocation rule and the equilibrium path of the winner. On the basis of lemmas 1 and 2, it transpires that the optimal mechanism always exists and is formulated as follows:

Theorem 1. *The mechanism that maximizes the seller's revenue is given as follows:*

(1) Allocation rule: $(\rho_p^*, \rho_r^*) = (1, 0)$*;*

(2) Resale price: $(P_p^*(s_p), P_r^*(s_p))$ *satisfies lemmas 1 and 2 for every* θ_r*;*

(3) Ex-ante price for buyer $i \in \{p, r\}$*:*

$$\begin{aligned} Q_i^*(\theta_r) = \ & E_{\theta_r, s_p}[\nu(\theta_p^*, s_p) G_r(\min\{s_r^*(s_p), \bar{s}\}) \\ & + \int_{\min\{s_r^*(s_p), \bar{s}\}}^{\bar{s}} (\nu(\theta_r, \xi) - \frac{\partial \nu(\theta_r, \xi)}{\partial \theta_r} \frac{1 - F_r(\theta_r)}{f_r(\theta_r)}) dG_r(\xi)] - U_r(\underline{\theta}, \underline{\theta}) \end{aligned}$$

Proof. Appendix □

As the result shows, the public buyer will acquire the object even if $\nu(\theta_p^*, s_p) = 0$ for all s_p (speculators) and, in the optimal mechanism, the optimal ex-ante price will be uniquely determined. This result has further properties as discussed in what follows. First, from the optimal resale price and lemmas 1 and 2, we obtain the following result:

Corollary 1. *In any optimal mechanism identified in Theorem 1, the seller will never reveal any information of the regular buyer to the resale market, which will reduce her expected revenue.*

From the lemmas 1 and 2, it can be easily verified that the resale price recommended from the seller to the public buyer satisfies incentive compatibility if and only if it is a pooling price. Consider if this recommended price is a separate price; in that case, the regular loser will always report false information to the market in order to obtain the lowest resale price so that price is not incentive-compatible. Furthermore, the regular loser's type can be conjectured if an inverse one-to-one function of price and type exists. In this case, the public winner will have an incentive to reject the recommended price but will require the regular loser to report the shock s_r, which may increase the public winner's payoff. However, because an ex-ante price mechanism that relies on the recommended price is designed, according to the optimal allocation rule, this feasible mechanism cannot achieve the upper bound revenue and, therefore, is not the optimal mechanism. Hence, in the optimal mechanism satisfying Theorem 1, the optimal recommended resale price for the public winner must be the pooling price, i.e., the seller never reveals any information to the public winner.

This corollary shows that in any optimal mechanism satisfying Theorem 1, the seller does not disclose any additional information to the resale market, and the feasible recommended resale price for the public winner is the pooling price. Then, we have the next corollary:

Corollary 2. *The optimal ex-ante price in Theorem 1 may be greater than the winner's highest ex-post valuation, i.e., the public speculator will certainly be the winner and will achieve the upper bound revenue.*

This result has been already discussed in numerous extant frameworks of mechanism design with resale, such as Zhang and Wang (2013) and Garratt and Tröger (2006). In the basic model herein, we assume that for any information parameter, the distribution must be mutually independent. Therefore, under the optimal ex-ante price mechanism, a lower type public buyer may face an ex-ante price higher than all of their ex-post valuations. Moreover, if we suppose that the distribution of the public buyer's shock is $G_p(\underline{s}) = 1$ and assume $\nu(\underline{\theta}, \underline{s}) = 0$, the optimal ex-ante price mechanism in Theorem 1, the public buyer with $(\underline{\theta}, \underline{s})$ must be the winner, but he is merely a pure speculator.

Through this analysis, we find that even if some post-contract uncertainty exists, the seller always prefers the resale market as a feasible mechanism that will achieve higher revenue. Moreover, in the optimal mechanism, if the speculator discloses true information to the market, he will be the winner. As Theorem 1 shows, Zhang and Wang (2013)'s result, which does not consider the bargaining power in the resale market, can exist as a simple case of our result. In addition, according to a comparison of the revenue rank in each market, the results show that asymmetric information in the initial market has more of an effect on the seller's expected revenue. This result was not discussed by Haile (2003) in their exploration of certainty and uncertainty in the resale market.

4.2 Ex-ante price mechanism design under limited liability

Now we characterize the seller's optimization problem when facing a limited liability buyer. The optimal mechanism for limited liability will be designed using the "resale recovery" and a similar analysis as per the unlimited liability case. By comparing the unlimited and limited liability mechanisms, the effects of changes in distribution or liability on seller preference and behavior are obtained. To solve the optimization problem, we invoke also a similar assumption to the previous subsection, but stronger:

Assumption 3. *For every interval* $\Theta_r \in [\underline{\theta}, \bar{\theta}]$ *and for all* $s_p \in [\underline{s}, \bar{s}]$*,*

$$E_{\Theta_r} \frac{1 - H(\nu|\theta_r)H(\nu|\theta_p)}{h(\nu|\theta_r)H(\nu|\theta_p) + H(\nu|\theta_r)h(\nu|\theta_p)}$$

is increasing in ν*.*

Note that this assumption implies that Assumption 2 and, hence, the argument, which will be discussed in the proof of Lemma 3, is satisfied as well.

In general, a feasible mechanism in limited liability, which has been defined in Section 2, contains quadruple parameters ρ_i, Q_i, P_i, and B_i. In Section 3, buyers' default strategies were analyzed and the unique bankruptcy transfers B_i^* by "resale recovery" were outlined. Therefore, the seller will choose ρ_i, Q_i, and P_i, depending on the individual rationality and incentive compatibility of each buyer affected by B_i^*, to maximize her revenue.

Suppose the regular buyer is the initial market winner and receives an ex-ante price Q_r from the seller. With a similar approach to the unlimited liability case, following Lemma 1, the uniquely optimal resale price satisfies:

$$P_r(s_p|Q_r) = \begin{cases} \nu(\theta_p^*, s_p^*) & \max\{\nu(\theta_r, s_r), Q_r\} < \nu(\theta_p^*, s_p^*) \\ > \nu(\theta_p^*, s_p^*) & \text{otherwise} \end{cases}. \tag{2}$$

If the regular winner's ex-post valuation is no less than the ex-ante price, i.e., $\nu(\theta_r, s_r) \geq Q_r$, then he must stay solvent from Definition 1 and Proposition 1. Therefore, $P_r(s_p|Q_r)$ is the optimal resale price and satisfies incentive compatibility and uniqueness. On the other hand, if there are no buyers, neither a winner nor a loser, whose ex-post valuation is higher than the ex-ante price Q_r, then the regular winner will truthfully declare bankruptcy. In this case, no matter what the recommended resale price $P_r(s_p) \leq Q_r$, the expected revenue of the seller is indifferent. As a result, the seller prefers to recommend the resale price above the ex-ante price, i.e., $P_r(s_p) \geq Q_r$. Note that if $\nu(\theta_r, s_r) < Q_r \leq \nu(\theta_p^*, s_p)$, then similarly under Lemma 1, the incentive-compatible recommended resale price must satisfy $P_r(s_p|Q_r) = \nu(\theta_p^*, s_p)$.

Next, suppose the initial market winner is the public buyer with θ_p^* and the ex-ante price is Q_p. If the public winner's ex-post valuation is higher than Q_p, the public winner will participate in the resale market only when he can benefit from the resale market, which can be expressed as follows:

$$\begin{aligned} & \nu(\theta_p^*, s_p)H(P_p(s_p, Q_p)|\theta_r) + P_p(s_p|Q_p)[1 - H(P_p(s_p, Q_p)|\theta_r)] - Q_p \\ \geq\ & \nu(\theta_p^*, s_p) - Q_p. \end{aligned}$$

Because $\nu(\theta_p^*, s_p) \geq Q_p$ implies $P_p(s_p|Q_p) \geq Q_p$, the default never occurs in this case. On the other hand, the public winner will declare bankruptcy if their ex-post valuation is below Q_p and they cannot resell the object. From Definition 1 and Proposition 1, the public bankrupt's payoff becomes zero. Thus, the individual rationality constraint of the public winner who faces the lower ex-post valuation in the resale market can be written as follows:

$$(P_p(s_p|Q_p) - Q_p)[1 - H(P_p(s_p|Q_p)|\theta_r)] \geq 0$$

Because $1 - H(P_p(s_p|Q_p)|\theta_r) \geq 0$, this equation can be simplified to $P_p(s_p|Q_p) \geq Q_p$. In that case, the optimization problem for the public winner in the resale market can be written as follows:

$$\begin{array}{ll} \max\limits_{P_p(s_p|Q_p(\theta_r))} & [P_p(s_p|Q_p(\theta_r)) - Q_p(\theta_r)][1 - H(P_p(s_p|Q_p(\theta_r))|\theta_r)] \\ \text{subject to} & P_p(s_p|Q_p(\theta_r)) \geq Q_p(\theta_r) \end{array}$$

if the seller receives θ_r. On the basis of the behavior of each winner and the optimal recommended resale price, the equilibrium path can be uniquely determined and, hence, the result can be summarized as follows:

Lemma 3. *(1) the resale price offer $P_p^*(s_p|Q_p(\theta_r))$ that satisfies*

$$\begin{aligned} & \int_{\max\{\underline{s}, \underline{s}_r^*\}}^{\min\{\bar{s}, \bar{s}_r^*\}} \Big(\frac{(\nu(\theta_r^*|\xi) - \max\{\nu(\theta_p^*, s_p), Q_p\}) f_r(\theta_r^*|\xi)}{\frac{\partial \nu(\theta_r^*|\xi)}{\partial \theta_r^*}} - [1 - F_r(\theta_r^*|\xi)]\Big) dG_r(\xi) \\ = \; & 1 - G_r(\min\{\bar{s}, \bar{s}_r^*\}) \end{aligned}$$

is incentive-compatible where $\nu(\theta_r^*|\xi) = P_p^*(s_p|Q_p(\theta_r))$;

(2) The resale price offered, $P_r^*(s_p|Q_r(\theta_r))$, is incentive-compatible if and only if it satisfies the following equation:

$$P_r^*(s_p, Q_r) = \begin{cases} \nu(\theta_p^*, s_p^*) & \max\{\nu(\theta_r, s_r), Q_r\} < \nu(\theta_p^*, s_p^*) \\ > \nu(\theta_p^*, s_p^*) & \textit{otherwise} \end{cases};$$

(3) the seller will recommend the resale price $(P_p^*(s_p|Q_p(\theta_r)), P_r^*(s_p|Q_r(\theta_r)))$ to each buyer when she receives the signal report as θ_r and sets the ex-ante price as $(Q_p(\theta_r), Q_r(\theta_r))$; each buyer will follow the recommendation.

Proof. Appendix □

This lemma can be directly derived from lemmas 1 and 2 and intuitively shows that the optimal resale price is the pooling price and is affected by an ex-ante price. As a result, the winner may declare bankruptcy only when there is no resale. Lemma 3 then deduces the equilibrium path of the incentive-compatible resale price and the behavior of each winner in a solvent state.

Now, consider that the winner of the initial market is a public buyer with θ_p^*, if the regular buyer reports θ_r and the seller offers $Q_p(\theta_r)$. In such a case, the individual rationality constraint of the public winner can be calculated as follows:

$$\begin{aligned} U_p(\theta_r) = \; & \max\{\nu(\theta_p^*, s_p), Q_p(\theta_r)\} H(P_p^*(s_p|Q_p(\theta_r))|\theta_r) \\ & + P_p^*(s_p|Q_p(\theta_r))[1 - H(P_p^*(s_p|Q_p(\theta_r))|\theta_r)] - Q_p(\theta_r) \\ \geq \; & 0. \qquad\qquad IRI_p \end{aligned}$$

Note: Lemma 3 shows that formulas such as

$$\begin{aligned} P_p^*(s_p|Q_p(\theta_r)) & \geq \max\{\nu(\theta_p^*, s_p), Q_p(\theta_r)\} \\ P_p^*(s_p|Q_p(\theta_r)) & = \max\{\nu(\theta_p^*, s_p), Q_p(\theta_r)\} \end{aligned}$$

hold only if $Q_p = \nu(\bar{\theta}, \bar{s})$ is established. Therefore, IRI_p is binding only if the seller sets an ex-ante price that prevents any resale possibility. On the other hand, if the public buyer is the loser in the initial market, they will participate in the resale market if and only if the following holds:

$$U_p(\theta_r, s_p) = \nu(\theta_p^*, s_p) - P_r^*(s_p|Q_r(\theta_r)) \geq 0, \quad IRR_p$$

when the seller offers an ex-ante price $Q_r(\theta_r)$ to the regular winner. Similar to the case of unlimited liability, not only the public buyer has no private information, but also the seller cannot monitor the public winner's resale behavior in the solvent state and hence the public buyer has no incentive compatibility constraints.

Now, suppose that the regular buyer with type $theta_r$ is the initial market winner and will truthfully report. In that case, the individual rationality constraint of the regular winner in the initial market is as follows:

$$\begin{aligned} U_r(\theta_r, \theta_r) = \; & \max\{\nu(\theta_r, s_r), Q_r(\theta_r)\}H(P_r^*(s_p|Q_r(\theta_r))|\theta_r) \\ & + P_r^*(s_p|Q_r(\theta_r))[1 - H(P_r^*(s_p|Q_r(\theta_r))|\theta_r)] - Q_r(\theta_r) \\ \geq \; & 0. \qquad IRI_r \end{aligned}$$

where the ex-ante price is given as $Q_r(\theta_r)$. If the regular buyer is the initial market loser, based on the result in Lemma 3, his individual rationality constraint in the resale market can be calculated as follows:

$$\begin{aligned} U_r(\theta_r, \theta_r) = \; & \max\{\nu(\theta_r, s_r) - P_p^*(s_p|Q_p(\theta_r)), 0\}[1 - H(P_p^*(s_p|Q_p(\theta_r))|\theta_r)] \\ \geq \; & 0, \qquad IRR_r \end{aligned}$$

where $Q_p(\theta_r)$ is the ex-ante price for the public winner. If the regular buyer reports $\theta_r^{'}$ instead of θ_r, his payoff is equivalent to the following:

$$\begin{aligned} U_r(\theta_r, \theta_r^{'}) = \; & \rho_p(\theta_r^{'}) \max\{\nu(\theta_r, s_r) - P_p^*(s_p|Q_p(\theta_r^{'})), 0\}[1 - H(P_p^*(s_p|Q_p(\theta_r^{'}))|\theta_r)] \\ & + \rho_r(\theta_r^{'})[\max\{\nu(\theta_r, s_r), Q_r(\theta_r^{'})\}H(P_r^*(s_p|Q_r(\theta_r^{'}))|\theta_r) \\ & + P_r^*(s_p|Q_r(\theta_r^{'}))(1 - H(P_r^*(s_p|Q_r(\theta_r^{'}))|\theta_r)] - Q_r(\theta_r^{'})] \end{aligned} \tag{3}$$

For simplification, the regular buyer's incentive compatibility constraint resolves to the following:

$$U_r(\theta_r, \theta_r) \geq U_r(\theta_r, \theta_r^{'}) \quad IC_r$$

According to each buyer's individual rationality and incentive compatibility constraint, the seller's optimization problem in the limited liability case can be specified as per the unlimited liability case. However, the limited liability and the liquidation rule show that in insolvent states, the seller's surplus gains must be equivalent to the public buyer's valuation. Thus, in the initial market, the seller's optimization problem is an expected function that comprises two parts-the maximization problem in the solvent state and the bankruptcy state. Using Proposition 1, the seller's optimization problem is characterized as follows:

$$\begin{aligned} \max_{(\rho_p,\rho_r),(Q_p,Q_r)} \quad & E_{\theta_r,s_p}[\rho_p(\theta_r)(\max\{Q_p(\theta_r), \nu(\theta_p^*, s_p)\}H(P_p^*(s_p|Q_p(\theta_r))|\theta_r) \\ & + Q_p(\theta_r)[1 - H(P_p^*(s_p|Q_p(\theta_r))|\theta_r)]) \\ & + \rho_r(\theta_r)(\max\{Q_r(\theta_r), \nu(\theta_p^*, s_p)\}H(P_r^*(s_p|Q_r(\theta_r^{'}))|\theta_r) \\ & + Q_r(\theta_r)[1 - H(P_r^*(s_p|Q_r(\theta_r^{'}))|\theta_r)])] \\ \text{subject to} \quad & IRI_p,\ IRI_r,\ IRR_p,\ IRR_r,\ IC_r, \\ & \rho_p(\theta_r) + \rho_r(\theta_r) \leq 1 \end{aligned}$$

Using the same logic as Theorem 1 and lemmas 1-3, the optimal ex-ante price mechanism, which will achieve the upper bound revenue, will be designed. The results are summarized below:

Theorem 2. *The mechanism that maximizes the seller's revenue is given as follows:*

(1) allocation rule: $(\rho_p^*, \rho_r^*) = (0.1)$*;*

(2) Liquidation monetary transfers: $B_i^* = \nu(\theta_p^*, s_p)$ *for all* $i \in \{p, r\}$

(3) resale price: $(P_p^*(\theta_r|s_p), P_r^*(s_p))$ *that satisfies Lemma 3 for every* θ_r*;*

(4) ex-ante price: $Q_p^*(\theta_r) = \nu(\theta_p^*, s_p^*)$ *that satisfies* $E_{s_r}\hat{\nu}(\theta_r|s_r, s_p) = 1 - G_p(s_p^*)G_r(s_r^*)$ *where* $P_p^*(Q_p^*) = \nu(\underline{\theta}, s_r^*)$ *and*

$$\hat{\nu}(\theta_r|s_r) = \begin{cases} \int_{\underline{s}}^{s_p^*} \left(\frac{[Q_p^* - \nu(\theta_p^*, \xi)] f_r(\theta_r|s_r)}{\frac{\partial \nu(\theta_r|s_r)}{\partial \theta_r}} \frac{\partial P_p^*}{\partial Q_p^*} - [1 - F_r(\theta_r|s_r)]\right) dG_p(\xi) & P_p^* \in [\nu(\theta_r, \underline{s}), \nu(\theta_r, \bar{s})] \\ 0 & \textit{otherwise} \end{cases}$$

(5) ex-ante price: $Q_r^* = \nu(\theta_p^*, s_p^{**}) = \nu(\underline{\theta}, s_r^{**})$ *that satisfies* $E_{s_r}\tilde{\nu}(\theta_r|s_r, s_p) = 1 - G_p(s_p^{**})G_r(s_r^{**})$ *where*

$$\tilde{\nu}(\theta_r|s_r) = \begin{cases} \int_{\underline{s}}^{s_p^{**}} \left(\frac{[Q_r^* - \nu(\theta_p^*, \xi)] f_r(\theta_r|s_r)}{\frac{\partial \nu(\theta_r|s_r)}{\partial \theta_r}} - [1 - F_r(\theta_r|s_r)]\right) dG_p(\xi) & Q_r^* \in [\nu(\theta_r, \underline{s}), \nu(\theta_r, \bar{s})] \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Appendix □

As Theorem 2 shows, if the ex-ante price is equivalent in each winner's case, the solvent winner is more likely to appear when the regular buyer is allocated the object in the initial market. Intuitively, if the probability of the solvent winner occurring is the same in each case, the optimal ex-ante price is higher when the regular buyer becomes the initial market winner. This shows that resale reduces the probability of the seller facing bankruptcy. In addition, the optimal ex-ante price mechanism, defined by Theorem 2, has more important properties than the result under the unlimited liability case and is able to explain the behaviors of sellers in the real market. These aspects will be described below.

It can be easily verified that in the optimal ex-ante price mechanism defined by Theorem 2, the result also implies Corollary 1. Intuitively, the reason behind this is that Lemma 3 is a direct extension of lemmas 1 and 2, and because the result of Corollary 1 is based on the regular loser's incentive compatibility constraint in the resale market. Thus, if the seller reveals any regular buyer's information to the public winner, although it can decrease the probability of default, the public winner will reject the recommended resale price and propose a new optimal resale price depending on the additional information that will reduce the seller's revenue. In addition, from Lemma 3, the next property is intuitive:

Corollary 3. *The seller cannot extract the entire surplus from the public winner if resale and limited liability are allowed simultaneously.*

In the resale market, because the seller cannot monitor the winner's resale behavior, the action of the public winner becomes private under the limited liability case. As a result, the public winner has more incentive to hide their resale action to gain more surpluses from the seller. Therefore, the seller cannot extract the entire surplus from the public winner through the optimal ex-ante price mechanism. This intuition shows that including endogenous bankruptcy recovery, the public winner becomes a regular winner to the seller and, through the unmonitored resale action, acquires some information rent from the seller. Therefore, even if the final allocation in solvency states achieves the Myerson allocation when the public buyer becomes the winner of the initial market, the seller will need to pay more information rent to the market.

If resale is prohibited in the solvent state, efficient or Myerson allocation may not be achievable. When resale is allowed, it will be effective in not only improving social welfare but also reducing the possibility of buyer bankruptcy. Therefore, Myerson allocation can be achieved only if it satisfies the efficient allocation in solvent states. Further, the next result will be derived by comparing the seller's revenue upper bound under the optimal mechanism of unlimited liability and limited liability:

Corollary 4. *The upper bound of the seller's revenue in the optimal mechanism is weakly higher under unlimited liability compared with that under limited liability.*

Proof. Appendix □

In general, it is easy to verify that given the ex-ante price Q_r^*, defined by Theorem 2, for every feasible shock s_p, the seller's expected revenue is weakly lower under the limited liability case. With the expectation of the revenue under all shocks, the upper bound of the seller's revenue will also be weakly lower. As confirmed by the proof, the expected Myerson allocation is achievable in the optimal mechanism defined in Theorem 2 when the resale market becomes certain because, in this case, the allocation satisfies efficiency in the solvent state, and social welfare losses are caused only by bankruptcy.

As the preceding properties show, the ex-ante price mechanism by Theorem 2 will maximize also the seller's revenue, which is equivalent to the revenue in the case of unlimited liability. As Corollary 4 shows, in such a mechanism, the seller prefers to prohibit the resale market through the mechanism and the revenue upper bound under unlimited liability is achievable by using endogenous recovery. Per this fact, the seller is averse to the resale in the case of limited liability. On the other hand, when only the ex-ante price mechanism is available, the seller prefers the resale market when she faces uncertain buyers because it will reduce the seller's loss in the event of the buyer's bankruptcy. This conclusion can be summed up as follows:

Corollary 5. *The seller prefers resale that is correlated with the distribution of shock in the limited liability case but is averse to the public speculator in all situations.*

Our conclusions show that the seller prefers to set up an ex-ante price to prohibit the resale market when the resale market becomes certain. Therefore, in such a case, the seller is averse to the speculator whether he is public or regular. When resale cannot be prohibited, in our results, the regular speculator may emerge but social welfare loss is prevented in solvent states. Moreover, the emergence of the regular speculator will also reduce the probability of bankruptcy. In addition, the probability of the occurrence of the regular speculator is higher when higher θ_p^* is observed. However, when the secondary market becomes certain, the seller will also prevent the emergence of the regular speculator through the optimal ex-ante price mechanism. This conclusion is deduced from

Corollary 4, i.e., the seller will prefer to set up an ex-ante price that is higher than every available resale price in order to prohibit the speculator's occurrence. This important corollary illustrates the reason why the seller is averse to the buyer's resale behaviors or the speculator's emergence in reality; and without the speculator, regardless of the limited liability or unlimited responsibility, resale will always enhance the seller's expect revenue.

5 Discussion and Conclusion

In this paper, a basic model has been posited under a symmetric information range and valuation function, where an optimal ex-ante price mechanism exists under each liability situation. Furthermore, optimal mechanism has been uniquely determined on the basis of assumptions 1-3. In general, these assumptions can be extended to more relaxed contexts. The first extension is the relaxation of the virtual valuation assumption in assumption 1, i.e., virtual utility may be negative. Notably, in our framework, only the regular buyer has virtual valuation and may be less than his valuation. Therefore, the seller will believe that the regular buyer, whose virtual valuation is negative, will never participate in any market. In addition, in the resale market, the public buyer will believe that only the regular buyer with a positive virtual valuation will accept a resale offer. On the other hand, although the seller's belief will be updated, the hazard rate never changes because the virtual valuation is irrelevant to it. Therefore, according to lemmas 1-3, the conclusion remains unchanged and optimal mechanism is uniquely determined in each liability case by assumption of the hazard rate. Next, supposing that assumptions 2 and 3 do not hold, it is easy to verify that the uniqueness of the optimal ex-ante price and resale price may not hold because the concavity of the maximization function may be satisfied. However, optimal mechanism is determined on the basis of the upper bound of the seller's revenue. Then, the optimal ex-ante price also uniquely exists in the unlimited liability situation because the revenue's upper bound is unique. On the other hand, although the allocation rule in Theorem 2 is uniquely determined, the ex-ante price may not be unique. Moreover, the result reveals that all the properties contained in each mechanism, as explained, still hold and that the optimal ex-ante price mechanism can be implemented in a more general situation.

The next extension is the relevance of each buyer's information. In the case of unlimited liability, the seller does not disclose any information to the buyer, which is based on Lemma 2 and Corollary 1. In other words, the recommended resale price is independent of the information reports but depends on the joint probability distribution. Hence, even if there is a non-trivial correlation between the buyer's type and shock, the resale price uniquely exists depending on Assumption 2. Furthermore, because the upper bound of the seller's revenue is only related to the virtual valuation of the buyer, the optimal ex-ante price mechanism will still achieve its upper bound. These arguments show that the result of Theorem 1 is independent of the correlation of buyer information. Similarly, in the case of limited liability, when Assumption 3 holds, the resale occurs only in the solvent states and thus the resale price is also directly depends on the joint probability distribution. When any winner declares bankruptcy, from Proposition 1, the bankruptcy transfers just depends on the ex-post valuation of the public buyer. Therefore, the optimal ex-ante price mechanism uniquely exists even if there is a correlation between buyer information.

This article considered an environment in which the seller is unable to prohibit reselling by unlimited or limited liability buyers. In each situation, the optimal ex-ante price mechanism is designed to maximize the expected revenue of the seller. In the unlimited liability case, the optimal ex-ante price mechanism designed uniquely achieves revenue maximization for the seller. Thus, resale will ensure the efficiency and increase the seller's revenue in the optimal mechanism. In the case of limited liability, we propose an endogenous recovery, i.e., "resale recovery", to prevent the seller from the strategic default. When resale cannot be prohibited in any feasible ex-ante price mechanism, the public buyer becomes a private buyer whose surplus cannot be extracted fully by the seller. Through the analysis of each buyer's incentive-compatible equilibrium path in the optimal mechanism of Theorem 2, the seller prefers to allocate the object to the regular buyer to obtain higher revenue while the resale reduces the bankruptcy loss. In addition, these results show that when the resale market tends to be certain, the seller prefers to prohibit the winner's resale behavior through endogenous recovery and the ex-ante price mechanism, and can achieve the same revenue as in the case of unlimited liability. Through these analyses, this paper shows that when faced with a limited liability buyer, the seller is averse to the speculator rather than the resale.

Appendix

Proof of Lemma 2

Proof. First, let the recommended resale price sequence defined as $\{P_p^*(\theta_r|s_p)\}_{\theta_r\in[\underline{\theta},\bar{\theta}]}$, and suppose, in this sequence, there are at least two different elements, that is, $P_p^*(\theta_r|s_p) \neq P_p^*(\theta_r^{'}|s_p)$ for any $\theta_r \neq \theta_r^{'}$. The above inequalities are divided into two cases, that is, $P_p^*(\theta_r|s_p) > P_p^*(\theta_r^{'}|s_p)$ and $P_p^*(\theta_r|s_p) < P_p^*(\theta_r^{'}|s_p)$. Then for the first case, there is following inequality:

$$\begin{aligned} & [\nu(\theta_r, s_r) - P_p^*(\theta_r^{'}|s_p)][1 - H(P_p(\theta_r^{'}|s_p)|\theta_r)] \\ > & [\nu(\theta_r, s_r) - P_p^*(\theta_r|s_p)][1 - H(P_p(\theta_r|s_p)|\theta_r)]. \end{aligned}$$

And the second case:

$$\begin{aligned} & [\nu(\theta_r^{'}, s_r) - P_p^*(\theta_r|s_p)][1 - H(P_p(\theta_r|s_p)|\theta_r^{'})] \\ > & [\nu(\theta_r^{'}, s_r) - P_p^*(\theta_r|s_p)][1 - H(P_p(\theta_r|s_p)|\theta_r^{'})]. \end{aligned}$$

It is easy to verify that both inequalities are contrary to the IC condition. Therefore, the recommended resale price sequence satisfies incentive compatibility if and only if $P_p^*(\theta_r|s_p) = P_p^*(\theta_r^{'}|s_p)$ for any $\theta_r \neq \theta_r^{'}$, that is, the optimal recommended resale price is independent with θ_r.

Next, consider the recommended resale price $P_p^*(s_p)$. Define the inverse functions as $s_r = \nu_{\theta_r}^{-1}(P_p(s_p))$ and $\theta_r = \nu_{s_r}^{-1}(P_p(s_p))$ if $\nu(\theta_r, s_r) = P_p(s_p)$, then the optimization problem becomes to:

$$\begin{aligned} \max_{P_p(s_p)} \quad & \int_{\underline{s}}^{\nu_{\bar{\theta}}^{-1}(P_p(s_p))} \nu(\theta_p^*, s_p) dG_r(\xi) + \int_{\nu_{\underline{\theta}}^{-1}(P_p(s_p))}^{\bar{s}} P_p(s_p) dG_r(\xi) \\ & + \int_{\nu_{\bar{\theta}}^{-1}(P_p(s_p))}^{\nu_{\underline{\theta}}^{-1}(P_p(s_p))} (\nu(\theta_p^*, s_p) F_r(\nu_{\xi}^{-1}(P_p(s_p))|\xi) \\ & + P_p(s_p)[1 - F_r(\nu_{\xi}^{-1}(P_p(s_p))|\xi)]) dG_r(\xi) \end{aligned}$$

Differentiate this function with respect to P_p, then

$$\frac{\nu(\theta_p^*, s_p) G_r(\underline{s}_r^*)}{\frac{\partial \nu(\bar{\theta}, \underline{s}_r^*)}{\partial \underline{s}_r^*}} + \frac{P_p^*(s_p) G_r(\bar{s}_r^*)}{\frac{\partial \nu(\underline{\theta}, \bar{s}_r^*)}{\partial \bar{s}_r^*}} - \frac{\nu(\theta_p^*, s_p) G_r(\underline{s}_r^*)}{\frac{\partial \nu(\bar{\theta}, \underline{s}_r^*)}{\partial \underline{s}_r^*}}$$

$$
\begin{aligned}
& + \int_{\underline{s}_r^*}^{\bar{s}_r^*} (\frac{\nu(\theta_p^*, s_p) f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r,\xi)}{\partial \theta_r}} - \frac{P_p^*(s_p) f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r,\xi)}{\partial \theta_r}} + [1 - F_r(\theta_r|\xi)]) dG_r(\xi) \\
& + [1 - G_r(\bar{s}_r^*)] - \frac{P_p^*(s_p) G_r(\bar{s}_r^*)}{\frac{\partial \nu(\underline{\theta}, \bar{s}_r^*)}{\partial \bar{s}_r^*}} \\
= & \int_{\underline{s}_r^*}^{\bar{s}_r^*} (\frac{\nu(\theta_p^*, s_p) f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r,\xi)}{\partial \theta_r}} - \frac{P_p^*(s_p) f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r,\xi)}{\partial \theta_r}} + [1 - F_r(\theta_r|\xi)]) dG_r(\xi) \\
& + [1 - G_r(\bar{s}_r^*)] \\
= & \ 0. \qquad (4)
\end{aligned}
$$

where $\underline{s}_r^* = \nu_{\bar{\theta}}^{-1}(P_p^*(s_p))$ and $\bar{s}_r^* = \nu_{\underline{\theta}}^{-1}(P_p^*(s_p))$. Rewrite equation (4), then

$$
\int_{\underline{s}_r^*}^{\bar{s}_r^*} (\frac{[P_p^*(s_p) - \nu(\theta_p^*, s_p)] f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r,\xi)}{\partial \theta_r}} - [1 - F_r(\theta_r|\xi)]) dG_r(\xi) = 1 - G_r(\bar{s}_r^*)
$$

Since $P_p^*(s_p)$ is independent with θ_r, then $P_p^*(s_p)$ is incentive compatible and the global maximum.

Finally, Athey (2001) shows that a pure strategy Nash equilibrium exists in non-decreasing strategies if $U_p(s_p, P_p(s_p))$ is log-supermodular in $(s_p, P_p(s_p))$. Differentiating the ICR_r with respect to θ_r and applying the envelope theorem, the regular loser's payoff in the secondary market becomes to:

$$
\begin{aligned}
U_p(s_p, \nu_r^*) = \ & E_{\theta_r}[\int_{\underline{s}}^{s_r^{-1}(\nu_r^*)} \nu(\theta_p^*, s_p) dG_r(s_r|\theta_r) \\
& + \int_{s_r^{-1}(\nu_r^*)}^{\bar{s}} [\nu(\theta_r, \xi) - \frac{\partial \nu(\theta, \xi)}{\partial \theta_r}] dG_r(s_r|\theta_r)] - U_r(\underline{\theta}, \underline{\theta}|\nu_r^*),
\end{aligned}
$$

where $\nu_r^* = P_p^*(s_p)$ and $s_r^{-1}(\cdot) : \max\{\nu(\theta_r, \underline{s}), \min\{\cdot, \nu(\theta_r, \bar{s})\}\} \mapsto s_r$. Suppose that these inequalities, $\nu_r^* > \nu_r^{'} = P_p^*(s_p^{'})$ and $s_p < s_p^{'}$, is hold, then since ν_r^* is the global maximum for s_p and given Assumption 1, we have:

$$
\begin{aligned}
U_p(\nu_r^*) - U_p(\nu_r^{'}) = \ & E_{\theta_r} \int_{s_r^{-1}(\nu_r^{'})}^{s_r^{-1}(\nu_r^*)} (\nu(\theta_p^*, s_p) - [\nu(\theta_r, \xi) - \frac{\partial \nu(\theta, \xi)}{\partial \theta_r})]) dG_r(s_r|\theta_r) \\
& - U_r(\underline{\theta}, \underline{\theta}|\nu_r^*) + U_r(\underline{\theta}, \underline{\theta}|\nu_r^{'}) \\
> \ & 0.
\end{aligned}
$$

On the other hand, $U_p(s_p^{'}, \nu_r^*) - U_p(s_p^{'}, \nu_r^{'})$ must be positive because $\nu(\theta_p^*, s_p) < \nu(\theta_p^*, s_p^{'})$. Hence this is contrary to the fact that $\nu_r^{'} = P_p^*(s_p^{'})$ is the global maximum when $s_p^{'}$ is observed. Therefore, $P_p^*(s_p)$ is increasing in s_p and $\frac{\partial P_p(s_p)}{\partial s_p} > 0$. Intuitively, $U_p(s_p, P_p^*(s_p))$ is also increasing in s_p.

Using these foregoing results, we can show that $U_p(s_p, P_p(s_p))$ is log-supermodular in $(s_p, P_p(s_p))$. By redefining the payoff function as:

$$
U_p(s_p, x) - \nu(\theta_p^*, s_p) = E_{\theta_r} \int_{\bar{s}}^{s_r^{-1}(\nu_r^*)} [\nu(\theta_p^*, s_p) - \nu(\theta_r, \xi) + \frac{\partial \nu(\theta, \xi)}{\partial \theta_r}] dG_r(s_r|\theta_r),
$$

where $x = \nu(\theta_p^*, s_p) - P_p(s_p)$. Taking the cross-partial verifies that the following equation:

$$
[U_p(s_p, x) - \nu(\theta_p^*, s_p)] \frac{\partial^2 U_p}{\partial x \partial s_p} - \frac{\partial U_p}{\partial s_p} \frac{\partial U_p}{\partial x} \qquad (5)
$$

is negative. First, take the boundary point $\bar{P}_p(s_p)$ such that $U_p(s_p,x)-\nu(\theta_p^*,s_p)$, then equation (5) equals $-\frac{\partial U_p}{\partial s_p}\frac{\partial U_p}{\partial x}$, which is negative. Therefore, taking the derivative at x in equation (5) when it equals 0, is just needed to confirm this differential equation is negative. Substituting for $U_p(s_p,x)-\nu(\theta_p^*,s_p)$ at equation (5) equals to 0, this equation's sign must equivalent to:

$$\frac{\partial[U_p(s_p,x)-\nu(\theta_p^*,s_p)]}{\partial s_p}(\frac{\partial U_p}{\partial x}\frac{\partial^3 U_p}{\partial^2 x\partial s_p}-\frac{\partial^2 U_p}{\partial x\partial s_p}\frac{\partial^2 U_p}{\partial^2 x}).$$

From the definition of hazard rate in Assumption 2, it is easy to verify that $\frac{\partial U_p}{\partial x}$ is log-submodular in (θ_r,x), and therefore:

$$\frac{\partial^2 \ln\frac{\partial U_p}{\partial x}}{\partial\theta_r\partial x}:\overset{\text{sgn}}{=}\frac{\partial U_p}{\partial x}\frac{\partial^3 U_p}{\partial\theta_r\partial^2 x}-\frac{\partial^2 U_p}{\partial\theta_r\partial x}\frac{\partial^2 U_p}{\partial^2 x}<0$$

Furthermore, since $\frac{\partial P_p}{\partial s_p}>0$, then

$$\frac{\partial^2 \ln\frac{\partial U_p}{\partial x}}{\partial s_p\partial x}=\frac{\partial^2 \ln\frac{\partial U_p}{\partial x}}{\partial\nu\partial x}\frac{\partial\nu}{\partial s_p}<0.$$

Thus, $U_p(s_p,P_p(s_p))$ is log-supermodular in $(s_p,P_p(s_p))$, and, hence, $P_p^*(s_p)$ is uniquely determined by s_p. □

Proof of Theorem 1

Proof. Differentiate the IC_r with respect to θ_r:

$$\begin{aligned}\frac{dU_r(\theta_r,\iota)}{d\theta_r} &= E_{s_p}(\rho_p(\iota)[\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_p^*(s_p)|\theta_r)]-Q_r(\iota)]\\ &\quad+\rho_r(\iota)\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_r^*(s_p)|\theta_r)]).\end{aligned}$$

Apply the envelope theorem:

$$\begin{aligned}U_r(\theta_r,\theta_r) = &\ E_{s_p}\int_{\underline{\theta}}^{\theta_r}([\rho_p(\iota)\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_p^*(s_p)|\theta_r)]-Q_r(\iota)]\\ &+\rho_r(\iota)\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_r^*(s_p)|\theta_r)])F_r(\iota)d\iota+U_r(\underline{\theta},\underline{\theta}).\end{aligned}$$

Therefore, the seller's optimization problem can be rewritten as follows:

$$\begin{aligned}\max_{\rho_p,\rho_r}\ & E_{\theta_r,s_p}[\rho_p(\theta_r)(\nu(\theta_p^*,s_p)H(P_p^*(s_p)|\theta_r)+P_p^*(s_p)[1-H(P_p^*(s_p)|\theta_r)])\\ &+\rho_r(\theta_r)[\nu(\theta_p^*,s_p)-P_r^*(s_p)]H(P_r^*(s_p)|\theta_r)\\ &+\rho_p(\theta_r)[\nu(\theta_r,s_r)-P_p^*(s_p)][1-H(P_p^*(s_p)|\theta_r)]\\ &+\rho_r(\theta_r)(P_r^*(s_p)H(P_r^*(s_p)|\theta_r)+\nu(\theta_r,s_r)[1-H(P_r^*(s_p)|\theta_r)])\\ &-\int_{\underline{\theta}}^{\theta_r}(\rho_p(\iota)\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_p^*(s_p)|\theta_r)]\\ &+\rho_r(\iota)\frac{\partial\nu(\iota,s_r)}{\partial\theta_r}[1-H(P_r^*(s_p)|\theta_r)])dF_r(\iota)]-U_r(\underline{\theta},\underline{\theta})\\ =\ & E_{\theta_r,s_p}[\rho_p(\theta_r)(\nu(\theta_p^*,s_p)H(P_p^*(s_p)|\theta_r)\end{aligned}$$

$$
\begin{aligned}
&+[\nu(\theta_r, s_r) - \frac{\partial \nu(\theta, s_r)}{\partial \theta_r}][1 - H(P_p^*(s_p)|\theta_r)]) \\
&+\rho_r(\theta_r)(\nu(\theta_p^*, s_p)H(P_r^*(s_p)|\theta_r) \\
&+[\nu(\theta_r, s_r) - \frac{\partial \nu(\theta, s_r)}{\partial \theta_r}][1 - H(P_r^*(s_p)|\theta_r)])] - U_r(\underline{\theta}, \underline{\theta})
\end{aligned}
$$

Since $P_p^*(s_p)$ is the global maximum and the virtual utility increases in θ_r, then for every s_p, there has:

$$
\begin{aligned}
& \nu(\theta_p^*, s_p)H(P_p^*(s_p)|\theta_r) + [\nu(\theta_r, s_r) - \frac{\partial \nu(\theta, s_r)}{\partial \theta_r}][1 - H(P_p^*(s_p)|\theta_r)] \\
> \; & \nu(\theta_p^*, s_p)H(P_r^*(s_p)|\theta_r) + [\nu(\theta_r, s_r) - \frac{\partial \nu(\theta, s_r)}{\partial \theta_r}][1 - H(P_r^*(s_p)|\theta_r)]
\end{aligned}
$$

Hence $(\rho_p^*(\theta_r), \rho_r^*(\theta_r)) = (1, 0)$. Finally, since each winner has unlimited liability, the optimal ex ante price must be equivalent to the upper bound of the seller's expected revenue. That is, the unique optimal ex ante price $(Q_p^*(\theta_r), Q_r^*(\theta_r))$ satisfies:

$$
\begin{aligned}
Q_i^*(\theta_r) = \; & E_{\theta_r, s_p}[\nu(\theta_p^*, s_p)G_r(\min\{s_r^*(s_p), \bar{s}\}) \\
& + \int_{\min\{s_r^*(s_p), \bar{s}\}}^{\bar{s}} (\nu(\theta_r, \xi) - \frac{\partial \nu(\theta_r, \xi)}{\partial \theta_r} \frac{1 - F_r(\theta_r)}{f_r(\theta_r)})dG_r(\xi)] - U_r(\underline{\theta}, \underline{\theta}).
\end{aligned}
$$

Hence this proof is completed. □

Proof of Lemma 3

Proof. Note that the second part of this lemma is intuitive, that is, the optimal resale price must be higher than the ex ante price and the public loser has no private information. To prove the part one, let us to divide the optimization problem of the public winner into two cases. The first case is that, when the public winner's ex post valuation is higher than Q_p, the optimization problem becames:

$$
\max_{P_p(s_p, Q_p)} \nu(\theta_p^*, s_p)H(P_p(s_p, Q_p)|\theta_r) + P_p(s_p, Q_p)[1 - H(P_p(s_p, Q_p)|\theta_r)].
$$

Equation (1) depends on the fact that $P_p^*(s_p, Q_p) \geq \nu(\theta_p^*, s_p) \geq Q_p$. Thus from Lemma 2, the result is optimal and incentive compatible. The next case is
$\nu(\theta_p^*, s_p) < Q_p$, then:

$$
\max_{P_p(s_p, Q_p)} (P_p(s_p, Q_p) - Q_p)[1 - H(P_p(s_p, Q_p)|\theta_r)].
$$

By Proposition 1, the bankrupt's payoff is zero, then this optimization problem can be rewritten as:

$$
\max_{P_p(s_p, Q_p)} Q_pH(P_p(s_p, Q_p)|\theta_r) + P_p(s_p, Q_p)[1 - H(P_p(s_p, Q_p)|\theta_r)].
$$

and, therefore, from Lemma 2, the optimal recommendation price must satisfy:

$$
\begin{aligned}
& \int_{\max\{\underline{s}, \underline{s}_r^*\}}^{\min\{\bar{s}, \bar{s}_r^*\}} (\frac{(\nu(\theta_r^*|\xi) - \max\{\nu(\theta_p^*, s_p), Q_p\})f_r(\theta_r^*|\xi)}{\frac{\partial \nu(\theta_r^*|\xi)}{\partial \theta_r^*}} - [1 - F_r(\theta_r^*|\xi)])dG_r(\xi) \\
= \; & 1 - G_r(\min\{\bar{s}, \bar{s}_r^*\}).
\end{aligned}
$$

This proof is completed. □

Proof of Theorem 2

Proof. Differentiate the equation (3) with respect to θ_r, and apply the envelope theorem:

$$\begin{aligned} U_r(\theta_r, \theta_r) = \quad & E_{s_p} \int_{\underline{\theta}}^{\theta_r} (\rho_p(\iota) \frac{\partial \nu(\iota, s_r)}{\partial \theta_r} [1 - H(P_p^*(s_p|Q_p(\iota))|\iota)] \\ & + \rho_r(\iota) \frac{\partial \nu(\iota, s_r)}{\partial \theta_r} [1 - H(P_r^*(s_p|Q_r(\iota))|\iota)]) + U_r(\underline{\theta}, \underline{\theta}) \end{aligned} \quad (6)$$

Equation (6) shows that the optimal ex ante price for each winner must be the pooling price.

To show the optimal ex ante price is uniquely determined, it needs to be verified that the seller's expected payoff is log-supermodular in (θ_p, Q_i). From Proposition 1, the seller's payoff is equivalent to the public buyer's valuation if any winner declares bankruptcy. The seller's payoff loss in insolvent states is $E_{\theta_r,(s_p,s_r)}[Q_i - \nu(\theta_p, s_p)]$, and the seller's payoff is:

$$U_s(\theta_p, Q_i) = Q_i - E_{\theta_r,(s_p,s_r)}[Q_i - \nu(\theta_p, s_p)]$$

DeMarzo et al. (2005) show that an increasing ex ante price function exists if and only if

$$\frac{\partial^2}{\partial \theta_p \partial Q_i} \log U_s(\theta_p, Q_i) \quad (7)$$

is log-supermodular. Assumption 3 shows that $\frac{\partial^2}{\partial^2 Q_i} \log U_s(\theta_p, Q_i) \frac{\partial Q_i}{\partial \theta_p}$ is negative at the point of $\frac{\partial^2}{\partial \theta_p \partial Q_i} \log U_s(\theta_p, Q_i) = 0$, that is, equation (7) is log-supermodular using the same logic in the proof of Lemma 2. Hence, there exists an increasing ex ante price function depending on θ_p.

Since the optimal ex ante price is determined uniquely, the price must solve the maximization problem of the seller. Now, suppose that the public buyer is the initial market winner, then by Proposition 1 and Lemma 3, the seller's optimization problem can be rewritten as

$$\begin{aligned} \max_{Q_p} \quad & \int_{\underline{s}}^{\nu_{\theta_p^*}^{-1}(Q_p)} [\int_{\underline{s}}^{\nu_{\underline{\theta}}^{-1}(P_p^{-1}(Q_p))} \nu(\theta_p^*, \iota) dG_r(\xi) + \int_{\nu_{\underline{\theta}}^{-1}(P_p^{-1}(Q_p))}^{\bar{s}} Q_p dG_r(\xi) \\ & + \int_{\nu_{\bar{\theta}}^{-1}(P_p^{-1}(Q_p))}^{\nu_{\underline{\theta}}^{-1}(P_p^{-1}(Q_p))} (\nu(\theta_p^*, \iota) F_r(\nu_\xi^{-1}(P_p^{-1}(Q_p))|\xi) \\ & + Q_p[1 - F_r(\nu_\xi^{-1}(P_p^{-1}(Q_p))|\xi)]) dG_r(\xi)] dG_p(\iota) + \int_{\nu_{\theta_p^*}^{-1}(Q_p)}^{\bar{s}} Q_p dG_p(\iota) \end{aligned}$$

Differentiate equation (8) with respect to Q_p, then

$$\begin{aligned} & \int_{\underline{s}}^{s_p^*} [\int_{\underline{s}_r^*}^{\bar{s}_r^*} (\frac{[Q_p^*(\theta_r) - \nu(\theta_p^*, \iota)] f_r(\theta_r|\xi)}{\frac{\partial \nu(\theta_r, \xi)}{\partial \theta_r}} \frac{\partial P_p^*}{\partial Q_p^*} - [1 - F_r(\theta_r|\xi)]) dG_r(\xi) \\ & -1 + G_r(\bar{s}_r^*)] dG_p(\iota) \\ = \quad & 1 - G_p(S_p^*) \end{aligned} \quad (8)$$

Similarly, if the regular buyer is the initial market winner, the seller's optimization problem is:

$$\max_{Q_r} \quad \int_{\underline{s}}^{\nu_{\theta_p^*}^{-1}(Q_r)} [\int_{\underline{s}}^{\nu_{\bar{\theta}}^{-1}(Q_r)} \nu(\theta_p^*, \iota) dG_r(\xi) + \int_{\nu_{\underline{\theta}}^{-1}(Q_r)}^{\bar{s}} Q_r dG_r(\xi)$$

$$
\begin{aligned}
&+\int_{\nu_{\bar{\theta}}^{-1}(Q_r)}^{\nu_{\underline{\theta}}^{-1}(Q_r)}(\nu(\theta_p^*,\iota)F_r(\nu_\xi^{-1}(Q_r|\xi)) \\
&+Q_r[1-F_r(\nu_\xi^{-1}(Q_r|\xi)])dG_r(\xi)]dG_p(\iota)+\int_{\nu_{\theta_p^*}^{-1}(Q_r)}^{\bar{s}}Q_rdG_p(\iota) \qquad (9)
\end{aligned}
$$

Differentiate equation (9) with respect to Q_r, then

$$
\begin{aligned}
&\int_{\underline{s}}^{s_p^{**}}[\int_{\underline{s}_r^*}^{\bar{s}_r^*}(\frac{[Q_r^*(\theta_r)-\nu(\theta_p^*,\iota)]f_r(\theta_r|\xi)}{\frac{\partial\nu(\theta_r,\xi)}{\partial\theta_r}}-[1-F_r(\theta_r|\xi)])dG_r(\xi) \\
&-1+G_r(\bar{s}_r^*)]dG_p(\iota) \\
=\;& 1-G_p(s_p^{**}) \qquad (10)
\end{aligned}
$$

Therefore, there exists a unique optimal ex ante price $Q_i^*(\theta_r)$ if and only if equations (8) and (10) hold.

Finally, to varify the optimal allocation rule, the expected revenue rank of upper bound under the optimal ex ante price in each case of different winners needs to be confirmed. Suppose that the optimal ex ante price (Q_p^*, Q_r^*) satisfies equations (8) and (10), yields from the IC_r, the seller's optimization problem becomes:

$$
\begin{aligned}
\max_{\rho_p,\rho_r}\; & E_{\theta_r}[\rho_p(\theta_r)(\nu(\theta_p^*,s_p^*)H(P_p^*(s_p)|\theta_r) \\
&+[\nu(\theta_r,s_r)-\frac{\partial\nu(\theta,s_r)}{\partial\theta_r}][1-H(P_p^*(s_p)|\theta_r)]) \\
&+\rho_r(\theta_r)(\nu(\theta_p^*,s_p^*)H(Q_r^*|\theta_r) \\
&+[\nu(\theta_r,s_r)-\frac{\partial\nu(\theta,s_r)}{\partial\theta_r}][1-H(Q_r^*|\theta_r)])]-U_r(\underline{\theta},\underline{\theta})
\end{aligned}
$$

Consider that there is a price Q_r such that $Q_r = P_p^* > Q_p^*$. Since $P_p^* > Q_p^*$, for each s_p, the difference of expected revenue from each winner is equivalent to:

$$
\begin{aligned}
&E_{\theta_r}[\int_{\underline{s}_r^*}^{\bar{s}_r^*}(\nu(\theta_p^*,s_p)H(P_p^*|\theta_r)+[\nu(\theta_r,\xi)-\frac{\partial\nu(\theta,\xi)}{\partial\theta_r}][1-H(P_p^*|\theta_r)] \\
&-\nu(\theta_p^*,s_p)H(Q_r|\theta_r)-[\nu(\theta_r,\xi)-\frac{\partial\nu(\theta,\xi)}{\partial\theta_r}][1-H(Q_r|\theta_r)])dG_r(\xi) \\
&+\int_{\bar{s}_r^*}^{\bar{s}}(\nu(\theta_r,\xi)-\frac{\partial\nu(\theta,\xi)}{\partial\theta_r}-\nu(\theta_r,\xi)+\frac{\partial\nu(\theta,\xi)}{\partial\theta_r})dG_r(\xi) \\
&+[\nu(\theta_p^*,s_p)-\nu(\theta_p^*,s_p)]G_r(\underline{s}_r^*)]-U_r^*(\underline{\theta},\underline{\theta})+U_r(\underline{\theta},\underline{\theta}) \\
=\;& -U_r^*(\underline{\theta},\underline{\theta})+U_r(\underline{\theta},\underline{\theta}) \\
\leq\;& 0
\end{aligned}
$$

The final inequality depends on the fact that $U_r^*(\underline{\theta},\underline{\theta}) \geq U_r(\underline{\theta},\underline{\theta})$ since $Q_p^* < Q_r$. That is, $U_r^*(\underline{\theta},\underline{\theta}) = 0$ implies $U_r(\underline{\theta},\underline{\theta}) = 0$. Since Q_r^* is the global maximum, then $(\rho_p^*, \rho_r^*) = (0, 1)$, and this proof is completed. □

Proof of Corollary 5

Proof. To shows this corollary, we need to verify that the upper bound of the expected revenue is lower under the limited liability case and at least, a distribution pair (G_i, g_i) exists such that the revenue is equivalent in each optimal mechanism. The first part is intuitive. The reason is that, under the unlimited liability case, the expected revenue upper bound in each s_p must be achieved, but it may not be achievable for every s_p in the limited liability case. To show the second part, now, suppose the emergence of public buyer's shock certainty, that is, there exists s_p^* such that

$$G_p(s_p) = \begin{cases} 0 & s_p < s_p^* \\ 1 & s_p \geq s_p^* \end{cases}.$$

From Theorem 1, the upper bound revenue in the unlimited liability case is equivalent to:

$$\begin{aligned} &E_{\theta_r}(\nu(\theta_p^*, s_p^*)H(P_p^*(s_p^*)|\theta_r) \\ &+[\nu(\theta_r, s_r) - \frac{\partial \nu(\theta_r, s_r)}{\partial \theta_r}\frac{1-F(\theta_r)}{f(\theta_r)}][1 - H(P_p^*(s_p^*)|\theta_r)]) - U_r(\underline{\theta}, \underline{\theta}). \end{aligned} \tag{11}$$

On the other hand, the upper bound revenue in the limited liability case is equivalent to:

$$\begin{aligned} &E_{\theta_r}(\nu(\theta_p^*, s_p^*)H(Q_r^*(s_p^*)|\theta_r) \\ &+[\nu(\theta_r, s_r) - \frac{\partial \nu(\theta_r, s_r)}{\partial \theta_r}\frac{1-F(\theta_r)}{f(\theta_r)}][1 - H(Q_r^*(s_p^*)|\theta_r)]) - U_r(\underline{\theta}, \underline{\theta}). \end{aligned} \tag{12}$$

Therefore, in this case, equations (11) and (12) are equivalent since $Q_r^*(s_p^*) = P_p^*(s_p^*)$. Hence this proof is completed. □

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